G. Independent sets and functions

This section is about applications to ultrafilter theory of the following combinatorial property of infinite sets.

Theorem 1. Let κ be an infinite cardinal number. There exists a family $\mathcal{I}$ of 2^κ functions from κ to κ which are independent in the following sense. For any finite subfamily $\mathcal{I}_0$ of $\mathcal{I}$ and any function $H: \mathcal{I}_0 \longrightarrow \kappa$, there exists an $\alpha \in \kappa$ such that $f(\alpha) = H(f)$ for all $f \in \mathcal{I}_0$.

~~In other words, for any finitely many functions in~~ $\cancel{\mathcal{I}_0}$, ~~so~~

In other words, for any finitely many functions $f_1, \dots, f_n \in \mathcal{I}$, the n-tuple $(f_1(\alpha), \dots, f_n(\alpha))$ ranges over all of κ^n as α ranges over κ.

Proof. It suffices to define 2^κ independent functions into κ from some set S of cardinality κ, for the domain can be changed to κ by simply composing the functions with a bijection from κ to S. Let S be the set of all pairs ~~(T, p) where T is a finite~~ (T, t) where T is a finite subset of κ and t is a function from the power set $\mathcal{P}(T)$ into κ. As κ is infinite, there are κ possible T's and, for each T, κ possible

t's, so S has cardinality κ. ~~The desired independent functions~~ For each subset A of κ, let $f_A : S \to \kappa$ be defined by

$$f_A(T,t) = t(T \cap A).$$

The proof will be completed by showing that these functions f_A are ~~independent~~ distinct (so there are 2^κ of them) and independent. For this it suffices to find, for any finitely many distinct $A_1, \ldots, A_n \in \kappa$ and any $\beta_1, \ldots, \beta_n \in \kappa$, an element (T,t) of S for which $f_{A_i}(T,t) = \beta_i$ for each i. Let the A_i's and β_i's be given, and let T be a finite subset of κ such that the n intersections $T \cap A_i$ are distinct. Such a T can be found by simply taking one ~~point~~ element from the symmetric difference of each pair of A_i's. Define t by setting $t(T \cap A_i) = \beta_i$, which makes sense as the $T \cap A_i$ are distinct, and extend t arbitrarily to $\mathcal{P}(T)$. Then, for each i,

$$f_{A_i}(T,t) = t(T \cap A_i) = \beta_i. \quad \square$$

Corollary 2. Let κ be an infinite cardinal number. There ~~are 2^κ sets~~ exists a family $\mathcal{J}$ of 2^κ subsets of κ that are independent in the sense that, for any ~~finite subfamily $\mathcal{J}_0$ of $\mathcal{J}$ and any~~ two disjoint finite subfamilies $\mathcal{J}^+$ and $\mathcal{J}^-$ of $\mathcal{J}$, there is an $\alpha \in \kappa$ such that $\alpha \in X$ for every $X \in \mathcal{J}^+$ but for no $X \in \mathcal{J}^-$.

In other words, the intersection of any finitely many members of $\mathcal{J}$ and the complements of any finitely many other members of $\mathcal{J}$ is nonempty.

Proof. Let $\mathcal{F}$ be a family of 2^κ independent functions as in Theorem 1. For each $f \in \mathcal{F}$ let $X(f) = \{\alpha \in \kappa \mid f(\alpha) = 0\}$. These $X(f)$'s are distinct, by the independence of $\mathcal{F}$ so there are 2^κ of them. To see that the family $\mathcal{J} = \{X(f) \mid f \in \mathcal{F}\}$ has the desired independence property, suppose $\mathcal{J}^+$ and $\mathcal{J}^-$ are given, and apply the independence of $\mathcal{F}$ as follows. Let $\mathcal{F}_0 = \{f \in \mathcal{F} \mid X(f) \in \mathcal{J}^+ \cup \mathcal{J}^-\}$, and, for $f \in \mathcal{F}_0$, set $H(f) = 0$ (resp. 1) if $X(f) \in \mathcal{J}^+$ (resp. $\mathcal{J}^-$). Let $\alpha \in \kappa$ be such that $f(\alpha) = H(f)$ for all $f \in \mathcal{F}_0$. Then $\alpha \in X(f)$ for each $X(f) \in \mathcal{J}^+$ and $\alpha \notin X(f)$ for each $X(f) \in \mathcal{J}^-$. $\square$

The first application of these results supersedes Proposition C4 by giving the precise number of ultrafilters on any infinite set.

