AXIOMS AND MODELS FOR AN EXTENDED SET THEORY

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Abstract. We present the axioms of extended set theory (XST) and the ideas underlying the axioms. We also present an interpretation of XST in ZFC plus "there exist arbitrarily large inaccessible cardinals," thereby proving the consistency of XST relative to this mild large-cardinal extension of ZFC.

1. Introduction

This paper presents an extended set theory (XST) and proves its consistency relative to the classical Zermelo-Fraenkel set theory with the axiom of choice (ZFC) and an axiom asserting the existence of arbitrarily large inaccessible cardinals (also known as Grothendieck's axiom of universes). The original motivation for this development of XST was to provide an improved, flexible mathematical foundation for certain constructions, like tuples and functions, that play important roles in computing. Subsequent developments revealed that XST also provides a smooth way of handling the set-class distinction that turns up especially in category theory. We shall comment on these matters in more detail after the axioms of XST have been introduced.

The extended set theory XST differs from Zermelo-Fraenkel set theory ZFC in three major ways (and several minor ways).

The biggest difference between XST and ZFC is that the fundamental membership relation of XST is a ternary relation, written $x \in_s y$ and read "x is an element of y with *scope* s." Most of the axioms of XST are analogs of traditional ZFC axioms, modified to account for membership scopes, and allowing atoms (also called urelements). This aspect of XST can be interpreted in ZFC quite straightforwardly.

The second major difference is that XST contains the so-called Klass axiom, which permits the formation of certain very large sets. Specifically, any sets in which (hereditarily) all scopes are below a particular bound can be collected into a set, with higher scopes. The natural

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interpretation of this in classical set theory, which we present below, uses the assumption of arbitrarily large inaccessible cardinals.

The third difference is that XST is agnostic concerning (the analog in the presence of scopes of) the axiom of regularity, also called the axiom of foundation. We shall present two versions of XST in this paper. One includes the axiom of regularity, suitably modified to work with the ternary membership relation. In this version, there are no infinite descending sequences with respect to membership or scopes. The other version permits (but does not require) the existence of infinite descending sequences whose members are all distinct, but it prohibits cycles.

In the absence of the Klass axiom, XST can be interpreted in ZFC by fairly standard methods. The interpretation is easy and natural because the axiom of regularity is assumed or at least not denied in the versions of XST considered here. (One can also design versions of XST that *require* the presence of non-well-founded sets; the interpretation of such versions is more complicated but in some cases very similar to the known interpretation of the so-called anti-foundation axiom in ZFC.)

When the Klass axiom is taken into account, the natural interpretations in classical set theory require the existence of arbitrarily large inaccessible cardinals. We shall exhibit such an interpretation for the first version. It is, of course, also an interpretation for the second version, since the second merely weakens the regularity axiom of the first. (Were we to add, to the second version, an axiom requiring the existence of some non-well-founded sets, then the interpretation would, of course, have to be altered.)

This paper is organized as follows. In Section 2, we present and discuss the axioms of XST. In Section 3, we show how to interpret all the XST axioms except the Klass axiom in ZFC. No large cardinal assumptions are needed for this interpretation. In Section 4, we modify the interpretation so as to verify the Klass axiom also. Finally, in Section 5, we briefly mention some applications of XST.

The original motivation for this work came from two of these applications, namely tuples (Subsections 5.1 and 5.2) and categories (Subsection 5.4). The desirability of a more natural set-theoretic representation of tuples was indicated by Skolem in [4]. The application to categories was suggested by the first author's discussion of set-theoretic foundations for category theory in [3, Section 1]; we provide a fourth option beyond the three listed in [3].

2. Axioms

In this section, we present the axioms of XST. As indicated above, there are actually two versions of these axioms, differing in their assumptions about foundation.

Some of our axioms are the result of importing into the XST context the axioms of ZFC, with suitable modifications to take scopes into account. Those modifications are influenced not only by the presence of scopes but also by the need for consistency with the Klass axiom. We shall discuss those aspects on an intuitive level as soon as they arise, even before the Klass Axiom is stated. For these intuitive discussions, it will be sufficient to keep in mind a rough summary of the relevant aspect of the Klass Axiom, namely that the "limitation of size" idea of classical set theory applies differently for different scopes. A set cannot have as many elements with a small scope as it can with a larger scope.

Our axiomatization of XST has three primitive symbols. The most important one is the ternary membership relation $\in$, used in the form $x \in_s y$ to mean that x is a member of y with scope s. The second primitive symbol is a constant $\varnothing$ denoting the empty set. Since atoms (also called urelements) are permitted in XST, we need a way to distinguish these atoms (which have no members) from the empty set; the constant $\varnothing$ is a simple way to do that. The third primitive symbol is a unary operation symbol $\mathcal{S}_*$, needed for technical reasons. Its intended meaning is that $\mathcal{S}_*(a)$ is the set of all scopes occurring in a, in its elements or scopes, in their elements or scopes, etc. More precisely, $\mathcal{S}_*(a)$ has all these scopes as elements, with themselves as their scopes.

