H. Limits, sums, and products

This section is devoted to three closely related ways of constructing new filters fro given ones. The most general of these is the limit construction, so called because of its topological aspects to be discussed in Chapter 3.

Definition 1. Let $\mathcal{F}$ be a filter on I, and, for each $i \in I$, let $\mathcal{G}_i$ be a filter on J. Then the limit with respect to $\mathcal{F}$ of the family $(\mathcal{G}_i)_{i\in I}$ is

$$\mathcal{F}\text{-}\lim_i \mathcal{G}_i = \{X \subseteq J \mid (\mathcal{F}\text{-}\forall i)\, X \in \mathcal{G}_i\}$$

$$= \{X \subseteq J \mid \{i \in I \mid X \in \mathcal{G}_i\} \in \mathcal{F}\}.$$

~~Proposition 2. (a) $\mathcal{F}$-$\lim_i \mathcal{G}_i$ is a filter on J~~

Proposition 2. Let $\mathcal{F}$, $\mathcal{G}_i$, I, and J be as in Definition 1.

(a) $\mathcal{F}\text{-}\lim_i \mathcal{G}_i$ is a filter on J.

(b) $(\mathcal{F}\text{-}\lim_i \mathcal{G}_i - \forall j)\, \varphi(j) \longleftrightarrow (\mathcal{F}\text{-}\forall i)(\mathcal{G}_i\text{-}\forall j)\, \varphi(j)$.

(c) $(\mathcal{F}\text{-}\lim_i \mathcal{G}_i - \exists j)\, \varphi(j) \longleftrightarrow (\mathcal{F}\text{-}\exists i)(\mathcal{G}_i\text{-}\exists j)\, \varphi(j)$

(d) If $\mathcal{F}$ and all the $\mathcal{G}_i$ are ultrafilters, then so is $\mathcal{F}\text{-}\lim_i \mathcal{G}_i$.

(e) If $\mathcal{F}$ is a principal ultrafilter concentrated at i_0, then $\mathcal{F}\text{-}\lim_i \mathcal{G}_i = \mathcal{G}_{i_0}$.

(f) If, for each $i \in I$, $\mathcal{G}_i$ is a principal ultrafilter concentrated at $f(i)$, so $f: I \to J$, then $\mathcal{F}\text{-}\lim_i \mathcal{G}_i = f(\mathcal{F})$.

(g) If $(\mathcal{G}'_i)$ is another family of filters on J indexed by I and if $(\mathcal{F}\text{-}\forall_i)\, \mathcal{G}_i = \mathcal{G}'_i$, then $\mathcal{F}\text{-}\lim_i \mathcal{G}_i = \mathcal{F}\text{-}\lim_i \mathcal{G}'_i$.

~~(h) If $f_i : J \to K$ for each $i \in I$, the~~

(h) If $f: J \to K$ then $\mathcal{F}\text{-}\lim_i f(\mathcal{G}_i) = f(\mathcal{F}\text{-}\lim_i \mathcal{G}_i)$.

<u>Proof</u>. All these assertions follow immediately from the definition and the quantifier-manipulation rules in D3, D6, E3, and E2(d). For example, to prove (f) observe that

$$
\begin{aligned}
X \in \mathcal{F}\text{-}\lim \mathcal{G}_i &\iff (\mathcal{F}\text{-}\forall_i)\ X \in \mathcal{G}_i && \text{by definition} \\
&\iff (\mathcal{F}\text{-}\forall_i)\ f(i) \in X && \text{by hypothesis of (f)} \\
&\iff (f(\mathcal{F})\text{-}\forall_j)\ j \in X && \text{by E3} \\
&\iff X \in f(\mathcal{F}) && \text{by Definition D1.} \quad \square
\end{aligned}
$$

<u>Proposition 3</u>. Let $\mathcal{F}$ be a filter on I, f a function $I \to K$, and $(\mathcal{G}_k)$ a K-indexed family of filters on J. Then $\mathcal{F}\text{-}\lim_i \mathcal{G}_{f(i)} = f(\mathcal{F})\text{-}\lim_k \mathcal{G}_k$.