Theorem 3. There are exactly 2^{2^κ} ultrafilters on any infinite set of cardinality κ.

<u>Proof</u>. There are at most 2^{2^κ} such ultrafilters because there are only 2^κ subsets, and therefore only 2^{2^κ} families of subsets, of a set of size κ. For the reverse inequality, it suffices to construct 2^{2^κ} ultrafilters on κ itself, since they can be transferred by a bijection to any set of cardinality κ. Let $\mathcal{J}$ be a family of 2^κ independent subsets of κ, as in Corollary 2. For each of the 2^{2^κ} subfamilies $\mathcal{Z}$ of $\mathcal{J}$, let

$$\mathcal{Z}' = \mathcal{Z} \cup \{\kappa - X \mid X \in \mathcal{J} - \mathcal{Z}\},$$

i.e. $\mathcal{J}$ with all the sets not in $\mathcal{Z}$ replaced by their complements. The independence of $\mathcal{J}$ means precisely that each such family $\mathcal{Z}'$ has the finite intersection property and can therefore be extended to an ultrafilter $\mathcal{U}(\mathcal{Z})$. If $\mathcal{Z}_0 \neq \mathcal{Z}_1$, then some $X \in \mathcal{J}$ belongs to one of the $\mathcal{Z}_i$ but not the other, so X belongs to one $\mathcal{Z}_i'$ and therefore to $\mathcal{U}(\mathcal{Z}_i)$ while $\kappa - X$ belongs to the other $\mathcal{U}(\mathcal{Z}_i)$. ~~Thus, the two $\mathcal{U}(\mathcal{Z}_i)$ are~~ ~~do~~ Thus $\mathcal{U}(\mathcal{Z}_0) \neq \mathcal{U}(\mathcal{Z}_1)$, and all 2^{2^κ} $\mathcal{Z}$'s yield distinct ultrafilters. □

<u>Remark 4</u>. Since there are only κ principal ultrafilters on κ (one concentrated at each point), Theorem 3 provides 2^{2^κ} ~~unifor~~ non-principal ultrafilters on κ. In fact, essentially the

same argument provides 2^{2^κ} uniform ultrafilters. To see this, notice first that in the conclusion of Corollary 2, "there is an $\alpha \in \kappa$" can be strengthened to "there are κ distinct values of $\alpha \in \kappa$" without changing the $\mathcal{J}$ in the proof, because one can add one more function $g \in \mathcal{I}$ to the $\mathcal{I}_0$ in the proof and specify $H(g)$ arbitrarily. From this improvement of Corollary 2, it follows that, in the proof of Theorem 3, any intersection of finitely many sets from $\mathcal{Z}'$ is not merely nonempty but of cardinality κ. Thus, ~~$\mathcal{Z}' \cup \{X \subseteq \kappa : |\kappa - X| < \kappa\}$ has~~ by Corollary B5, $\mathcal{Z}'$ is included in an ultrafilter $\mathcal{U}(\mathcal{Z}')$ consisting only of sets of cardinality κ. Then, as $\mathcal{Z}'$ varies, the $\mathcal{U}(\mathcal{Z}')$ are 2^{2^κ} distinct uniform ultrafilters on κ.

<u>Proposition 5</u>. For any 2^κ given ultrafilters on κ, there exists an ultrafilter on κ that is above all the given ones in the RK-ordering.