We now begin listing the axioms of XST, interspersing them with comments to clarify their meaning and with notational conventions to simplify subsequent axioms and comments. In stating the axioms, we adopt the common convention that any free variables in an axiom are understood to be universally quantified, with the quantifier governing the whole axiom.

Axiom 1 (Empty Set)**.** *The empty set has no members:*

$$\neg(x \in_s \varnothing).$$

We use the word "atom" to mean anything that has no members (with any scope) but is distinct from $\varnothing$. It will be convenient to have a short notation for "nonempty set" and "set".

Definition 1.

- $\mathcal{X}(a)$ abbreviates $(\exists x, s)\, x \in_s a$.
- $\mathcal{X}_\varnothing(a)$ abbreviates $\mathcal{X}(a) \vee a = \varnothing$.

- $\mathcal{X}(a, b, \ldots, r)$ abbreviates $\mathcal{X}(a) \wedge \mathcal{X}(b) \wedge \cdots \wedge \mathcal{X}(r)$.

So the empty set axiom can be formulated as $\neg\mathcal{X}(\varnothing)$.

Axiom 2 (Extensionality)**.** *Sets with the same members are equal:*

$$[\mathcal{X}(x, y) \wedge (\forall z, s)\,(z \in_s x \iff z \in_s y)] \implies x = y.$$

Although the formal statement of the extensionality axiom refers only to nonempty sets x and y, the more general case where x or y or both can be $\varnothing$ follows immediately. Of course, the axiom cannot apply when x and y are atoms, since all atoms have the same members (as each other and as $\varnothing$).

Definition 2. A set a is a *subset* of b, written $a \subseteq b$, if

$$(\forall x, s)\,(x \in_s a \implies x \in_s b).$$

Note that, in this definition, b is not required to be a set, so the empty set counts as a subset of any atom. This will be convenient later.

Axiom 3 (Pairing)**.**

$$(\exists a)(\forall x)\,(x \in_s a \iff (x = p \vee x = q) \wedge s = \varnothing)$$

Although this axiom provides only for pairs where both scopes are $\varnothing$, pairs that we write $\{p^\varnothing, q^\varnothing\}$, it will follow, from the axiom of replacement, introduced below, that pairs $\{p^s, q^s\}$ (i.e., with the scopes being s rather than $\varnothing$) exist for arbitrary s. When $p = q$, we use the notation $\{p^s\}$. When $p \neq q$, replacement also gives $\{p^s, q^t\}$. (When $p = q$ but $s \neq t$, we still get $\{p^s, q^t\}$, but for this we need, in addition to pairing and replacement, the axiom of binary union.)

Axiom 4 (Separation schema)**.**

$$(\exists a)(\forall x, s)\,[x \in_s a \iff (x \in_s b \wedge \varphi)],$$

where φ is any formula in which a is not free.

The set a asserted to exist in this axiom is written $\{x^s \in b : \varphi\}$.

Not surprisingly, we abbreviate $\{x^s \in b : x \in_s c\}$ as $b \cap c$. Note that the intersection defined this way contains only those common elements that have the same scope in b and in c.

Axiom 5 (Binary union)**.**

$$(\exists a)(\forall x, s)\,[x \in_s a \iff (x \in_s p \vee x \in_s q)].$$

The set a asserted to exist in this axiom is written $p \cup q$. Note that elements and scopes from p and from q are used in $p \cup q$, so both p and q are subsets of $p \cup q$ (if they are sets).

Axiom 6 (Infinity)**.**

$$(\exists a)\,[\varnothing \in_s a \wedge (\forall x)\,(x \in_s a \implies x \cup \{x^s\} \in_s a)].$$

Like any variable free in an axiom, s is to be understood as universally quantified here. For any fixed s, the smallest a of the sort required in the axiom of infinity is a version of the von Neumann natural numbers with all scopes being s.

Axiom 7 (Replacement schema)**.**

$$\begin{aligned}[(\forall x, s, y, t, y', t')\big((x \in_s b \wedge \varphi(x, s, y, t) \wedge \varphi(x, s, y', t')) \implies \\ (y = y' \wedge t = t' \wedge s \subseteq t)\big)] \implies \\ (\exists a)(\forall y, t)\,[y \in_t a \iff (\exists x, s)\,(x \in_s b \wedge \varphi(x, s, y, t))].\end{aligned}$$

Most of this axiom schema is what one might expect on the basis of the replacement schema of ZFC. We can, in any set b, replace elements x and their scopes s by other elements y and scopes t. There is, however, one possibly surprising clause, namely the requirement that $s \subseteq t$. Scopes can be replaced only by larger ones (in the sense of $\subseteq$). To understand the reason for this requirement, recall that the Klass axiom will allow sets to have more elements with large scopes than with small scopes. That arrangement would be ruined if we could take a set with a great many elements, allowed with large scopes, and then replace those scopes by small ones.

Notice that, if all members of b have scope $\varnothing$, then the requirement $s \subseteq t$ is satisfied, no matter what t is. (Recall that the empty set counts as a subset even of atoms.) So, in this case, there is no special restriction on replacement.

