

G. Independent sets and functions

This section is about applications to ultrafilter theory of the following combinatorial property of infinite sets.

Theorem 1. Let κ be an infinite cardinal number. There exists a family \mathcal{A} of 2^κ functions from κ to κ which are independent in the following sense. For any finite subfamily \mathcal{A}_0 of \mathcal{A} and any function $H: \mathcal{A}_0 \rightarrow \kappa$, there exists an $\alpha \in \kappa$ such that $f(\alpha) = H(f)$ for all $f \in \mathcal{A}_0$.

~~In other words, for any finitely many functions in \mathcal{A}_0 , see~~

In other words, for any finitely many functions $f_1, \dots, f_m \in \mathcal{A}$, the m -tuple $(f_1(\alpha), \dots, f_m(\alpha))$ ranges over all of κ^m as α ranges over κ .

Proof. It suffices to define 2^κ independent functions ^{into κ from} some set S of cardinality κ , for the domain can be changed to κ by simply composing the functions with a bijection from κ to S . Let S be the set of all pairs ~~(T, t) where T is a finite~~
 (T, t) where T is a finite subset of κ and t is a ~~function~~ function from the power set $\mathcal{P}(T)$ into κ . As κ is infinite, there are κ possible T 's and, for each T , κ possible

t 's, so S has cardinality κ . ~~the desired independent functions~~ For each subset A of κ , let $f_A: S \rightarrow \kappa$ be defined by

$$f_A(T, t) = t(T \cap A).$$

The proof will be completed by showing that these functions f_A are ~~independent~~ distinct (so there are 2^κ of them) and independent. For this it suffices to find, for any finitely many ~~distinct~~ $A_1, \dots, A_n \in \kappa$ and any $\beta_1, \dots, \beta_n \in \kappa$, an element (T, t) of S for which $f_{A_i}(T, t) = \beta_i$ for each i . Let the A_i 's and β_i 's be given, and let T be a finite subset of κ such that the n intersections $T \cap A_i$ are distinct. Such a T can be found by simply taking one ~~element~~ element from the symmetric difference of each pair of A_i 's. Define t by setting $t(T \cap A_i) = \beta_i$, which makes sense as the $T \cap A_i$ are distinct, and extend t arbitrarily to $\mathcal{P}(T)$. Then, for each i ,

$$f_{A_i}(T, t) = t(T \cap A_i) = \beta_i. \quad \square$$

Corollary 2. Let κ be an infinite cardinal number. There ~~are 2^κ such~~

exists a family of 2^κ subsets of κ that are independent in the sense that, for any ~~finite subfamily \mathcal{I}_0 of \mathcal{I} and any two disjoint finite subfamilies \mathcal{I}^+ and \mathcal{I}^- of \mathcal{I} ,~~

there is an $\alpha \in K$ such that $\alpha \in X$ for every $X \in \mathcal{J}^+$ but for no $X \in \mathcal{J}^-$.

In other words, the intersection of any finitely many members of \mathcal{J} and the complements of any finitely many other members of \mathcal{J} is nonempty.

Proof. Let \mathcal{d} be a family of independent functions as in Theorem 1. For each $f \in \mathcal{d}$ let $X(f) = \{\alpha \in K \mid f(\alpha) = 0\}$. These $X(f)$'s are distinct, by the independence of \mathcal{d} so there are 2^k of them. To see that the family $\mathcal{J} = \{X(f) \mid f \in \mathcal{d}\}$ has the desired independence property, suppose \mathcal{J}^+ and \mathcal{J}^- are given, and apply the independence of \mathcal{d} as follows. Let $\mathcal{d}_0 = \{f \in \mathcal{d} \mid X(f) \in \mathcal{J}^+ \cup \mathcal{J}^-\}$, and, for $f \in \mathcal{d}_0$, set $H(f) = 0$ (resp. 1) if $X(f) \in \mathcal{J}^+$ (resp. \mathcal{J}^-). Let $\alpha \in K$ be such that $f(\alpha) = H(f)$ for all $f \in \mathcal{d}_0$. Then $\alpha \in X(f)$ for each $X(f) \in \mathcal{J}^+$ and $\alpha \notin X(f)$ for each $X(f) \in \mathcal{J}^-$. \square

The first application of these results supersedes Proposition C4 by giving the precise number of ultrafilters on any infinite set.