<u>Proof</u>. Fix an independent family $\mathcal{I}$ of functions from κ to κ, as in Theorem 1, and index the given ultrafilters by elements of $\mathcal{I}$; write $\mathcal{U}_f$ for the ultrafilter associated to $f \in \mathcal{I}$. ~~The desired ultrafilter $\mathcal{V}$ will be constructed so such that $f(\mathcal{V}) = \mathcal{U}_f$ for each f. To~~
To complete the proof of the proposition, it suffices to show that the family
~~ee that such $\mathcal{V}$ exists [illegible]~~

$$\mathcal{S} = \bigcup_f \{f^{-1}(X) \mid X \in \mathcal{U}_f\}$$

has the finite intersection property, for any ultrafilter $\mathcal{V}$ that includes $\mathcal{S}$ will satisfy $f(\mathcal{V}) = \mathcal{U}_f$ and therefore $\mathcal{V} \geq \mathcal{U}_f$ for each f. So consider any finite subfamily of $\mathcal{S}$, say $\{f_1^{-1}(X_1), \ldots, f_n^{-1}(X_n)\}$, where $X_i \in \mathcal{U}_{f_i}$ for each i. For the purpose of proving that the intersection of this subfamily is nonempty, there is no loss of generality in assuming that the functions f_i are distinct, for if $f_i = f_j$ then $f_i^{-1}(X_i)$ and $f_j^{-1}(X_j)$ could be replaced with $f_i^{-1}(X_i \cap X_j)$ without altering the intersection. So assume that the f_i are distinct, and let x_i be an arbitrary element of X_i. (Of course, $X_i \neq \emptyset$ as $X_i \in \mathcal{U}_{f_i}$.) By the independence of $\mathcal{L}$, there is $\alpha \in \kappa$ such that $f_i(\alpha) = x_i$ for each i. Thus, $\alpha \in f_1^{-1}(X_1) \cap \ldots \cap f_n^{-1}(X_n)$, as desired. $\square$

<u>Corollary 6</u>. The RK-ordering of ultrafilters on κ has no maximal elements.

<u>Proof</u>. Let $\mathcal{U}$ be any ultrafilter on κ. There are ~~at most~~ 2^κ ultrafilters on κ that are $\leq \mathcal{U}$, for these have the form $f(\mathcal{U})$ and there are only 2^κ f's. But there are 2^{2^κ} ultrafilters on κ by Theorem. So let $\mathcal{U}'$ be an ultrafilter on κ that is not $\leq \mathcal{U}$. By Proposition 5, let $\mathcal{V}$ be an ultrafilter on κ that is $\geq \mathcal{U}$ and also $\geq \mathcal{U}'$. The latter implies that $\mathcal{V} \not\equiv \mathcal{U}$, and then the former implies $\mathcal{V} > \mathcal{U}$. $\square$

This corollary could also have been proved directly, without the use of independent functions. Another, more "explicit" proof is in .

<u>Corollary 7</u>. The RK-ordering of (isomorphism types of) ultrafilters on κ contains increasing well-ordered sequences of order type $(2^\kappa)^+$ but no longer ones.

<u>Proof</u>. To define an increasing sequence $(\mathcal{U}_\alpha)_{\alpha<(2^\kappa)^+}$, proceed by induction on α, starting with an arbitrary $\mathcal{U}_0$. At successor stages, use Corollary 6; at limit stages $\alpha<(2^\kappa)^+$ use Proposition 5.

There is no such sequence of order type $(2^\kappa)^+ + 1$ because, as was pointed out in the proof of Corollary 6, the isomorphism type of an ultrafilter on κ has at most 2^κ RK-predecessors. $\square$

~~The final application in this section~~
This section's final application of independent sets involves a combination of this technique with the inductive technique of B7 and F8. Although the result is stated for ultrafilters on ω, it is also true for uniform ultrafilters on any infinite κ, by an easy modification of the proof along the lines of Remark 4. (The corresponding

result for not necessarily uniform ultrafilters follows trivially from the result for ultrafilters on ω, via any embedding of ω in κ.) This result shows that $P(\underset{\sim}{c})$ is not needed for the last statement in Corollary F10.

<u>Theorem 8</u>. There exist two ultrafilters on ω neither of which is $\geq$ the other in the RK-ordering. That is, the RK-ordering of ultrafilters on ω is not linear.

Proof.