Axiom 8 (Union)**.**

$$\begin{aligned}[(\forall x, t, y, u)\,((x \in_t y \wedge y \in_u b) \implies t \cup u \subseteq w)] \implies \\ (\exists a)(\forall x, s)\,[x \in_s a \iff (s = w \wedge (\exists y, t, u)\,(x \in_t y \wedge y \in_u b))\end{aligned}$$

The a asserted to exist in this axiom is the union of the family b in the sense that its elements are the elements of elements of b, but the scopes of these elements in a are all w. Here w can be any set that includes, as subsets, all the supports of elements of b and all the supports of their elements. It might seem more reasonable to take, as the scope of a member x of a, not some fixed w but the same scope t that x has as an element of y; that way, each element y of b would be a subset of the union a. Unfortunately, that form of the union axiom would conflict with the Klass axiom. When there are many y's, with correspondingly large u's, while each y contains only a few x's, each with a small t, then the union could contain many x's (a few from each of many y's) with small scopes. This together with the Klass axiom would lead to a contradiction.

Intuitively, the union would have many elements if either b has many elements (with large scopes u) or those elements have many elements (with large scopes t). For coherence with the Klass axiom, the scopes in the union must be correspondingly large, and that is what our requirement on w is designed to accomplish.

Notice that, although the axiom of union allows a good deal of freedom in choosing w, as long as it is large enough, any use of the axiom requires that we first produce some appropriate w, presumably by invoking appropriate other axioms. The same comment applies to the power set axiom, which we state next.

Axiom 9 (Power set)**.**

$$\begin{aligned}&[(\forall y, u)\,(y \in_u b \implies u \subseteq w)] \implies \\ &\qquad\qquad (\exists a)(\forall x, s)\,[x \in_s a \iff (s = w \wedge x \subseteq b)]\end{aligned}$$

Axiom 10 (Choice)**.**

$$\begin{aligned}&[(\forall x, y)\,((x \in_s a \wedge y \in_s a) \implies \\ &\qquad (\mathcal{X}(x,y) \wedge (x = y \vee x \cap y = \varnothing)))] \implies \\ &\qquad\qquad (\exists c)(\forall z)[z \in_s a \implies (\exists! x)\,(x \in_s c \wedge (\exists v)\,x \in_v z)]\end{aligned}$$

The set c in this axiom "chooses" one element from each of the pairwise disjoint sets z that are elements of a with scope s. The chosen elements are in c with scope s.

To formulate the first version of the foundation axiom, it is convenient to abbreviate "element or scope" as "ingredient". More formally, we say that x is an *ingredient* of a set y, and we write $\mathcal{I}(x, y)$, if there is some z such that either $x \in_z y$ or $z \in_x y$.

Axiom 11 (Strong foundation)**.**

$$\mathcal{X}(a) \implies (\exists x)\,[\mathcal{I}(x, a) \wedge \neg(\exists y)(\mathcal{I}(y, x) \wedge \mathcal{I}(y, a))].$$

The second version of the foundation axiom says that no set is an ingredient of itself, or an ingredient of an ingredient of itself, or .... To formalize this (without an infinite disjunction), we use the following definitions.

Definition 3. A set a is *inductive* if $\varnothing \in_\varnothing a$ and, whenever $x \in_\varnothing a$, then also $x \cup \{x^\varnothing\} \in_\varnothing a$. (In other words, a is as in the axiom of infinity with $s = \varnothing$.) A *natural number* is a set that is a member of every inductive set. The set whose elements are exactly the natural numbers, all with scope $\varnothing$, is denoted by ω; it exists by the axioms of infinity and separation. If n is a natural number, then an *n-tuple* is a set t such that

- $(\forall x, s)\,[x \in_s t \implies s \in_\varnothing n]$, and
- $(\forall s)\,[s \in_\varnothing n \implies (\exists! x)\, x \in_s t]$.

That is, an n-tuple has exactly n members, one with each scope that is a smaller natural number than n. An *ingredient cycle* for a set a is an n-tuple t such that

- $a \in_\varnothing t$,
- $(\forall x, y, k)\,[(x \in_k t \wedge y \in_{k\cup\{k^\varnothing\}} t) \implies \mathcal{I}(y, x)]$, and
- $a \in_k t$ for some $k \neq \varnothing$.

Axiom 12 (Weak foundation)**.** *No set has an ingredient cycle.*

This completes the importation of ZFC axioms, with modifications to deal with scopes. It remains to present the Klass axiom, but in order to do so we first need to specify the last of the primitive notions of XST, the nested-scope operator $\mathcal{S}_*$.

Axiom 13 (Nested scope set)**.**

$$(\forall x, s)\,\big(x \in_s \mathcal{S}_*(a) \iff$$
$$[(x = s \wedge (\exists y)\, y \in_s a) \vee (\exists b, t)\,(b \in_t a \wedge (x \in_s \mathcal{S}_*(b) \vee x \in_s \mathcal{S}_*(t)))]\big).$$

We note in passing that the (non-nested, simple) scope set

$$\mathcal{S}(a) = \{s^s : (\exists x)\, x \in_s a\}$$

can be obtained from $\mathcal{S}_*(a)$ by applying a separation axiom.