Theorem 3. There are exactly 2^{2^k} ultrafilters on any infinite set of cardinality

Proof. There are at most 2^{2^k} such ultrafilters because there are ^{only 2^k subsets, and therefore} 2^{2^k} families of subsets, of a set of size k . For the reverse inequality, it suffices to construct 2^{2^k} ultrafilters on k itself, since they can be transferred by a bijection to any set of cardinality k . Let \mathcal{I} be a family of 2^k independent subsets of k , as in Corollary 2. For each of the 2^{2^k} subfamilies \mathcal{J} of \mathcal{I} , let

$$\mathcal{J}' = \mathcal{J} \cup \{k - X \mid X \in \mathcal{I} - \mathcal{J}\},$$

i.e. \mathcal{I} with all the sets not in \mathcal{J} replaced by their complements. The independence of \mathcal{I} means precisely that each such family \mathcal{J}' has the finite intersection property and can therefore be extended to an ultrafilter $\mathcal{U}(\mathcal{J})$. If $\mathcal{J}_0 \neq \mathcal{J}_1$, then some $X \in \mathcal{I}$ belongs to one of the \mathcal{J}_i but not the other, so X belongs to one \mathcal{J}'_i and therefore to $\mathcal{U}(\mathcal{J}'_i)$ while $k - X$ belongs to the other $\mathcal{U}(\mathcal{J}'_i)$. ~~Thus, the two $\mathcal{U}(\mathcal{J}'_i)$ are~~
~~Thus~~ $\mathcal{U}(\mathcal{J}'_0) \neq \mathcal{U}(\mathcal{J}'_1)$, and all 2^{2^k} \mathcal{J}'_i yield distinct ultrafilters. \square

Remark 4. Since there are only k principal ultrafilters on k (one concentrated at each point), Theorem 3 provides 2^{2^k} ^{non-principal} ~~ultrafilters~~ ultrafilters on k . In fact, essentially the

$$\mathcal{S} = \bigcup_f \{f^{-1}(X) \mid X \in \mathcal{U}_f\}$$

has the finite intersection property, for any ultrafilter \mathcal{V} that includes \mathcal{S} will satisfy $f(\mathcal{V}) = \mathcal{U}_f$ and therefore $\mathcal{V} \geq \mathcal{U}_f$ for each f . So consider any finite subfamily ~~of \mathcal{S} , say~~ ~~of \mathcal{S} , say~~ $\{f_1^{-1}(X_1), \dots, f_m^{-1}(X_m)\}$, where $X_i \in \mathcal{U}_{f_i}$ for each i . For the purpose of proving that the intersection of this subfamily is nonempty, there is no loss of generality in assuming that the functions f_i are distinct, for if $f_i = f_j$ then $f_i^{-1}(X_i)$ and $f_j^{-1}(X_j)$ could be replaced with $f_i^{-1}(X_i \cap X_j)$ without altering the intersection. So assume that the f_i are distinct, and let x_i be an arbitrary element of X_i . (Of course, $X_i \neq \emptyset$ as $X_i \in \mathcal{U}_{f_i}$.) By the independence of \mathcal{S} , there is $\alpha \in K$ such that $f_i(\alpha) = x_i$ for each i . Thus, $\alpha \in f_1^{-1}(X_1) \cap \dots \cap f_m^{-1}(X_m)$, as desired. \square

Corollary 6. The RK-ordering of ultrafilters on κ has no maximal elements.

Proof. Let \mathcal{U} be any ultrafilter on κ . There are at most 2^κ ultrafilters on κ that are $\leq \mathcal{U}$, for these have the form $f(\mathcal{U})$ and there are only 2^κ f 's. ~~But there are 2^{2^κ} ultrafilters on κ by Theorem.~~ So let \mathcal{U}' be an ultrafilter on κ that is not $\leq \mathcal{U}$. By Proposition 5, let \mathcal{V} be an ultrafilter on κ that is $\geq \mathcal{U}$ and also $\geq \mathcal{U}'$. ~~The latter implies that $\mathcal{V} \neq \mathcal{U}$, and then the former implies $\mathcal{V} > \mathcal{U}$.~~

This corollary could also have been proved directly, without the use of independent families. Another, more "explicit" proof is in

Corollary 7. The RK-ordering of (isomorphism types of) ultrafilters on κ contains increasing well-ordered sequences of order type $(2^\kappa)^+$ but no longer ones.

