

EXISTENCE OF BASES IMPLIES THE AXIOM OF CHOICE

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ABSTRACT. The axiom of choice follows, in Zermelo-Fraenkel set theory, from the assertion that every vector space has a basis.

Halpern [2] deduced the axiom of choice from the assertion that, in every vector space, every generating set includes a basis. He conjectured that the axiom of choice cannot be deduced from the weaker assertion that every vector space has a basis, even if one makes the additional assumption that all bases of a single vector space have the same cardinality. The purpose of this note is to present the following partial refutation of this conjecture.

THEOREM 1. In Zermelo-Fraenkel set theory (ZF), the axiom of choice is deducible from the assertion that every vector space has a basis.

The reason that this theorem only partially refutes Halpern's conjecture is that Halpern worked with a weaker theory than ZF, in which the axiom of regularity is omitted and the axiom of extensionality is weakened to admit atoms (urelements); we shall call this theory WZF. It is known (see [3, Chapter 9]) that the axiom of choice is deducible in ZF, but not in WZF, from the axiom of multiple choice, which asserts, for every family of nonempty sets, the existence of a function assigning to each set in the family a finite nonempty subset. Thus, Theorem 1 will be established once we prove the following result.

THEOREM 2. In WZF, the axiom of multiple choice is deducible from the assertion that every vector space has a basis.

1980 Mathematics Subject Classification 03E25

¹Partially supported by NSF grant MCS 8101560

PROOF. We work in WZF with the additional assumption that every vector space has a basis. Let $\{X_i | i \in I\}$ be a family of non-empty sets; we must find a family $\{F_i | i \in I\}$ of nonempty finite sets $F_i \subseteq X_i$. We assume, without loss of generality, that the sets X_i are pairwise disjoint. Adjoin all the elements of $X = \bigcup_{i \in I} X_i$ as indeterminates to some (arbitrary) field k , obtaining the field $k(X)$ of rational functions of the "variables" in X . For each $i \in I$, we define the i -degree of a monomial to be the sum of the exponents of members of X_i in that monomial. A rational function $f \in k(X)$ is called i -homogeneous of degree d if it is the quotient of two polynomials such that all monomials in the denominator have the same i -degree n while all those in the numerator have i -degree $n+d$. The rational functions that are i -homogeneous of degree 0 for all $i \in I$ constitute a subfield K of $k(X)$. Thus, $k(X)$ is a vector space over K , and we let V be the subspace spanned by the set X .

By assumption, the K -vector space V has a basis; we fix such a basis B , and we use it to explicitly define the desired finite sets F_i . For each $i \in I$ and each $x \in X_i$, we can express x as a (finite) K -linear combination of elements of B :

$$(1) \quad x = \sum_{b \in B(x)} \alpha_b(x) \cdot b,$$

where $B(x)$ is a finite subset of B and $\alpha_b(x)$ is, for $b \in B(x)$, a non-zero element of K . If y is another element of the same X_i as x , then we have on the one hand

$$y = \sum_{b \in B(y)} \alpha_b(y) \cdot b$$

and on the other hand, after multiplying (1) by the element y/x in K ,

$$y = \sum_{b \in B(x)} (y/x) \alpha_b(x) \cdot b.$$

Comparing these two expressions for y and using the fact that B is a basis, we infer that $B(x) = B(y)$ and $\alpha_b(y) = (y/x) \cdot \alpha_b(x)$. This means that the finite subset $B(x)$ of B and the elements $\alpha_b(x)/x$ of $k(X)$ depend only on i , not on the particular $x \in X_i$; we therefore call them B_i and β_{bi} . Observe that, since $\alpha_b(x) \in K$, β_{bi} is i -homogeneous of degree -1 (and j -homogeneous

of degree 0 for $j \neq i$), so, when β_{bi} is written as a quotient of polynomials in reduced form, some variables from X_i must occur in the denominator. Define F_i to be the set of those members of X_i that occur in the denominator of β_{bi} (in reduced form) for some $b \in B_i$. Then F_i is a nonempty finite subset of X_i , as required. \square

In view of Halpern's theorem, quoted in the introduction, it is natural to ask whether the axiom of choice is deducible in WZF from the assertion that in every vector space every independent set is included in a basis. Multiple choice is deducible (and therefore choice is deducible in ZF) either by Theorem 2 above or by Lemma 2 of [1], but the question as stated is, to the best of my knowledge, open. It is also unknown to me whether results like Theorem 2 can be obtained if the existence of bases is assumed only for vector spaces over some specific fields, say the rationals. Note that the results of Bleicher [1] hold under such a restriction, while Halpern's proof [2] and mine seem to depend crucially on building some combinatorial complexity into the field of scalars.

I wish to thank Dr. R. Harting for asking me the question that Theorem 1 answers and also for pointing out the wording of Bleicher's [1] citation of Halpern's result; the wording suggested attention to the construction of the ground field, a key step in finding the desired proof. I also wish to thank Professor and Mrs. G. H. Müller for their kind hospitality during my visit to Heidelberg where the result presented here was obtained.

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