The idea of the proof is to construct the two ultrafilters ~~inductively~~ $\mathcal{U}$ and $\mathcal{V}$ inductively, defining by induction on $\alpha < \underset{\sim}{c}$ filters $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$, to approximate $\mathcal{U}$ and $\mathcal{V}$ respectively. The requirements to be met by this construction are those of B7

Requirement $(X,0)$: $X \in \mathcal{U}$ or $\omega - X \in \mathcal{U}$

and the analogs for $\mathcal{V}$

Requirement $(X,1)$: $X \in \mathcal{V}$ or $\omega - X \in \mathcal{V}$

for all $X \subseteq \omega$, plus, for each $f : \omega \to \omega$, requirements to ensure that $f(\mathcal{V}) \neq \mathcal{U}$ and $f(\mathcal{U}) \neq \mathcal{V}$, namely

Requirement $(f,0)$: There is a set $X \in \mathcal{U}$ with $\omega - f^{-1}(X) \in \mathcal{V}$

Requirement $(f,1)$: There is a set $X \in \mathcal{V}$ with $\omega - f^{-1}(X) \in \mathcal{U}$.

~~Let all these requirements be~~

There are $\mathfrak{c}$ requirements; fix an enumeration of them in order type $\mathfrak{c}$. The filters $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$ will be defined inductively for $\alpha \leq \mathfrak{c}$ so as to satisfy the following inductive hypotheses:

(a) If $\alpha \leq \beta$ then $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$ and $\mathcal{G}_\alpha \subseteq \mathcal{G}_\beta$.

(b) $\mathcal{F}_0 = \mathcal{G}_0 =$ cofinite filter.

(c) If β is a limit ordinal, then $\mathcal{F}_\beta = \bigcup_{\alpha<\beta} \mathcal{F}_\alpha$ and $\mathcal{G}_\beta = \bigcup_{\alpha<\beta} \mathcal{G}_\beta$.

(d0) If $\alpha < \beta$ and the α^{th} requirement is $(X, 0)$ then $\mathcal{F}_\beta$ contains X or $\omega - X$.

(d1) If $\alpha < \beta$ and the α^{th} requirement is $(X, 1)$ then $\mathcal{G}_\beta$ contains X or $\omega - X$.

(e0) If $\alpha < \beta$ and the α^{th} requirement is $(f, 0)$ then $\mathcal{F}_\beta$ contains an X with $f^{-1}(\omega - X) \in \mathcal{G}_\beta$.

(e1) If $\alpha < \beta$ and the α^{th} requirement is $(f, 1)$ then $\mathcal{G}_\beta$ contains an X with $f^{-1}(\omega - X) \in \mathcal{F}_\beta$.

As usual, (a), (d), and (e) follow in general if they hold when $\beta = \alpha + 1$, and, as usual, ~~this can proceed with~~ ~~But as ab~~ the induction steps for $\beta = 0$, β a limit ordinal, and $\beta = \alpha + 1$ with the α^{th} requirement being $(X, 0)$ or $(X, 1)$ are straightforward. The new element in this proof is the definition of $\mathcal{F}_{\alpha+1}$ and $\mathcal{G}_{\alpha+1}$ when the α^{th} requirement is $(f, 0)$ or $(f, 1)$. To satisfy, say, $(f, 0)$, it is necessary to find an $X \subseteq \omega$ such that X can be adjoined to $\mathcal{F}_\alpha$ (i.e., $\mathcal{F}_\alpha \cup \{X\}$ has the FIP) and $\omega - f^{-1}(X)$ can be adjoined to $\mathcal{G}_\alpha$. Unfortunately, such an X need not exist, unless additional precautions were taken at previous stages of the induction; indeed, it is conceivable that $\mathcal{F}_\alpha$ is an ultrafilter and $\mathcal{F}_\alpha = f(\mathcal{G}_\alpha)$, which would imply that no such X exists. (If $P(\underset{\sim}{c})$ were assumed, then $\mathcal{F}_\alpha$ could not be an ultrafilter, since it is generated by fewer than $\underset{\sim}{c}$ sets, and then X would exist. But this information is useless since the non-linearity of the RK-ordering under the assumption $P(\underset{\sim}{c})$ is already given by Corollary F10.) It is for taking the necessary precautions that independent families will be useful.

Definition 9. A family $\mathcal{J}$ of subsets of ω is *independent modulo* a filter $\mathcal{F}$ on ω if, for any two disjoint finite subsets $\mathcal{J}^+$ and $\mathcal{J}^-$ of $\mathcal{J}$, $(\mathcal{F}-\exists i)$ i belongs to every set in $\mathcal{J}^+$ and to no set in $\mathcal{J}^-$.