Proof. To define an increasing sequence $(U_\alpha)_{\alpha < (2^\kappa)^+}$, proceed by induction on α , starting with an arbitrary U_0 . At successor stages, use Corollary 6; at limit stages $\alpha < (2^\kappa)^+$ use Proposition 5.

There is no such sequence of order type $(2^\kappa)^+ + 1$ because, as was pointed out in the proof of Corollary 6, the isomorphism type of an ultrafilter on κ has at most 2^κ RK-predecessors. \square

~~The final application in this section~~
This section's final application of independent sets involves a combination of this technique with the inductive technique of B7 and F8. Although the result is stated for ultrafilters on ω , it is also true for uniform ultrafilters on any infinite κ , by an easy modification of the proof along the lines of Remark 4. (The corresponding

result for not necessarily uniform ultrafilters follows trivially from the result for ultrafilters on ω , via any embedding of ω in κ .) This result shows that $P(\underline{\omega})$ is not needed for the last statement in Corollary F10.

Theorem 8. There exist two ultrafilters on ω neither of which is \geq the other in the RK-ordering. That is, the RK-ordering of ultrafilters on ω is not linear.

Proof.

The idea of the proof is to construct the two ultrafilters ~~inductively~~ \mathcal{U} and \mathcal{V} inductively, defining by induction on $\alpha < \omega_1$ filters \mathcal{F}_α and \mathcal{G}_α , to approximate \mathcal{U} and \mathcal{V} respectively. The requirements to be met by this construction are those of B7

Requirement (X, 0): $X \in \mathcal{U}$ ~~or~~ or $\omega - X \in \mathcal{U}$

and the analogs for \mathcal{V}

Requirement (X, 1): $X \in \mathcal{V}$ or $\omega - X \in \mathcal{V}$

for all $X \subseteq \omega$, plus, for each $f: \omega \rightarrow \omega$, requirements to ensure that $f(\mathcal{V}) \neq \mathcal{U}$ and $f(\mathcal{U}) \neq \mathcal{V}$, namely

Requirement (f, 0): There is a set $X \in \mathcal{U}$ with ~~\mathcal{V}~~ $\omega - f^{-1}(X) \in \mathcal{V}$

Requirement (f, 1): There is a set $X \in \mathcal{V}$ with $\omega - f^{-1}(X) \in \mathcal{U}$.

~~Let all these requirements be~~

There are ϵ requirements; fix an enumeration of them in order type ϵ . The filters

\mathcal{F}_α and \mathcal{G}_α will be defined inductively for $\alpha \leq \epsilon$ so as to satisfy the following inductive hypotheses:

- (a) If $\alpha \leq \beta$ then $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$ and $\mathcal{G}_\alpha \subseteq \mathcal{G}_\beta$.
- (b) $\mathcal{F}_0 = \mathcal{G}_0 =$ cofinite filter.
- (c) If β is a limit ordinal, then $\mathcal{F}_\beta = \bigcup_{\alpha < \beta} \mathcal{F}_\alpha$ and $\mathcal{G}_\beta = \bigcup_{\alpha < \beta} \mathcal{G}_\alpha$.
- (d0) If $\alpha < \beta$ and the α^{th} requirement is $(X, 0)$ then \mathcal{F}_β contains X or $w - X$.
- (d1) If $\alpha < \beta$ and the α^{th} requirement is $(X, 1)$ then \mathcal{G}_β contains X or $w - X$.
- (e0) If $\alpha < \beta$ and the α^{th} requirement is $(f, 0)$ then \mathcal{F}_β contains an X with $f^{-1}(wX) \in \mathcal{G}_\beta$.
- (e1) If $\alpha < \beta$ and the α^{th} requirement is $(f, 1)$ then \mathcal{G}_β contains an X with $f^{-1}(wX) \in \mathcal{F}_\beta$.

As usual, (a), (d), and (e) follow in general if they hold when $\beta = \alpha + 1$, and, as usual,

~~there is no problem with~~

~~But it is~~ the induction steps for $\beta = 0$, β a limit ordinal, and $\beta = \alpha + 1$ with the α^{th} requirement being $(X, 0)$ or $(X, 1)$ are straightforward. The new element in this

proof is the definition of $\mathcal{F}_{\alpha+1}$ and $\mathcal{G}_{\alpha+1}$ when the α^{th} requirement is $(f, 0)$ or $(f, 1)$.