Proof. According to Definition 1, what must be shown is that

$$(\mathcal{F}\text{-}\forall i)\, X \in \mathcal{G}_{f(i)} \iff (f(\mathcal{F})\text{-}\forall k)\, X \in \mathcal{G}_k.$$

But this is true by Proposition E3. $\square$

Proposition 4. Let $\mathcal{F}$ be a filter on I, $(\mathcal{G}_i)$ an I-indexed family of filters on J, and $(\mathcal{H}_j)$ a J-indexed family of filters on K. Then $(\mathcal{F}\text{-}\lim_i \mathcal{G}_i)\text{-}\lim_j \mathcal{H}_j = \mathcal{F}\text{-}\lim_i (\mathcal{G}_i\text{-}\lim_j \mathcal{H}_j)$.

Proof.

$$\begin{aligned}
X \in (\mathcal{F}\text{-}\lim_i \mathcal{G}_i)\text{-}\lim_j \mathcal{H}_j &\iff (\mathcal{F}\text{-}\lim_i \mathcal{G}_i - \forall j)\, X \in \mathcal{H}_j && \text{by Definition 1} \\
&\iff (\mathcal{F}\text{-}\forall i)(\mathcal{G}_i\text{-}\forall j)\, X \in \mathcal{H}_j && \text{by Proposition 2(b.} \\
&\iff (\mathcal{F}\text{-}\forall i)\, X \in \mathcal{G}_i\text{-}\lim_j \mathcal{H}_j && \text{by Definition 1} \\
&\iff X \in \mathcal{F}\text{-}\lim_i (\mathcal{G}_i\text{-}\lim_j \mathcal{H}_j) && \text{by Definition 1.}
\end{aligned}$$

Insert Ⓐ from p. $3\frac{1}{2}$

Definition 5. Let $\mathcal{F}$ be a filter on I, $(\mathcal{G}_i)$ an I-indexed family of filters on J. The *sum* with respect to $\mathcal{F}$ of the family $(\mathcal{G}_i)$ is

$$\begin{aligned}
\mathcal{F}\text{-}\textstyle\sum_i \mathcal{G}_i &= \{X \subseteq I \times J \mid (\mathcal{F}\text{-}\forall i)(\mathcal{G}_i\text{-}\forall j)\ (i,j) \in X\} \\
&= \{X \subseteq I \times J \mid \{i \in I \mid \{j \in J \mid (i,j) \in X\} \in \mathcal{G}_i\} \in \mathcal{F}\}.
\end{aligned}$$

Sums and limits are obviously closely related. The following propositions make the relationship explicit.

(A) Insert on p. 3

The next construction is the $\mathcal{F}$-sum of a family of filters $\mathcal{G}_i$. It should be viewed as a limit of isomorphic copies of the $\mathcal{G}_i$, the isomorphisms being chosen so as to put pairwise disjoint sets into the images of the $\mathcal{G}_i$. This viewpoint is made precise in Proposition 6(b,c) below.

Return to p. 3.

~~(B) Insert on p. 4~~

~~Proposition 6½. Suppose that Let $\mathcal{F}$ and $(\mathcal{G}_i)$ be as in Definitions 1 and 5, and suppose there exist pairwise disjoint sets A_i such that $A_i \in \mathcal{G}_i$ for all $i \in I$. Then π_2 is an isomorphism from $\mathcal{F}$-$\sum_i \mathcal{G}_i$ to $\mathcal{F}$-$\lim_i \mathcal{G}_i$~~

~~Proof. In view~~

(B) Insert on p. 4

(c) If there are pairwise disjoint sets A_i such that $A_i \in \mathcal{G}_i$ for each $i \in I$, then π_2 is an isomorphism from $\mathcal{F}$-$\sum_i \mathcal{G}_i$ to $\mathcal{F}$-$\lim_i \mathcal{G}_i$.

Return to p. 4

(C) Insert on p. 4

(c) In view of (b) and Proposition E10, it suffices to observe that, because the A_i are pairwise disjoint, π_2 is one-to-one on $\{(i,j) \in I \times J \mid j \in A_i\}$, which is in $\mathcal{F}$-$\sum_i \mathcal{G}_i$. □

Return to p. 4.

<u>Proposition 6</u>. Let $\mathcal{F}$ and $(\mathcal{G}_i)$ be as in Definitions 1 and 5, let $\pi_2 : I \times J \to J$ be the projection to the second factor, and, for each i, let $\varepsilon_i : J \to I \times J$ be the inclusion of the i^{th} fiber, $\varepsilon_i(j) = (i,j)$. Then

(a) $\mathcal{F}\text{-}\lim_i \mathcal{G}_i = \pi_2(\mathcal{F}\text{-}\sum_i \mathcal{G}_i)$

(b) $\mathcal{F}\text{-}\sum_i \mathcal{G}_i = \mathcal{F}\text{-}\lim_i \varepsilon_i(\mathcal{G}_i)$.