Thus, if $\mathcal{J}$ is independent modulo $\mathcal{F}$ then, for any $\mathcal{Z} \subseteq \mathcal{J}$ and $\mathcal{Z}'$ defined as in the proof of Theorem 3, $\mathcal{F} \cup \mathcal{Z}'$ has the FIP. Independence as in Corollary 2 is independence modulo the filter consisting of just ω. The usefulness of independence in the proof of Theorem 8 is that, if a nonempty family $\mathcal{J}$ is independent modulo $\mathcal{F}$, then $\mathcal{F}$ is not an ultrafilter, for if $A \in \mathcal{J}$ then independence modulo

$\mathcal{F}$ requires both $(\mathcal{F}\text{-}\exists_i)\ i \in A$ and $(\mathcal{F}\text{-}\exists_i)\ i \notin A$. Thus, by maintaining independent families $\mathcal{J}_\alpha$ modulo $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$, one can ensure that these filters do not become ultrafilters prematurely and thus ensure that the new requirements $(f,0)$ and $(f,1)$ can be satisfied. More specifically, the inductive process whereby $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$ are defined is expanded to also define families $\mathcal{J}_\alpha$ ~~such that~~ satisfying the additional inductive hypotheses

(f) If $\alpha \leq \beta$ then $\mathcal{J}_\alpha \supseteq \mathcal{J}_\beta$

(g) $\mathcal{J}_\alpha$ is independent modulo $\mathcal{F}_\alpha$ and modulo $\mathcal{G}_\alpha$.

(h) $\mathcal{J}_0$ has cardinality $\underset{\sim}{c}$.

(i) If β is a limit ordinal, then $\mathcal{J}_\beta = \bigcap_{\alpha<\beta} \mathcal{J}_\alpha$.

(j) $\mathcal{J}_\alpha - \mathcal{J}_{\alpha+1}$ is finite.

Observe that the last three of these induction hypotheses imply that, for $\alpha < \underset{\sim}{c}$, $\mathcal{J}_\alpha$ has cardinality κ; indeed, $\mathcal{J}_0 - \mathcal{J}_\alpha$ has cardinality $\leq |\alpha| \cdot \aleph_0$. In particular, $\mathcal{J}_\alpha \neq \emptyset$ for $\alpha < \underset{\sim}{c}$. (But $\mathcal{J}_{\underset{\sim}{c}}$ can be empty.)

The rest of the proof is devoted to the inductive construction of $\mathcal{F}_\beta$, $\mathcal{G}_\beta$, and $\mathcal{J}_\beta$ satisfying all these inductive conditions. Of course, it is necessary to re-examine the steps where $\mathcal{F}_\beta$ and $\mathcal{G}_\beta$ were easy to obtain, to see that $\mathcal{J}_\beta$ can also be obtained.

~~For $\beta = 0$, $\mathcal{J}_\beta$ must be a family~~

$\beta = 0$: This case requires the following lemma.

<u>Lemma 10</u>. If $\mathcal{J}$ is an infinite family of independent sets, then $\mathcal{J}$ is independent modulo the cofinite filter.

<u>Proof</u>. Independence means that, if $\mathcal{J}^+$ and $\mathcal{J}^-$ are disjoint finite subsets of $\mathcal{J}$ then the intersection $I(\mathcal{J}^+, \mathcal{J}^-)$ of the sets in $\mathcal{J}^+$ and the complements of the sets in $\mathcal{J}^-$ is nonempty. Independence modulo the cofinite filter means that each such $I(\mathcal{J}^+, \mathcal{J}^-)$ is infinite. Suppose that the former held but the latter failed for some infinite $\mathcal{J}$. Choose $\mathcal{J}^+$ and $\mathcal{J}^-$ as above so that $I(\mathcal{J}^+, \mathcal{J}^-)$ is finite and of the smallest possible cardinality. As $\mathcal{J}$ is infinite, let $A \in \mathcal{J} - (\mathcal{J}^+ \cup \mathcal{J}^-)$. Then, by independence, both $I(\mathcal{J}^+ \cup \{A\}, \mathcal{J}^-) = I(\mathcal{J}^+, \mathcal{J}^-) \cap A$ and $I(\mathcal{J}^+, \mathcal{J}^- \cup \{A\}) = I(\mathcal{J}^+, \mathcal{J}^-) - A$ are nonempty, so both have smaller cardinality than $I(\mathcal{J}^+, \mathcal{J}^-)$, contradicting the choice of $\mathcal{J}^+$ and $\mathcal{J}^-$. $\square$