Axiom 14 (Klass)**.**

$$(\exists a)(\forall x)\,[\mathcal{S}_*(x) \subseteq h \implies x \in_h a].$$

3. A ZFC Interpretation Without the Klass Axiom

We begin by interpreting in ZFC the idea of sets with scopes. This interpretation can be used to establish the relative consistency of XST minus the Klass axiom. (The Klass axiom will be treated in the next section.) The interpretation is quite straightforward. An XST set a is very similar to a binary relation in ZFC; the primitive relation $x \in_s a$ of XST corresponds to the (non-primitive) $\langle x, s\rangle \in a$ of classical set theory. Accordingly, our interpretation will simply model in ZFC a cumulative hierarchy of binary relations.

We shall begin our construction of this hierarchy with any number of atoms. Strictly speaking, we could do without atoms, since XST does not require the existence of any atoms. Nevertheless, it permits the existence of atoms, and they might be useful for some purposes. Since no serious additional work is needed to model them, we do so.

Within ZFC, we shall use the standard Kuratowski coding of ordered pairs, $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$. We also use the standard von Neumann coding of ordinal numbers, where each ordinal number is the set of all strictly smaller ordinal numbers. In particular, the empty set $\varnothing$ is identified with the ordinal 0, and it is a member of every non-zero ordinal. We note that an ordered pair neither equals nor contains $\varnothing$, so no ordered pair is an ordinal number. Furthermore, no relation (i.e., set of ordered pairs) except $\varnothing$ is an ordinal number.

Let $\mathcal{A}$ be an arbitrary set of non-zero ordinal numbers. The *cumulative hierarchy of relations* over $\mathcal{A}$ is defined in ZFC as follows.

$$\begin{aligned} C_0 &= \mathcal{A} \\ C_{\alpha+1} &= \mathcal{A} \cup \mathcal{P}(C_\alpha \times C_\alpha) \\ C_\lambda &= \bigcup_{\alpha<\lambda} C_\alpha && \text{for limit } \lambda \\ C &= \bigcup_{\text{ordinals } \alpha} C_\alpha && . \end{aligned}$$

Here $\times$ is the usual cartesian product of two sets; $X \times Y$ is the set of ordered pairs $\langle x, y\rangle$ with $x \in X$ and $y \in Y$. We also used the usual notation $\mathcal{P}$ for power set, the collection of all subsets of a set. Thus, $\mathcal{P}(C_\alpha \times C_\alpha)$ is the set of all binary relations on C_α.

For each ordinal α, C_α is a set, but C, the union over all α, is a proper class. It is easy to check, by transfinite induction, that $C_\alpha \subseteq C_\beta$ whenever $\alpha < \beta$. For each $x \in C - \mathcal{A}$, there is a unique ordinal α such that $x \in C_{\alpha+1} - C_\alpha$. This α will be called the *height* of x.

We adopt the convention that atoms (if any exist) have height -1. Here -1 is to be regarded as a formal symbol strictly smaller than all ordinals. Thus, height is defined for all elements of C. The elements of C_α are exactly the elements of C of height $< \alpha$.

An equivalent way to describe C (which we won't need, but which may be helpful for intuition) is that it is the smallest class that has, among its elements, all the members of $\mathcal{A}$ and all sets of ordered pairs whose two components are in C.

Our interpretation of XST minus the Klass axiom has C as its universe of discourse, and the primitive ternary predicate $\in$ is interpreted as

$$x \in_s a \text{ is } \langle x, s\rangle \in a.$$

The $\varnothing$ of XST can (conveniently) be taken to be the $\varnothing$ of ZFC, the empty relation, which is of course in C_1 and thus in C.

Recalling that the elements of $\mathcal{A}$, being non-zero ordinal numbers, are not binary relations, we see that they serve as the atoms in our interpretation of XST.

It remains to give the interpretation of the nested scope operation $\mathcal{S}_*$. The idea here is simply to read the Nested Scope Set axiom, Axiom 13, as a recursive definition of $\mathcal{S}_*$. More explicitly, we invoke the familiar metatheorem on definitions by recursion in ZFC to obtain a definable operation $\mathcal{S}_*$ such that (provably in ZFC), for all $a \in C$:

- If $a \in C_0 = \mathcal{A}$ then $\mathcal{S}_*(a) = \varnothing$.
- If $a \in C_{\alpha+1} - C_\alpha$ then

$$\mathcal{S}_*(a) = \{\langle s, s\rangle : \langle x, s\rangle \in a\} \cup \bigcup_{\langle x,s\rangle \in a} \big(\mathcal{S}_*(x) \cup \mathcal{S}_*(s)\big).$$

With these definitions for the universe of discourse and the primitive notions of XST, it is straightforward to work out the defined terms of XST and to check that the axioms of XST, except for the Klass axiom, hold in our interpretation. Explicitly, this means that, when these axioms are translated into the language of ZFC, using the definitions above, the translated assertions are theorems of ZFC. We shall not carry out the details of this verification, because the Klass axiom fails in this interpretation. Instead, we shall, in the next section, modify the interpretation in such a way that all the XST axioms, including the Klass axiom, hold.