Insert B from p. $3\frac{1}{2}$.

<u>Proof</u>. (a) $X \in \pi_2(\mathcal{F}\text{-}\sum_i \mathcal{G}_i) \iff \pi_2^{-1}(X) \in \mathcal{F}\text{-}\sum_i \mathcal{G}_i$

$$\iff (\mathcal{F}\text{-}\forall i)(\mathcal{G}_i\text{-}\forall j)\ (i,j) \in \pi_2^{-1}(X)$$

$$\iff (\mathcal{F}\text{-}\forall i)(\mathcal{G}_i\text{-}\forall j)\ j \in X$$

$$\iff X \in \mathcal{F}\text{-}\lim_i \mathcal{G}_i.$$

(b) $X \in \mathcal{F}\text{-}\lim_i \varepsilon_i(\mathcal{G}_i) \iff (\mathcal{F}\text{-}\forall i)(\varepsilon_i(\mathcal{G}_i)\text{-}\forall x)\ x \in X$

$$\iff (\mathcal{F}\text{-}\forall i)(\mathcal{G}_i\text{-}\forall j)\ \varepsilon_i(j) \in X$$

$$\iff X \in \mathcal{F}\text{-}\sum_i \mathcal{G}_i.$$

Insert C from p. $3\frac{1}{2}$

The formal properties of filter sums listed in the next proposition can be proved either directly from Definition 5 or by reducing them to properties of limits via Proposition 6.

<u>Proposition 7.</u> Let $\mathcal{F}$ be a filter on I, $(\mathcal{G}_i)$ an I-indexed family of filters on J.

(a) $\mathcal{F}\text{-}\sum_i \mathcal{G}_i$ is a filter on $I\times J$.

(b) $(\mathcal{F}\text{-}\sum_i \mathcal{G}_i\text{-}\forall x)\,\varphi(x) \iff (\mathcal{F}\text{-}\forall i)(\mathcal{G}_i\text{-}\forall j)\;\varphi(i,j)$.

(c) $(\mathcal{F}\text{-}\sum_i \mathcal{G}_i\text{-}\exists x)\,\varphi(x) \iff (\mathcal{F}\text{-}\exists i)(\mathcal{G}_i\text{-}\exists j)\;\varphi(i,j)$.

(d) If $\mathcal{F}$ and all the $\mathcal{G}_i$ are ultrafilters, then so is $\mathcal{F}\text{-}\sum_i \mathcal{G}_i$.

(e) If $\mathcal{F}$ is a principal ultrafilter concentrated at i_0, then $\mathcal{F}\text{-}\sum_i \mathcal{G}_i = \varepsilon_{i_0}(\mathcal{G}_{i_0})$.

(f) If, for each $i\in I$, $\mathcal{G}_i$ is a principal ultrafilter concentrated at $f(I)$, then $\mathcal{F}\text{-}\sum_i \mathcal{G}_i = f'(\mathcal{F})$, where $f': I \to I\times J$ is defined by $f'(i) = (i, f(i))$.

(g) If $(\mathcal{G}'_i)$ is another family of filters on J and if $(\mathcal{F}\text{-}\forall i)\;\mathcal{G}_i = \mathcal{G}'_i$, then $\mathcal{F}\text{-}\sum_i \mathcal{G}_i = \mathcal{F}\text{-}\sum_i \mathcal{G}'_i$

~~Proposition 8. Let $\mathcal{F}$ be a filter on~~

~~(h) If $f: J \to K$, define $\mathrm{id}\times f: I\times J \to I\times K$ by $(\mathrm{id}\times f)(i,j) = (i, f(j))$~~

so $\varepsilon_i f_i = f'\varepsilon_i$.