In view of this lemma, $\mathcal{J}_0$ can be defined to be the independent set from Corollary 2 (with $\kappa=\omega$), and (g) and (h) will be satisfied.

β a limit: Because of condition (i), there is no choice about the definition of $\mathcal{J}_\beta$. It must be proved that, if $\mathcal{J}_\alpha$ is independent modulo $\mathcal{F}_\alpha$ for every $\alpha<\beta$ then $\mathcal{J}_\beta = \bigcap_{\alpha<\beta} \mathcal{J}_\alpha$ is independent modulo $\mathcal{F}_\beta = \bigcup_{\alpha<\beta} \mathcal{F}_\alpha$. (Of course the same is true with $\mathcal{G}$ in place of $\mathcal{F}$.) But this is easy. Given $\boxed{\text{finite}}$ disjoint subsets $\mathcal{J}^+$ and $\mathcal{J}^-$ of $\mathcal{J}_\beta$ and given any $X \in \mathcal{F}_\beta$, find an $\alpha<\beta$ such that $X \in \mathcal{F}_\alpha$, and use the fact that $\mathcal{J}^+$ and $\mathcal{J}^-$ are subsets of $\mathcal{J}_\alpha$ to conclude that X meets $I(\mathcal{J}^+, \mathcal{J}^-)$. (The $(\mathcal{F}_\alpha\text{-}\exists i)$ in the definition of independence modulo $\mathcal{F}_\alpha$ has been interpreted using Proposition D3(f).) Therefore, $\mathcal{J}_\beta$ is independent modulo $\mathcal{F}_\beta$.

~~$\beta = \alpha+1$ and the α^{th} requirement is $(X, 0)$: $\mathcal{F}_{\alpha+1}$ is formed from $\mathcal{F}_\alpha$ by adjoining X or $\omega - X$; suppose without loss of generality that X was adjoined, so $\mathcal{F}_{\alpha+1}$ is generated by $\mathcal{F}_\alpha \cup \{X\}$. Thus each set in $\mathcal{F}_{\alpha+1}$ includes one of the form $A \cap X$ with $A \in \mathcal{F}_\alpha$. Induction hypotheses (g) and (j) demand that the independent modulo $\mathcal{F}_\alpha$ family $\mathcal{J}_\alpha$ be made independent~~

$\beta = \alpha+1$ and the α^{th} requirement is $(X, 0)$: As in previous constructions, $\mathcal{F}_{\alpha+1}$ will be generated by $\mathcal{F}_\alpha \cup \{X\}$ or by $\mathcal{F}_\alpha \cup \{\omega - X\}$, so that (a) and (d0) are satisfied, but the choice between X and $\omega - X$ must be made with (g) and (j) in mind. These demand that the independent modulo $\mathcal{F}_\alpha$ family $\mathcal{J}_\alpha$ be made independent modulo $\mathcal{F}_{\alpha+1}$ by the removal of only finitely many members of $\mathcal{J}_\alpha$. It will be useful to have a characterization of independence modulo filters described in terms of generators.

<u>Lemma 11</u>. Let $\mathcal{F}$ be a filter on ω, X a subset of ω, and $\mathcal{J}$ a family of subsets of ω. The following are equivalent.

(a) $\mathcal{F} \cup \{X\}$ generates a filter (i.e., it has the FIP) and $\mathcal{J}$ is independent modulo this filter.

(b) For any disjoint finite subsets $\mathcal{J}^+$ and $\mathcal{J}^-$ of $\mathcal{J}$ and for any $A \in \mathcal{F}$, $I(\mathcal{J}^+, \mathcal{J}^-) \cap A \cap X \neq \emptyset$, where $I(\mathcal{J}^+, \mathcal{J}^-)$ is the intersection of the sets in $\mathcal{J}^+$ and the complements of the sets in $\mathcal{J}^-$.