4. Interpreting the Klass Axiom

If we tried to verify the Klass axiom in the interpretation from the preceding section, we would need (in ZFC)

$$a = \{\langle x, h\rangle : \mathcal{S}_*(x) \subseteq h\},$$

but this a is a proper class, so it is not available to serve as a witness. We must therefore modify the interpretation, and the following seems to be the most natural way to accomplish this.

We work in the theory ZFC+, which is ZFC plus the assumption that there are arbitrarily large inaccessible cardinals. This assumption is equivalent to the assumption that every set is a member of some Grothendieck universe; in this form it is often used in category theory and its applications to algebraic geometry. From a set-theorist's point of view, it is a rather mild large-cardinal assumption. From now on, we work in the theory ZFC+.

Notation 4. Fix an enumeration $\langle i_\alpha \rangle$, in increasing order, of some inaccessible cardinals such that $i_\alpha > |\mathcal{A}|$ and $i_\alpha > \alpha$, for all α.

Note that the inequalities imposed on i_α can both be achieved by just skipping some elements in the increasing enumeration of all the inaccessible cardinals. The following lemma is a useful consequence of these inequalities.

Lemma 5. *The cardinality of C_α is strictly smaller than i_α.*

Proof. We proceed by induction on α. If $\alpha = 0$, we must prove $|C_0| < i_0$. Since $C_0 = \mathcal{A}$, this is covered by our requirements on the enumeration $\langle i_\alpha \rangle$.

Suppose $\alpha = \beta + 1$. By induction hypothesis, $|C_\beta| < i_\beta < i_\alpha$. Since i_α is inaccessible, we can infer that

$$|C_\alpha| = |\mathcal{A} \cup \mathcal{P}(C_\beta \times C_\beta)| < i_\alpha.$$

Finally, suppose α is a limit cardinal. Then C_α is, by definition, the union of an increasing α-sequence of sets, each having, by induction hypothesis, cardinality strictly smaller than i_α. Since $\alpha < i_\alpha$, the inaccessibility of i_α gives us $|C_\alpha| < i_\alpha$, as required. □

Definition 6. An element a of C is *scope-bounded* if, for each ordinal α, the number of $\langle x, s \rangle \in a$ with s of height $\leq \alpha$ is $< i_{2\alpha+1}$.

So a scope-bounded set could have arbitrarily many elements, but, if the number of elements is large enough, then most of them must have scopes of great height. Note that every element of $\mathcal{A}$ is scope-bounded, because it has no elements of the form $\langle x, s \rangle$.

The following lemma says that, when verifying that a set is scope-bounded, we do not need to consider all scopes s of height $\leq \alpha$ together but can consider each s separately.

Lemma 7. *Suppose $a \in C$ and, for each ordinal α and each s of height α, the number of x for which $\langle x, s \rangle \in a$ is $< i_{2\alpha+1}$. Then a is scope-bounded.*

Proof. For any ordinal α, the set of s's of height $\leq \alpha$ is $C_{\alpha+1}$, which has, by Lemma 5, cardinality $< i_{\alpha+1} \leq i_{2\alpha+1}$. Since each of these s's contributes, by assumption, $< i_{2\alpha+1}$ pairs $\langle x, s \rangle$ to a, and since $i_{2\alpha+1}$ is inaccessible, we obtain that all these s's together contribute $< i_{2\alpha+1}$ pairs to a, as required by the definition of "scope-bounded". □

Let B be the subclass of C consisting of the hereditarily scope-bounded elements. Here a set is called *hereditarily scope-bounded* if not only it but also all members of its transitive closure are scope-bounded.

We define a new interpretation of the language of XST in ZFC+ by taking B as its universe of discourse. (So we are shrinking the universe from C down to B.) The primitive, ternary, membership predicate of XST is interpreted as it was in C, and so is the operation $\mathcal{S}_*$. Notice that $\mathcal{S}_*(a)$ contains, for each s, at most one pair of the form $\langle x, s\rangle$, namely $\langle s, s\rangle$. It follows by Lemma 7 that $\mathcal{S}_*(a)$ is scope-bounded. If, furthermore, a is hereditarily scope-bounded, then so is $\mathcal{S}_*(a)$, because all elements of the transitive closure of $\mathcal{S}_*(a)$, except for $\mathcal{S}_*(a)$ itself, either have size at most two or are in the transitive closure of a. Thus, B is closed under the operation $\mathcal{S}_*$, and therefore we have a legitimate interpretation of $\mathcal{S}_*$ with universe B.

The rest of this section is devoted to the verification that this interpretation "works", i.e., that the interpretations of the axioms of XST are provable in ZFC+. Some of the axioms of XST involve, in addition to the primitive concepts $\in$, $\varnothing$ and $\mathcal{S}_*$, various defined concepts. So we shall need to use the interpretations of the primitive concepts to produce the resulting interpretations of those defined concepts. We describe those interpretations between the verifications of the axioms, in the same order as in Section 2.