(k) If, for each $i\in I$, $f_i : J \to K$, define $f': I\times J \to I\times K$ by $f'(i,j) = (i, f_i(j))$,

Then $\mathcal{F}\text{-}\sum_i f_i(\mathcal{G}_i) = f'(\mathcal{F}\text{-}\sum_i \mathcal{G}_i)$. $\square$

<u>Proposition 8</u>. If $\mathcal{F}$ is a filter on I, f is a function from I to K, $(\mathcal{G}_k)$ is a (K-indexed) family of filters on J, and $f' : I \times J \to K \times J$ is defined by $f'(i,j) = (f(i), j)$, then

$$f'(\mathcal{F}\text{-}\textstyle\sum_i \mathcal{G}_{f(i)}) = f(\mathcal{F})\text{-}\textstyle\sum_k \mathcal{G}_k .$$

<u>Proof</u>. Notice that $f'\varepsilon_i = \varepsilon_{f(i)}$, where $\varepsilon_i : J \to I \times J$ and $\varepsilon_{f(i)} : J \to K \times J$. Thus,

$$f'(\mathcal{F}\text{-}\textstyle\sum_i \mathcal{G}_{f(i)}) = f'(\mathcal{F}\text{-}\lim_i \varepsilon_i(\mathcal{G}_{f(i)})) = \mathcal{F}\text{-}\lim_i f'\varepsilon_i(\mathcal{G}_{f(i)})$$

$$= \mathcal{F}\text{-}\lim_i \varepsilon_{f(i)}(\mathcal{G}_{f(i)}) = f(\mathcal{F})\text{-}\lim_k \varepsilon_k(\mathcal{G}_k) = f(\mathcal{F})\text{-}\textstyle\sum_k \mathcal{G}_k ,$$

by Propositions 6 and 3. $\square$

<u>Proposition 9</u>. Let $\mathcal{F}$ be a filter on I, $(\mathcal{G}_i)$ an I-indexed family of filters on J, and $(\mathcal{H}_{ij})$ an $I \times J$-indexed family of filters on K. Then the bijection $g : (I \times J) \times K \to I \times (J \times K)$, given by $g((i,j),k) = (i,(j,k))$ is an isomorphism from $(\mathcal{F}\text{-}\sum_i \mathcal{G}_i)\text{-}\sum_{ij} \mathcal{H}_{ij}$ to $\mathcal{F}\text{-}\sum_i (\mathcal{G}_i - \sum_j \mathcal{H}_{ij})$.

<u>Proof</u>. Since g is a bijection, it suffices, by Proposition E10, to check that it maps the one filter to the other. Observe that

$$\cancel{g((\mathcal{F}\text{-}\textstyle\sum_i \mathcal{G}_i)\text{-}\sum_{ij} \mathcal{H}_{ij})}$$

$$X \in (\mathcal{F}\text{-}\textstyle\sum_i \mathcal{G}_i)\text{-}\textstyle\sum_{ij} \mathcal{H}_{ij} \iff (\mathcal{F}\text{-}\textstyle\sum_i \mathcal{G}_i\text{-}\forall x)(\mathcal{H}_x\text{-}\forall k)\ (x,k)\in X$$
$$\iff (\mathcal{F}\text{-}\forall i)(\mathcal{G}_i\text{-}\forall j)(\mathcal{H}_{ij}\text{-}\forall k)\ ((i,j),k)\in X.$$

Similarly,

$$X \in \mathcal{F}\text{-}\textstyle\sum_i (\mathcal{G}_i\text{-}\textstyle\sum_{j} \mathcal{H}_{ij}) \iff (\mathcal{F}\text{-}\forall i)(\mathcal{G}_i\text{-}\textstyle\sum_j \mathcal{H}_{ij}\text{-}\forall y)\ (i,y)\in X$$
$$\iff (\mathcal{F}\text{-}\forall i)(\mathcal{G}_i\text{-}\forall j)(\mathcal{H}_{ij}\text{-}\forall k)\ (i,(j,k))\in X.$$

These equivalences and the definition of g clearly imply the proposition. $\square$

Usually, $(I\times J)\times K$ and $I\times(J\times K)$ are identified via g (and called simply $I\times J\times K$). Then $(\mathcal{F}\text{-}\sum_i \mathcal{G}_i)\text{-}\sum_{ij}\mathcal{H}_{ij}$ and $\mathcal{F}\text{-}\sum_i(\mathcal{G}_i\text{-}\sum_{ij}\mathcal{H}_{ij})$ are not merely isomorphic but equal.