<u>Proof</u>. The special case of (b) where $\mathcal{J}^+$ and $\mathcal{J}^-$ are both empty says that X meets every set in $\mathcal{F}$, so $\mathcal{F} \cup \{X\}$ has the FIP. Independence of $\mathcal{J}$ modulo this filter means that every $I(\mathcal{J}^+, \mathcal{J}^-)$ as in (b) meets every set in this filter. But every set in this filter

has a subset of the form $A\cap X$ with $A\in\mathcal{F}$, and, conversely, every such $A\cap X$ is in the filter generated by $\mathcal{F}\cup\{A\}$. Thus, for a set to meet every set in this filter is the same as for it to meet every such $A\cap X$. $\square$

If $\mathcal{F}_\alpha\cup\{\omega-X\}$ generates a filter and $\mathcal{J}_\alpha$ is independent modulo this filter, then let $\mathcal{F}_{\alpha+1}$ be this filter and let $\mathcal{J}_{\alpha+1}=\mathcal{J}_\alpha$. Otherwise, the lemma gives disjoint finite subsets $\mathcal{J}^+$ and $\mathcal{J}^-$ of $\mathcal{J}_\alpha$ and $A\in\mathcal{F}_\alpha$ such that $I(\mathcal{J}^+,\mathcal{J}^-)\cap A\cap(\omega-X)=\emptyset$; i.e. $I(\mathcal{J}^+,\mathcal{J}^-)\cap A\subseteq X$. Then let $\mathcal{F}_{\alpha+1}$ be the filter generated by $\mathcal{F}_\alpha\cup\{X\}$ and let $\mathcal{J}_{\alpha+1}=\mathcal{J}_\alpha-(\mathcal{J}^+\cup\mathcal{J}^-)$. It remains to check that $\mathcal{J}_{\alpha+1}$ is independent modulo $\mathcal{F}_{\alpha+1}$, which, by the lemma, amounts to checking that $I(\mathcal{K}^+,\mathcal{K}^-)\cap B\cap X\neq\emptyset$ for all disjoint finite $\mathcal{K}^+,\mathcal{K}^-\subseteq\mathcal{J}$ and for all $B\in\mathcal{F}_\alpha$. But

$$I(\mathcal{K}^+,\mathcal{K}^-)\cap B\cap X\supseteq I(\mathcal{K}^+,\mathcal{K}^-)\cap B\cap I(\mathcal{J}^+,\mathcal{J}^-)\cap A$$
$$=I(\mathcal{J}^+\cup\mathcal{K}^+,\mathcal{J}^-\cup\mathcal{K}^-)\cap(A\cap B),$$

and the last of these sets is nonempty, because $\mathcal{J}^+\cup\mathcal{K}^+$ and $\mathcal{J}^-\cup\mathcal{K}^-$ are disjoint finite subsets of $\mathcal{J}_\alpha$, an independent family modulo the filter $\mathcal{F}_\alpha$, which contains $A\cap B$. This completes the construction in the case at hand, except for the trivial matter of setting $\mathcal{G}_{\alpha+1} = \mathcal{G}_\alpha$.

Of course, the case where the α^{th} requirement is $(X, 1)$ is handled analogously

$\beta = \alpha + 1$ and the α^{th} requirement is $(f, 0)$. Choose an arbitrary element X of $\mathcal{J}_\alpha$. Lemma 11 shows that $\mathcal{F}_\alpha \cup \{X\}$ and $\mathcal{F}_\alpha \cup \{\omega - X\}$ both generate filters and $\mathcal{J}_\alpha - \{X\}$ is independent modulo both of them; indeed, in the notation of part (b) of the lemma, $I(\mathcal{J}^+, \mathcal{J}^-) \cap A \cap X = I(\mathcal{J}^+ \cup \{X\}, \mathcal{J}^-) \cap A$ and $I(\mathcal{J}^+, \mathcal{J}^-) \cap A \cap (\omega - X) = I(\mathcal{J}^+, \mathcal{J}^- \cup \{X\}) \cap A$, and both of these are nonempty because $\mathcal{J}_\alpha$ is independent modulo $\mathcal{F}_\alpha$.