To satisfy, say, $(f, 0)$, it is necessary to find an $X \subseteq w$ such that X can be adjoined

to \mathcal{F}_α (i.e., $\mathcal{F}_\alpha \cup \{X\}$ has the FIP) and $\omega - f^{-1}(X)$ can be adjoined to \mathcal{F}_α . Unfortunately, such an X need not exist, unless additional precautions ~~are~~ ^{were} taken at previous stages of the induction; indeed, it is conceivable that \mathcal{F}_α is an ultrafilter and $\mathcal{F}_\alpha = f(\mathcal{F}_\alpha)$, which would imply that no such X exists. (If $P(\underline{c})$ were assumed, then \mathcal{F}_α could not be an ultrafilter, since it ~~is~~ is generated by fewer than \underline{c} sets, and then X would exist. But this information is useless since ~~the non-linearity of the RK-ordering under the assumption $P(\underline{c})$ is already given by Corollary F(10).~~ It is for taking the necessary precautions that independent families will be useful.

Definition 9. A family \mathcal{J} of subsets of ω is independent modulo a filter \mathcal{F} on ω if, for any two disjoint ^{finite} subsets \mathcal{J}^+ and \mathcal{J}^- of \mathcal{J} , $(\mathcal{F} - \exists i)$ belongs to every set in \mathcal{J}^+ and to no set in \mathcal{J}^- .

(if \mathcal{J} is independent modulo \mathcal{F} then)

Thus, for any $\mathcal{J} \subseteq \mathcal{J}$ and \mathcal{J}' defined as in the proof of Theorem 3, $\mathcal{F} \cup \mathcal{J}'$ has the FIP. Independence as in Corollary 2 is independence modulo the filter consisting of just ω . The usefulness of independence in the proof of Theorem 8 is that, if a nonempty family \mathcal{J} is independent modulo \mathcal{F} , then \mathcal{F} is not an ultrafilter, for if $A \in \mathcal{F}$ then independence modulo

\mathcal{F} requires both $(\mathcal{F} - \exists i)_{i \in A}$ and $(\mathcal{F} - \exists i)_{i \notin A}$. Thus, by maintaining independent families \mathcal{J}_α modulo \mathcal{F}_α and \mathcal{G}_α , one can ensure that these filters do not become ultrafilters prematurely and thus ensure that the new requirements $(\mathcal{f}, 0)$ and $(\mathcal{f}, 1)$ can be satisfied. More specifically, the inductive process whereby \mathcal{F}_α and \mathcal{G}_α are defined is expanded to also define families \mathcal{J}_α ~~such that~~ satisfying the additional inductive hypotheses

- (f) ~~\mathcal{J}_α~~ If $\alpha \leq \beta$ then $\mathcal{J}_\alpha \supseteq \mathcal{J}_\beta$
- (g) \mathcal{J}_α is independent modulo \mathcal{F}_α and modulo \mathcal{G}_α .
- (h) \mathcal{J}_0 has cardinality \aleph_κ .
- (i) If β is a limit ordinal, then $\mathcal{J}_\beta = \bigcap_{\alpha < \beta} \mathcal{J}_\alpha$.
- (j) $\mathcal{J}_\alpha - \mathcal{J}_{\alpha+1}$ is finite.

Observe that the last three of these induction hypotheses imply that, for $\alpha < \underline{\kappa}$, \mathcal{J}_α has cardinality κ ; indeed, $\mathcal{J}_0 - \mathcal{J}_\alpha$ has cardinality $\leq |\alpha| \cdot \aleph_0$. ~~In~~ In particular, $\mathcal{J}_\alpha \neq \emptyset$ for $\alpha < \underline{\kappa}$. (But $\mathcal{J}_{\underline{\kappa}}$ can be empty.)

The rest of the proof is devoted to the inductive construction of \mathcal{F}_β , \mathcal{G}_β , and \mathcal{J}_β satisfying all these inductive conditions. Of course, it is necessary to re-examine the

steps where \mathcal{F}_β and \mathcal{G}_β were easy to obtain, to see that \mathcal{G}_β can also be obtained.

~~For $\beta=0$, \mathcal{G}_β must be a family~~

$\beta=0$: This case requires the following lemma.

Lemma 10. If \mathcal{J} is an infinite family of independent sets, then \mathcal{J} is independent modulo the cofinite filter.