Observe that, in Proposition 7(k), if each f_i is an isomorphism from $\mathcal{G}_i$ to $f_i(\mathcal{G}_i)$, then f' is an isomorphism from $\mathcal{F}\text{-}\sum_i \mathcal{G}_i$ to $\mathcal{F}\text{-}\sum_i f_i(\mathcal{G}_i)$. Indeed, if each f_i is one-to-one on a set $A_i \in \mathcal{G}_i$, then f' is one-to-one on the set $\{(i,j)\mid j\in A_i\}\in \mathcal{F}\text{-}\sum_i \mathcal{G}_i$. Similarly, in Proposition 8, if f is an isomorphism from $\mathcal{F}$ to $f(\mathcal{F})$, then f' is an isomorphism from $\mathcal{F}\text{-}\sum_i \mathcal{G}_{f(i)}$ to $f(\mathcal{F})\text{-}\sum_i \mathcal{G}_k$.

A remarkable theorem of Rudin [ ], to be proved in Chapter 2, asserts that, when all the filters involved are ultrafilters, any isomorphism between two sums can be obtained by composing isomorphisms obtained in this way from Propositions 7(b) and 8 (and an isomorphism given by Proposition 9, if one does not identify $(I\times J)\times K$ with $I\times(J\times K)$). The ~~next few ... results ...~~ next ~~few~~ results will be used in the proof of that theorem but are also of interest in their own right. The first of them complements Proposition 6(a).

<u>Proposition 10.</u> Let $\pi_1: I\times J \to I$ be the first projection. $\pi_1(\mathcal{F}\text{-}\sum_i \mathcal{G}_i) = \mathcal{F}$.

<u>Proof.</u> $X \in \pi_1(\mathcal{F}\text{-}\sum_i \mathcal{G}_i) \iff (\mathcal{F}\text{-}\forall_i)(\mathcal{G}_i\text{-}\forall_j)\ (i,j)\in \pi_1^{-1}(X)$

$$\iff (\mathcal{F}\text{-}\forall_i)(\mathcal{G}_i\text{-}\forall_j)\ \ i\in X \quad \iff (\mathcal{F}\text{-}\forall_i)\ i\in X. \ \square$$

<u>Proposition 11.</u> Let $\mathcal{F}, \mathcal{F}'$ be filters on I and let $(\mathcal{G}_i), (\mathcal{G}_i')$ be I-indexed families of filters on J. $\mathcal{F}\text{-}\sum_i \mathcal{G}_i \subseteq \mathcal{F}'\text{-}\sum_i \mathcal{G}_i'$, if and only if both $\mathcal{F}\subseteq\mathcal{F}'$ and $(\mathcal{F}'\text{-}\forall_i)\ \mathcal{G}_i \subseteq \mathcal{G}_i'$.

<u>Proof.</u> Assume $\mathcal{F}\subseteq\mathcal{F}'$ and $(\mathcal{F}'\text{-}\forall_i)\ \mathcal{G}_i\subseteq\mathcal{G}_i'$ and $X\in\mathcal{F}\text{-}\sum_i\mathcal{G}_i$. So $(\mathcal{F}\text{-}\forall_i)(\mathcal{G}_i\text{-}\forall_j)\ (i,j)\in X$. Apply Proposition D4, first to $\mathcal{F}$ and $\mathcal{F}'$ and then to $\mathcal{G}_i$ and

$$\mathcal{G}_i'' = \begin{cases} \mathcal{G}_i' & \text{if } \mathcal{G}_i\subseteq\mathcal{G}_i' \\ \mathcal{G}_i & \text{otherwise} \end{cases}$$

to get $X \in \mathcal{F}'\text{-}\sum_i \mathcal{G}_i''$. Then invoke Proposition 7(g) to replace $\mathcal{G}_i''$ with $\mathcal{G}_i'$.

Conversely, assume $\mathcal{F}\text{-}\sum_i \mathcal{G}_i \subseteq \mathcal{F}'\text{-}\sum_i \mathcal{G}_i'$. Apply π_1 to both sides; the inclusion relationship is preserved by Proposition E4, and by Proposition 10 the result is $\mathcal{F} \subseteq \mathcal{F}'$. Suppose that not $(\mathcal{F}'\text{-}\forall i)\ \mathcal{G}_i \subseteq \mathcal{G}_i'$. So every set in $\mathcal{F}'$ meets the set