The interpretation of the axiom of empty set says that $\varnothing$ has no members of the form $\langle x, s\rangle$ with $x, s \in B$. It holds because (it is provable in ZFC+ that) $\varnothing$ has no members at all.

The interpretation of the symbol $\mathcal{X}(a)$ says, for a in our universe B, that a contains a pair $\langle x, s\rangle$ (i.e., that $a \notin \mathcal{A} \cup \{\varnothing\}$). Similarly, the interpretation of $\mathcal{X}_\varnothing(a)$ says that $a \notin \mathcal{A}$.

The interpretation of the axiom of extensionality says that, if $x, y \in B - \mathcal{A}$ and if x and y contain the same pairs $\langle z, s\rangle$, then $x = y$. This follows from the extensionality axiom of ZFC+, because all elements of x and y are such pairs $\langle z, s\rangle$. Technically, it should be noted here that the relevant z and s are also in B, since the quantifiers in the interpretation of the axiom of extensionality range only over elements of B. Here we invoke the "hereditarily" part of the definition of B, which guarantees that, if $a \in B$ and $\langle z, s\rangle \in a$, then $z, s \in B$.

The definition of $a \subseteq b$ gives, in our interpretation, that $a \in B - \mathcal{A}$, $b \in B$, and $a \subseteq b$ (as sets in the ZFC+ sense).

For the axiom of pairing, we need to check that, if $p, q \in B$ and $a = \{\langle p, \varnothing\rangle, \langle q, \varnothing\rangle\}$ then $a \in B$. It suffices to check that a is scope-bounded, since the elements of its transitive closure either have size at most two or are covered by the hypothesis that p and q are hereditarily scope-bounded. But the scope-boundedness of a is clear, as 2 is smaller than any inaccessible cardinal.

The interpretation of the separation schema is also clearly true, since all subsets of hereditarily scope-bounded sets are hereditarily scope-bounded.

To check the interpretation of the axiom of binary union, it suffices to check that the union of two hereditarily scope-bounded sets is hereditarily scope-bounded, and this is again clear as every inaccessible cardinal is closed under (binary) addition.

The interpretation of the axiom of infinity is true because $\aleph_0$ is smaller than every inaccessible cardinal.

The verification of (any instance of) the replacement schema boils down to checking that, when $b \in B$ and the (interpretations of the) hypotheses of the schema are satisfied, then the set

$$a = \{\langle y, t\rangle : y, t \in B \wedge (\exists\langle x, s\rangle \in b)\, \tilde{\varphi}(x, s, y, t)\}$$

is in B, where $\tilde{\varphi}$ is the interpretation, in ZFC+, of the XST-formula φ. Note first that (ZFC+ proves that) a is a set, by replacement in ZFC+. To show that $a \in B$, it suffices to check scope-boundedness, since the elements of the transitive closure are either pairs or covered by the fact that $y, t \in B$. So consider any t of some ordinal rank α; according to Lemma 7, we need only show that fewer than $i_{2\alpha+1}$ pairs of the form $\langle y, t\rangle$ are in a. Each such pair comes from a pair $\langle x, s\rangle \in b$, where $s \subseteq t$ and therefore s has height at most α. Since $b \in B$, there are fewer than $i_{2\alpha+1}$ such pairs $\langle x, s\rangle$. Furthermore, the hypotheses in the replacement schema ensure that each $\langle x, s\rangle \in b$ contributes at most one $\langle y, t\rangle$ to a, so the proof is complete.

To verify the interpretation of the axiom of union, let b and w be in B and satisfy the interpretation of the hypothesis of the axiom. The conclusion will be satisfied by

$$a = \{\langle x, w\rangle : x \in B \wedge (\exists y, t, u)\,(\langle x, t\rangle \in y \wedge \langle y, u\rangle \in b)\}$$

if this set is in B. As before, the verification reduces to checking that a is support-bounded, which means, since all pairs in a have the same second component w, that $|a| < i_{2\alpha+1}$, where α is the height of w. Each element $\langle x, w\rangle$ of a comes from some y, t, u where $\langle x, t\rangle \in y$, $\langle y, u\rangle \in b$, and therefore (by the hypothesis in the axiom) t and u have heights $\leq \alpha$. But then the scope-boundedness of b and all the relevant y's (which are in the transitive closure of b) implies that there are only $< i_{2\alpha+1}$ y's and, for each such y, only $< i_{2\alpha+1}$ x's. Since $i_{2\alpha+1}$ is inaccessible, this makes $< i_{2\alpha+1}$ x's altogether, as required.

For the axiom of power set, suppose $b, w \in B$ are as in (the interpretation of) the hypothesis of the axiom, and let

$$a = \{\langle x, w\rangle : x \in B \wedge x \subseteq b\}.$$

As before, it suffices to check that, if α is the height of w, then there are fewer than $i_{2\alpha+1}$ subsets x of b in B. Each element of b has the form $\langle y, u\rangle$ with $u \subseteq w$, so u has height at most α. As b is scope-bounded, there are $< i_{2\alpha+1}$ y's such that $\langle y, u\rangle \in b$. Furthermore, by Lemma 5, there are $< i_{2\alpha+1}$ u's, and therefore, as $i_{2\alpha+1}$ is inaccessible, we get $|b| < i_{2\alpha+1}$. Invoking again the inaccessibility of $i_{2\alpha+1}$, we get that b has $< i_{2\alpha+1}$ subsets x, and the proof is complete.