Proof. Independence ~~modulo the~~ means that, if ~~some~~ \mathcal{J}^+ and \mathcal{J}^- are disjoint finite subsets of \mathcal{J} then the intersection $I(\mathcal{J}^+, \mathcal{J}^-)$ of the sets in \mathcal{J}^+ and the complements of the sets in \mathcal{J}^- is nonempty. Independence modulo the cofinite filter means that each such $I(\mathcal{J}^+, \mathcal{J}^-)$ is infinite. Suppose that the former held but the latter failed for some infinite \mathcal{J} . Choose \mathcal{J}^+ and \mathcal{J}^- as above so that $I(\mathcal{J}^+, \mathcal{J}^-)$ is finite and of the smallest possible cardinality. As \mathcal{J} is infinite, let $A \in \mathcal{J} - (\mathcal{J}^+ \cup \mathcal{J}^-)$. Then, by independence, both $I(\mathcal{J}^+ \cup \{A\}, \mathcal{J}^-) = I(\mathcal{J}^+, \mathcal{J}^-) \cap A$ and $I(\mathcal{J}^+, \mathcal{J}^- \cup \{A\}) = I(\mathcal{J}^+, \mathcal{J}^-) - A$ are nonempty, so both have smaller cardinality than $I(\mathcal{J}^+, \mathcal{J}^-)$, contradicting the ~~choice~~ choice of \mathcal{J}^+ and \mathcal{J}^- . \square

In view of this lemma, \mathcal{I}_0 can be defined to be the independent set from Corollary 2 (with $\kappa = \omega$), and (g) and (h) will be satisfied.

β a limit: Because of condition (i), there is no choice about the definition of \mathcal{I}_β . It must be proved that, if \mathcal{I}_α is independent modulo \mathcal{F}_α for every $\alpha < \beta$ then $\mathcal{I}_\beta = \bigcap_{\alpha < \beta} \mathcal{I}_\alpha$ is independent modulo $\mathcal{F}_\beta = \bigcup_{\alpha < \beta} \mathcal{F}_\alpha$. (Of course the same is true with \mathcal{I} in place of \mathcal{F} .) But this is easy. Given finite disjoint subsets \mathcal{I}^+ and \mathcal{I}^- of \mathcal{I}_β and given any $X \in \mathcal{F}_\beta$, find an $\alpha < \beta$ such that $X \in \mathcal{F}_\alpha$, and use the fact that \mathcal{I}^+ and \mathcal{I}^- are subsets of \mathcal{I}_α to conclude that X meets $I(\mathcal{I}^+, \mathcal{I}^-)$. (The $(\mathcal{F}_\alpha - \mathcal{I}_\alpha)$ in the definition of independence modulo \mathcal{F}_α has been interpreted using Proposition D3(f).) ~~Therefore, \mathcal{I}_β is independent modulo \mathcal{F}_β .~~

~~$\beta \neq \alpha + 1$ and the α^{th} requirement is (X, \emptyset) : $\mathcal{F}_{\alpha+1}$ is formed from \mathcal{F}_α by adjoining X or $\omega - X$; suppose without loss of generality that X was adjoined, so $\mathcal{F}_{\alpha+1}$ is generated by $\mathcal{F}_\alpha \cup \{X\}$. Thus each set in $\mathcal{F}_{\alpha+1}$ includes one of the form $A \cap X$ with $A \in \mathcal{F}_\alpha$. Induction hypotheses (g) and (j) demand that the independent modulo \mathcal{F}_α family \mathcal{I}_α be made independent~~

$\beta = \alpha + 1$ and the α^{th} requirement is $(X, 0)$: As in previous constructions, $\mathcal{F}_{\alpha+1}$ will be generated by $\mathcal{F}_{\alpha} \cup \{X\}$ or by $\mathcal{F}_{\alpha} \cup \{\omega - X\}$, so that (a) and (d0) are satisfied, but the choice between X and $\omega - X$ must be made with (g) and (g') in mind. These demand that the independent modulo \mathcal{F}_{α} family \mathcal{I}_{α} be made independent modulo $\mathcal{F}_{\alpha+1}$ by the removal of only finitely many members of \mathcal{I}_{α} . It will be useful to have a characterization of independence modulo filters described in terms of generators.

Lemma 11. Let \mathcal{F} be a filter on ω , X a subset of ω , and \mathcal{I} a family of subsets of ω .

The following are equivalent.

(a) $\mathcal{F} \cup \{X\}$ generates a filter (i.e., it has the FIP) and \mathcal{I} is independent modulo this filter.