$$A = \{ i \in I \mid \mathcal{G}_i \not\subseteq \mathcal{G}_i' \}.$$

For $i \in A$, let $B_i \in \mathcal{G}_i - \mathcal{G}_i'$. For $i \notin A$, let $B_i = J$, so $B_i \in \mathcal{G}_i$ for all i. Thus, the set $\{(i,j) \in I \times J \mid j \in B_i\}$ belongs to $\mathcal{F}\text{-}\sum_i \mathcal{G}_i$, hence also to $\mathcal{F}'\text{-}\sum_i \mathcal{G}_i'$. So $(\mathcal{F}'\text{-}\forall i)(\mathcal{G}_i'\text{-}\forall j)\ j \in B_i$. But, by definition of B_i, $(\mathcal{G}_i'\text{-}\forall j)\ j \in B_i$ if and only if $i \notin A$. So $(\mathcal{F}'\text{-}\forall i)\ i \notin A$. This contradicts the fact that every set in $\mathcal{F}'$ meets A. $\square$

<u>Corollary 12</u>. With notation as in Proposition 11, $\mathcal{F}\text{-}\sum_i \mathcal{G}_i = \mathcal{F}'\text{-}\sum_i \mathcal{G}_i'$ if and only if $\mathcal{F} = \mathcal{F}'$ and $(\mathcal{F}\text{-}\forall i)\ \mathcal{G}_i = \mathcal{G}_i'$. $\square$

<u>Proposition 13</u>. ~~Suppose that~~ Let $\mathcal{F}$ be a filter on I, $(\mathcal{G}_i)$ an I-indexed family of filters on J, and $h : I \times J \to I \times K$ a function such that $\pi_1 h = \pi_1$ at $\mathcal{F}\text{-}\sum_i \mathcal{G}_i$-almost all points. Then $h(\mathcal{F}\text{-}\sum_i \mathcal{G}_i) = \mathcal{F}\text{-}\sum_i \mathcal{H}_i$ for certain filters $\mathcal{H}_i$ on K, and h agrees, at $\mathcal{F}\text{-}\sum_i \mathcal{G}_i$-most points, with an f' obtained as in

Proposition 7(h) from some maps f_i of $\mathcal{G}_i$ to $\mathcal{H}_i$.

Proof. By hypothesis, h is $\mathcal{F}$-$\sum_i \mathcal{G}_i$-almost everywhere equal to a function h' that satisfies $\pi_1 h' = \pi_1$ everywhere. Thus, for all $(i,j) \in I \times J$, $h'(i,j)$ has the form $(i, g(i,j))$. Set $f_i(j) = g(i,j)$; then h' agrees everywhere with the map f' of Proposition 7(h). Set $\mathcal{H}_i = f_i(\mathcal{G}_i)$. Then, as h and f' agree $\mathcal{F}$-$\sum_i \mathcal{G}_i$-almost everywhere,

$$h(\mathcal{F}\text{-}\textstyle\sum_i \mathcal{G}_i) = f'(\mathcal{F}\text{-}\sum_i \mathcal{G}_i) = \mathcal{F}\text{-}\sum_i f_i(\mathcal{G}_i) = \mathcal{F}\text{-}\sum_i \mathcal{H}_i. \quad \square$$

An important special case of sums is the case where all the summands are equal.

Definition 14 Let $\mathcal{F}$ and $\mathcal{G}$ be filters on I and J, respectively. Their (tensor) product is

$$\mathcal{F} \otimes \mathcal{G} = \mathcal{F}\text{-}\textstyle\sum_i \mathcal{G} = \{X \subseteq I \times J \mid (\mathcal{F}\text{-}\forall i)(\mathcal{G}\text{-}\forall j)\ (i,j) \in X\}.$$

Of course, the previous results about sums are all applicable to this special case.

Corollary 15. Let $\mathcal{F}$ and $\mathcal{G}$ be filters on I and J, respectively.

(a) $\mathcal{F} \otimes \mathcal{G}$ is a filter.

(b) $(\mathcal{F} \otimes \mathcal{G}\text{-}\forall x)\ \varphi(x) \iff (\mathcal{F}\text{-}\forall i)(\mathcal{G}\text{-}\forall j)\ \varphi(i,j)$.

(c) $(\mathcal{F}\otimes\mathcal{G}\text{-}\exists x)\,\varphi(x) \iff (\mathcal{F}\text{-}\exists i)(\mathcal{G}\text{-}\exists j)\,\varphi(i,j)$.

(d) If $\mathcal{F}$ and $\mathcal{G}$ are ultrafilters, then so is $\mathcal{F}\otimes\mathcal{G}$.