To verify the interpretation of the axiom of choice, suppose that a and s are elements of B satisfying (the interpretation of) the hypothesis of that axiom. Using the axiom of choice in ZFC, let q be a set containing exactly one member $\langle x, v\rangle$ from each z such that $\langle z, s\rangle \in a$. This application of the axiom of choice is legitimate because distinct such z's have no members of the form $\langle x, v\rangle$ in common. Now let $c = \{\langle x, s\rangle : (\exists v)\, \langle x, v\rangle \in q\}$. We must show that this c is in B; that it satisfies the (interpretation of the) conclusion of the axiom of choice is then clear. As usual, it suffices to check that c is scope-bounded, since everything in its transitive closure either is a pair or comes from the transitive closure of a. Since all pairs $\langle x, s\rangle$ in c have the same second component s, we need only check that c has fewer than $i_{2\alpha+1}$ elements, where α is the height of s. But this is clear, since we have at most one such element for every $\langle z, s\rangle$ in a, and a is scope-bounded. (In the preceding sentence, we have "at most" rather than "exactly" because several elements $\langle x, v\rangle$ of q can lead to the same $\langle x, s\rangle$ in c.)

The interpretation of the strong foundation axiom is true simply because the ingredients of an element of B have lower height than that element, and the heights, being ordinals, are well-ordered. We need not concern ourselves with the weak foundation axiom, as it is a consequence of the strong one.

The nested scope axiom is clearly true with our interpretation of the primitive symbol $\mathcal{S}_*$, and we have already checked that B is closed under that operation.

It remains to verify the interpretation of the Klass axiom. To this end, let an arbitrary $h \in B$ be given, let α be its height, and let

$$a = \{\langle x, h\rangle : x \in B \text{ and } \mathcal{S}_*(x) \subseteq h\}.$$

Recall that the subset relation used here, $\subseteq$, also serves as the interpretation of the subset relation of XST. It follows easily that the a defined here will, if it is in B, witness the interpretation of the Klass axiom. So we must verify that a is hereditarily scope-bounded. Since the elements $\langle x, h\rangle$ of a have x and h in B, we need only check that a is scope-bounded. So we must check that the number of $x \in B$ with $\mathcal{S}_*(x) \subseteq h$ is $< i_{2\alpha+1}$.

Lemma 8. *If $x \in B$ and $\mathcal{S}_*(x) \subseteq h$ with h of height α, then $x \in C_{i_{2\alpha}}$.*

Proof. We fix h and (therefore) α and proceed by induction on the height of x. Given x, consider an arbitrary $\langle z, s\rangle \in x$. We have $\mathcal{S}_*(z) \subseteq \mathcal{S}_*(x) \subseteq h$, so by the induction hypothesis $z \in C_{i_{2\alpha}}$; similarly $s \in C_{i_{2\alpha}}$. Furthermore, $s \in_s h$ (by definition of $\mathcal{S}_*$), and so the number of possible s's is bounded by $|h| < i_\alpha \leq i_{2\alpha}$ (since i_α is inaccessible and strictly greater than the height α of h).

Temporarily fix one such s and ask how many z's there might be with $\langle z, s\rangle \in x$. Since $s \in_s h$, the height of s is some $\beta < \alpha$. Since x is scope-bounded, the number of z's under consideration is $< i_{2\beta+1}$.

Now un-fix s. The β of the preceding paragraph depends on s, but, since the number of possible s's is smaller than $i_{2\alpha}$ and the latter is a regular cardinal, the total number of pairs $\langle z, s\rangle$ in x is strictly below $i_{2\alpha}$.

Since $i_{2\alpha}$ is a limit ordinal (because it is an infinite cardinal), $C_{i_{2\alpha}}$ is the union of the sets C_ξ for $\xi < i_{2\alpha}$. So each $\langle z, s\rangle \in x$ is in some such C_ξ, and a single $\xi < i_{2\alpha}$ will work for all of them, since there are fewer than $i_{2\alpha}$ of them and $i_{2\alpha}$ is regular. (In the degenerate case that x is a set of atoms so ξ could be 0, use $\xi = 1$ instead.) Then $x \in C_{\xi+1} \subseteq C_{i_{2\alpha}}$, as required. □

In view of the lemma, the number of $x \in B$ with $\mathcal{S}_*(x) \subseteq \tau$ is bounded above by the number of elements of $C_{i_{2\alpha}}$. That is strictly less than any inaccessible cardinal larger than $i_{2\alpha}$, in particular $i_{2\alpha+1}$. This completes the verification that the B interpretation satisfies the Klass axiom of XST.

5. Applications

In this section, we sketch some ideas for using XST to give simpler or more natural formulations of some common concepts in mathematics.