~~If $\mathcal{J}_\alpha - \{X\}$~~

If $\mathcal{G}_\alpha \cup \{f^{-1}(\omega - X)\}$ generates a filter and $\mathcal{J}_\alpha - \{X\}$ is independent modulo it, then let ~~the filter~~ $\mathcal{G}_{\alpha+1}$ be this filter, let $\mathcal{F}_{\alpha+1}$ be generated by $\mathcal{F}_\alpha \cup \{X\}$, and let $\mathcal{J}_{\alpha+1} = \mathcal{J}_\alpha - \{X\}$. Then all the inductive hypotheses are preserved; in particular, (e0) holds.

Otherwise, use Lemma 11 to find disjoint finite subsets $\mathcal{J}^+$ and $\mathcal{J}^-$ of $\mathcal{J}_\alpha - \{X\}$ and to find $A \in \mathcal{G}_\alpha$ such that $I(\mathcal{J}^+, \mathcal{J}^-) \cap A \cap f^{-1}(\omega - X) = \emptyset$, i.e.,

$I(\mathcal{J}^+,\mathcal{J}^-)\cap A \subseteq f^{-1}(X)$. Let $\mathcal{G}_{\alpha+1}$ be the filter generated by $\mathcal{G}_\alpha \cup \{f^{-1}(X)\}$, let $\mathcal{F}_{\alpha+1}$ be the filter generated by $\mathcal{F}_\alpha \cup \{\omega - X\}$, and let $\mathcal{J}_{\alpha+1} = \mathcal{J}_\alpha - (\{X\}\cup\mathcal{J}^+\cup\mathcal{J}^-)$. Then (e0) holds with $\omega - X$ in the role of X, and (j) is clear, so it remains to check (g). As remarked above, $\mathcal{J}_\alpha - \{X\}$ is independent modulo the filter $\mathcal{F}_{\alpha+1}$ generated by $\mathcal{F}_\alpha \cup \{X\}$; a fortiori, so is the subset $\mathcal{J}_{\alpha+1}$. Finally, to check that $\mathcal{J}_{\alpha+1}$ is independent modulo $\mathcal{G}_{\alpha+1}$, it suffices by Lemma 11 to check that $I(K^+,K^-)\cap B\cap f^{-1}(X)\neq\emptyset$ for all disjoint finite subsets K^+ and K^- of $\mathcal{J}_{\alpha+1}$ and all $B\in\mathcal{G}_\alpha$. But

$$
\begin{aligned}
I(K^+,K^-)\cap B\cap f^{-1}(X) &\supseteq I(K^+,K)\cap B\cap I(\mathcal{J}^+,\mathcal{J}^-)\cap A \\
&= I(\mathcal{J}^+\cup K^+,\mathcal{J}^-\cup K^-)\cap(A\cap B),
\end{aligned}
$$

and the last of these is nonempty because $\mathcal{J}_\alpha$ is independent modulo $\mathcal{G}_\alpha$. This completes the construction in the case at hand.

Of course, the last remaining case, where the α^{th} requirement is $(f,1)$, is handled analogously. Thus, the construction of $\mathcal{F}_\alpha$, $\mathcal{G}_\alpha$, and $\mathcal{J}_\alpha$ is complete. Inductive conditions (a) and (d) ensure that $\mathcal{U}=\mathcal{F}_{\underset{\sim}{c}}$ and $\mathcal{V}=\mathcal{G}_{\underset{\sim}{c}}$ are ultrafilters. Inductive condition (e) ensures that $\mathcal{U}$ and $\mathcal{V}$ are incomparable in the RK-ordering. $\square$

The method in the preceding proof can be extended without difficulty to show that there are $\mathfrak{c}$ pairwise RK-incomparable ultrafilters on ω and, more generally, 2^κ pairwise RK-incomparable uniform ultrafilters on κ for any infinite cardinal κ. It is in fact true that there are 2^{2^κ} pairwise ~~incomparable~~ RK-incomparable uniform ultrafilters on κ, but the proof of this [ ] is more difficult.