(e) If $\mathcal{F}$ is a principal ultrafilter concentrated at i_0, then $\mathcal{F}\otimes\mathcal{G} = \varepsilon_{i_0}(\mathcal{G})$.

(f) If $\mathcal{G}$ is a principal ultrafilter concentrated at j_0, then $\mathcal{F}\otimes\mathcal{G} = \varepsilon'_{j_0}(\mathcal{F})$, where $\varepsilon'_j : I \to I\times J$ sends each i to (i,j).

((g). Omitted, since the analog of Proposition 7(g) in the present context is trivial.)

(h) If $f: J\to K$, define $f' : I\times J \to I\times K$ by $f'(i,j)=(i,f(j))$. Then $\mathcal{F}\otimes f(\mathcal{G}) = f'(\mathcal{F}\otimes\mathcal{G})$.

~~(i) $\pi_1(\mathcal{F}\otimes\mathcal{G})=\mathcal{F}$ and $\pi_2(\mathcal{F}\otimes\mathcal{G})=\mathcal{G}$~~

(i) If $f: I\to K$, define $f' : I\times J\to K\times J$ by $f'(i,j)=(f(i),j)$. Then $f(\mathcal{F})\otimes\mathcal{G} = f'(\mathcal{F}\otimes\mathcal{G})$.

(j) $\pi_1(\mathcal{F}\otimes\mathcal{G})=\mathcal{F}$; $\pi_2(\mathcal{F}\otimes\mathcal{G})=\mathcal{G}$.

(k) If $\mathcal{H}$ is a filter on K then $(\mathcal{F}\otimes\mathcal{G})\otimes\mathcal{H}$ is isomorphic to $\mathcal{F}\otimes(\mathcal{G}\otimes\mathcal{H})$ by the canonical bijection from $(I\times J)\times K$ to $I\times(J\times K)$.

(l) If $\mathcal{G}'$ is another filter on J, then $\mathcal{F}\otimes\mathcal{G}\subseteq\mathcal{F}'\otimes\mathcal{G}'$ if and only if $\mathcal{F}\subseteq\mathcal{F}'$ and $\mathcal{G}\subseteq\mathcal{G}'$. Hence, $\mathcal{F}\otimes\mathcal{G}=\mathcal{F}'\otimes\mathcal{G}'$ if and only if $\mathcal{F}=\mathcal{F}'$ and $\mathcal{G}=\mathcal{G}'$. $\square$

The product construction can be used to extend to filters any binary operation defined on their underlying sets. This will be especially useful for ultrafilters.

<u>Definition 16</u>. Let $* : I \times I \to I$ be any binary operation on I, and let $\mathcal{F}$ and $\mathcal{G}$ be filters on I. Then

$$\mathcal{F} * \mathcal{G} = *(\mathcal{F} \otimes \mathcal{G}) = \{ X \subseteq I \mid (\mathcal{F}\text{-}\forall i)(\mathcal{G}\text{-}\forall j)\ i * j \in X \}.$$

<u>Proposition 18</u>. Let $*, \mathcal{F}, \mathcal{G}$ be as in Definition 17.

(a) $\mathcal{F} * \mathcal{G}$ is a filter on I. If $\mathcal{F}$ and $\mathcal{G}$ are ultrafilters, then so is $\mathcal{F} * \mathcal{G}$.

(b) $(\mathcal{F} * \mathcal{G}\text{-}\forall i)\ \varphi(i) \iff (\mathcal{F}\text{-}\forall i)(\mathcal{G}\text{-}\forall j)\ \varphi(i * j)$;

$(\mathcal{F} * \mathcal{G}\text{-}\exists i)\ \varphi(i) \iff (\mathcal{F}\text{-}\exists i)(\mathcal{G}\text{-}\exists j)\ \varphi(i * j)$

(c) For each $i \in I$, let $t_i : I \to I$ be the "translation" defined by $t_i(j) = i * j$. Then $\mathcal{F} * \mathcal{G} = \mathcal{F}\text{-}\lim_i t_i(\mathcal{G})$.

(d) If $\mathcal{F}$ and $\mathcal{G}$ are principal ultrafilters concentrated at i and j, then $\mathcal{F} * \mathcal{G}$ is the principal ultrafilter concentrated at $i * j$.

(e) If the operation $*$ is associative on I, then it is also associative on filters, i.e., $(\mathcal{F} * \mathcal{G}) * \mathcal{H} = \mathcal{F} * (\mathcal{G} * \mathcal{H})$.