5.1. **Tuples.** Traditional set theory has several ways of coding ordered tuples $\langle a_1, a_2, \ldots, a_n\rangle$ as sets, none of which is really canonical; see for example [4]. One is to iterate the Kuratowski pair construction, so that, for example, an ordered quadruple would be coded as $\langle\langle\langle a_1, a_2\rangle, a_3\rangle, a_4\rangle$; or one could proceed from the other end, $\langle a_1, \langle a_2, \langle a_3, a_4\rangle\rangle\rangle$. Another traditional approach is to regard a tuple as the function sending each index i to the corresponding component a_i; functions are, in turn, represented as sets of ordered pairs.

XST provides a simple and natural way to represent tuples of this sort, namely to use the indices as scopes. Thus, a tuple $\langle a_1, a_2, \ldots, a_n\rangle$ is identified with the (extended) set $\{{a_1}^1, {a_2}^2, \ldots, {a_n}^n\}$. The natural numbers used here as scopes can be represented by the traditional von Neumann coding (with scope $\varnothing$),

$$0 = \varnothing \quad \text{and} \quad n + 1 = n \cup \{n^{\varnothing}\}.$$

Concatenation of tuples is easy to define in this set-up, once one has defined addition of natural numbers. Indeed, the concatenation of an m-tuple $\vec{a}$ and an n-tuple $\vec{b}$ is just the union $a \cup b'$, where b' is obtained (via the replacement axioms) by adding m to all the scopes in b.

5.2. **Generalized tuples.** Instead of indexing the components of a tuple by (consecutive) natural numbers, one could index them by arbitrary, distinct labels. The same XST representation still works; use the labels as scopes.

This provides, for example, a convenient way to deal with what are often called *records* in computing. Records have fields, in which data are inserted. We can represent them set-theoretically by taking the field names as scopes with the data as elements. Similarly, we can represent relations, in the sense of relational databases, by sets of generalized tuples, one for each row of the relation. The scopes for such a generalized tuple would be the attribute names, while the corresponding elements would be the values, in that row, of the attributes.

In principle, one could treat functions in the same way, using elements of the function's domain as scopes and using the corresponding values of the function as the associated elements. That is, a function f with domain D (in the sense of traditional set theory) could be represented by the XST set $\{f(x)^x : x \in D\}$. In the next subsection, we shall indicate a more flexible way of treating functions with possibly many inputs and many outputs.

5.3. **Functions and multi-functions.** In general, a set f of generalized tuples, all having the same scope set d, can be regarded as describing several (possibly multi-valued) operations, called the *behaviors* of f as follows. Suppose σ and τ are two subsets of d. (Often, but not always, they will be disjoint.) They determine the following behavior. An acceptable input is a generalized tuple x, with scope set σ, that is a subset of some member y of f; the corresponding output is the generalized tuple with scope set τ obtained as a subset of y by keeping only those elements of y whose scopes are in τ.

In general, such a behavior will produce several outputs for the same input x, since it may have several elements y that include x as a subset. In order for the behavior to be single-valued, one must require f to be appropriately functional: Any two members of f with the same elements of scope in σ must also have the same elements with scopes in τ.

In the preceding discussion, only the elements of σ and τ mattered, not their scopes. We might as well have taken all the scopes in σ and τ to be $\varnothing$ (or some other fixed value). But this notion of function behavior can be generalized to take into account non-trivial scopes in σ and τ. The idea is that the inputs and outputs of the new, generalized behavior of f (with respect to σ and τ) would still be tuples indexed by the elements of σ and τ but, instead of matching those elements to the scopes occurring in elements y of f, we would match the corresponding scopes in σ and τ to the scopes in such y's.

5.4. **Categories.** It is well known that category theory presents some difficulties for traditional set-theoretic foundations. One wants to consider such things as the category of all sets, or the category of all groups, or the category of all topological spaces, but none of these are sets. One can treat them as proper classes, in theories like Morse-Kelley or von-Neumann-Bernays-Gödel class theories, but difficulties soon reappear because the category of all functors between two such "large" categories might not even be a class; one needs hyperclasses (collections of classes), hyper-hyper-classes, etc. One traditional approach to overcoming such difficulties is Grothendieck's axiom of universes. This postulates the existence of an ample supply of sets, called (Grothendieck) universes that are rich enough to accommodate all the usual set-theoretic constructions. Instead of working with the category of all sets, one works with the category of those sets that belong to certain universe. This category, and functor categories involving it, etc., can then be constructed within a larger universe.

The Klass axiom is designed to handle the same issue. (So it is perhaps not surprising that the consistency proof for the Klass axiom relied on an ample supply of inaccessible cardinals, which are essentially equivalent to Grothendieck universes.) Given a set τ, we can consider, for example, all sets x (in the sense of XST) whose nested scope set $\mathcal{S}_*(x)$ is a subset of τ. (This would be the analog of considering, in classical set theory, all sets x in a particular universe.) According to the Klass axiom, all these sets x can be collected into a set, all with scope τ. (This corresponds to passing to a larger universe in the classical theory.) So, as long as we are willing to increase scopes when needed, we can handle the category of all such sets, functor categories involving it, etc.

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