(b) For any disjoint finite ^{subsets} \mathcal{I}^+ and \mathcal{I}^- of \mathcal{I} and for any $A \in \mathcal{F}$, $I(\mathcal{I}^+, \mathcal{I}^-) \cap A \cap X \neq \emptyset$ where $I(\mathcal{I}^+, \mathcal{I}^-)$ is the intersection of the sets in \mathcal{I}^+ and the complements of the sets in \mathcal{I}^- .

Proof. The special case of (b) where \mathcal{I}^+ and \mathcal{I}^- are both empty says that X meets every set in \mathcal{F} , so $\mathcal{F} \cup \{X\}$ has the FIP. Independence of \mathcal{I} modulo this filter means that every $I(\mathcal{I}^+, \mathcal{I}^-)$ as in (b) meets every set in this filter. But every set in this filter

has a subset of the form $A \cap X$ with $A \in \mathcal{F}$, and, conversely, every such $A \cap X$ is in the filter generated by $\mathcal{F} \cup \{A\}$. Thus, ^{for a set} to meet every set in this filter is the same as for it to meet every such $A \cap X$. \square

If $\mathcal{F}_\alpha \cup \{\omega - X\}$ generates a ~~filter~~ filter and \mathcal{I}_α is independent modulo this filter, then let $\mathcal{F}_{\alpha+1}$ be this filter and let $\mathcal{I}_{\alpha+1} = \mathcal{I}_\alpha$. Otherwise, the lemma gives disjoint finite subsets \mathcal{J}^+ and \mathcal{J}^- of \mathcal{I}_α and $A \in \mathcal{F}_\alpha$ such that $I(\mathcal{J}^+, \mathcal{J}^-) \cap A \cap X = \emptyset$, i.e. $I(\mathcal{J}^+, \mathcal{J}^-) \cap A \subseteq X$. Then let $\mathcal{F}_{\alpha+1}$ be the filter generated by $\mathcal{F}_\alpha \cup \{X\}$ and let $\mathcal{I}_{\alpha+1} = \mathcal{I}_\alpha - (\mathcal{J}^+ \cup \mathcal{J}^-)$. It remains to check that $\mathcal{I}_{\alpha+1}$ is independent modulo $\mathcal{F}_{\alpha+1}$, which, by the lemma, amounts to checking that $I(\mathcal{K}^+, \mathcal{K}^-) \cap B \cap X \neq \emptyset$ for all disjoint finite $\mathcal{K}^+, \mathcal{K}^- \subseteq \mathcal{I}_{\alpha+1}$ and for all $B \in \mathcal{F}_\alpha$. But

$$\begin{aligned} I(\mathcal{K}^+, \mathcal{K}^-) \cap B \cap X &\supseteq I(\mathcal{K}^+, \mathcal{K}^-) \cap B \cap I(\mathcal{J}^+, \mathcal{J}^-) \cap A \\ &= I(\mathcal{J}^+ \cup \mathcal{K}^+, \mathcal{J}^- \cup \mathcal{K}^-) \cap (A \cap B), \end{aligned}$$

and the last of these sets is nonempty, because $\mathcal{J}^+ \cup \mathcal{K}^+$ and $\mathcal{J}^- \cup \mathcal{K}^-$ are disjoint finite subsets of \mathcal{I}_α , an independent family modulo the filter \mathcal{F}_α , which contains $A \cap B$. This completes the construction in the case at hand, except for the

trivial matter of setting $\mathcal{I}_{\alpha+1} = \mathcal{I}_{\alpha}$.

Of course, the case where the α^{th} requirement is $(X, 1)$ is handled analogously $\beta = \alpha + 1$ and the α^{th} requirement is $(f, 0)$. Choose an arbitrary element A of \mathcal{I}_{α} . Lemma 11 shows that $\mathcal{F}_{\alpha} \cup \{X\}$ and $\mathcal{F}_{\alpha} \cup \{\omega - X\}$ both generate filters and $\mathcal{I}_{\alpha} - \{X\}$ is independent modulo both of them; indeed, in the notation of part (b) of the lemma, $I(\mathcal{I}_{\alpha}^{+}, \mathcal{I}_{\alpha}^{-}) \cap A \cap X = I(\mathcal{I}_{\alpha}^{+} \cup \{X\}, \mathcal{I}_{\alpha}^{-}) \cap A$ and $I(\mathcal{I}_{\alpha}^{+}, \mathcal{I}_{\alpha}^{-}) \cap A \cap (\omega - X) = I(\mathcal{I}_{\alpha}^{+}, \mathcal{I}_{\alpha}^{-} \cup \{X\}) \cap A$, and both of these are nonempty because \mathcal{I}_{α} is independent modulo \mathcal{F}_{α} .

~~If $\mathcal{I}_{\alpha} - \{X\}$~~
 If $\mathcal{I}_{\alpha} \cup \{f^{-1}(\omega - X)\}$ generates a filter and $\mathcal{I}_{\alpha} - \{X\}$ is independent modulo it, then let ~~the filter~~ $\mathcal{I}_{\alpha+1}$ be this filter, let $\mathcal{F}_{\alpha+1}$ be generated by $\mathcal{F}_{\alpha} \cup \{X\}$, and let $\mathcal{I}_{\alpha+1} = \mathcal{I}_{\alpha} - \{X\}$. Then all the inductive hypotheses are preserved; in particular, (e0) holds.

Otherwise, use Lemma 11 to find disjoint finite subsets \mathcal{I}^{+} and \mathcal{I}^{-} of $\mathcal{I}_{\alpha} - \{X\}$ and to find $A \in \mathcal{I}_{\alpha}$ such that $I(\mathcal{I}^{+}, \mathcal{I}^{-}) \cap A \cap f^{-1}(\omega - X) = \emptyset$, i.e.,

$I(j^+, j^-) \cap A \subseteq f^{-1}(X)$. Let $\mathcal{G}_{\alpha+1}$ be the filter generated by $\mathcal{G}_\alpha \cup \{f^{-1}(X)\}$,
 let $\mathcal{F}_{\alpha+1}$ be the filter generated by $\mathcal{F}_\alpha \cup \{\omega - X\}$, and let $\mathcal{J}_{\alpha+1} = \mathcal{J}_\alpha - \{X\} \cup j^+ \cup j^-$.
 Then (20) holds with $\omega - X$ in the role of X , and (j) is clear, so it remains to check (g). As
 remarked above, $\mathcal{J}_\alpha - \{X\}$ is independent modulo the filter $\mathcal{F}_{\alpha+1}$ generated by $\mathcal{F}_\alpha \cup \{X\}$;
2 fortiori, so is the subset $\mathcal{J}_{\alpha+1}$. Finally, to check that $\mathcal{J}_{\alpha+1}$ is independent modulo
 $\mathcal{G}_{\alpha+1}$, it suffices by Lemma 11_o to check that $I(K^+, K^-) \cap B \cap f^{-1}(X) \neq \emptyset$ for all
 disjoint finite subsets K^+ and K^- of $\mathcal{J}_{\alpha+1}$ and all $B \in \mathcal{G}_\alpha$. But

$$\begin{aligned}
 I(K^+, K^-) \cap B \cap f^{-1}(X) &\supseteq I(K^+, K^-) \cap B \cap I(j^+, j^-) \cap A \\
 &= I(j^+ \cup K^+, j^- \cup K^-) \cap (A \cap B),
 \end{aligned}$$

and the last of these is nonempty because \mathcal{J}_α is independent modulo \mathcal{G}_α . This
 completes the construction in the case at hand.

Of course, the ~~last~~ last remaining case, where the α^{th} requirement is $(f, 1)$, is
 handled analogously. Thus, the construction of \mathcal{F}_α , \mathcal{G}_α , and \mathcal{J}_α is complete. Inductive
 conditions (a) and (d) ensure that $\mathcal{U} = \bigcap_{\alpha} \mathcal{F}_\alpha$ and $\mathcal{V} = \bigcap_{\alpha} \mathcal{G}_\alpha$ are ultrafilters. Inductive
 condition (c) ensures that \mathcal{U} and \mathcal{V} are incomparable in the RK-ordering. \square

The method in the preceding proof can be extended without difficulty to show that there are \mathfrak{c} ^{pairwise} RK-incomparable ultrafilters on ω and, more generally, 2^{κ} ^{RK-}pairwise ^{uniform} incomparable ultrafilters on κ for any infinite cardinal κ . It is in fact true that there are $2^{2^{\kappa}}$ pairwise ~~incomparable~~ RK-incomparable uniform ultrafilters on κ , but the proof of this [] is more difficult.