BAER MEETS BAIRE: APPLICATIONS OF CATEGORY ARGUMENTS AND DESCRIPTIVE SET THEORY TO $\mathbb{Z}^{\aleph_0}$

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Abstract. We apply the Baire category theorem and other classical results of descriptive set theory to the study of the structure of the group $\mathbb{Z}^{\aleph_0}$ of infinite sequences of integers and some of its subgroups.

1. Introduction

Let $\Pi = \mathbb{Z}^{\aleph_0}$ be the Baer-Specker group of all infinite sequences of integers, the group operation being componentwise addition. Our purpose in this paper is to show how certain classical results of descriptive set theory can be combined with known facts about Π to produce new information about the structure of Π and some of its subgroups. Several of our theorems are about the possibility (or impossibility) of expressing these groups as unions of chains of subgroups with specified properties. Others are concerned with the possible quotients of these groups.

We begin by listing the definitions and classical results that we shall need from descriptive set theory. General references for this material are the books of Kuratowski [10], Moschovakis [14], and Kechris [9]; we shall give specific references to [14] for the results we use.

Definition. A set in a metric space is *meager* (or *of first (Baire) category*) if it can be covered by countably many closed sets with empty interiors. A set in a metric space has the *Baire property* if it differs from some open set by a meager set.

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Baire Category Theorem [2], [14, 2H2]. *In a complete metric space, no non-empty open set is meager.*

The union of countably many meager sets is meager, hence cannot cover a non-empty open set in a complete metric space. Thus, it is reasonable to regard meager sets as very small. A set with the Baire property is then very nearly an open set. In particular, if such a set is not itself meager, then it contains most (i.e., all but a meager subset) of some non-empty open set.

Definition. A *Borel* set is a set obtainable from open sets by repeated (possibly transfinitely repeated) formation of complements and countable unions. An *analytic* set is the image, under a continuous function, of a Borel set in a complete separable metric space.

The definition of "analytic" in [14] (page 43) seems more restrictive, but it follows from [14, 1E2 and 1E5] that it is actually equivalent.

Analytic sets share much of the well-known good behavior of Borel sets. For example, in Euclidean spaces, they are Lebesgue measurable. We shall need the following two results of this sort.

Lusin-Sierpiński Theorem [12], [14, 2H5]. *Every analytic set in a complete separable metric space has the Baire property.*

Suslin's Perfect Set Theorem [11], [14, 2C3]. *Every uncountable analytic set in a complete separable metric space has cardinality* $\mathfrak{c}$.

Here and throughout this paper, $\mathfrak{c}$ denotes the cardinality of the continuum, $2^{\aleph_0}$.

Notice that this last theorem says that an analytic set can never be a counterexample to the continuum hypothesis. The name of the theorem derives from the fact that the harder part of its proof establishes that an uncountable analytic set has a perfect (i.e., closed, non-empty, and without isolated points) subset; the easier part then infers the cardinality result.

We shall apply these results to the complete separable metric space Π. Its metric is defined by

$$\rho(x, y) = 2^{-n} \quad \text{where } n \text{ is the first number with } x(n) \neq y(n),$$

whenever $x \neq y$. To verify completeness, notice that, if (x_k) is a Cauchy sequence in Π then for each fixed n the sequence $(x_k(n))$ of integers is eventually constant; call its final value $y(n)$ and check that x_k converges to y. To verify separability, observe that the elements of Π that have only finitely many non-zero components form a countable dense set. The topology induced by the metric ρ is the product topology on $\mathbb{Z}^{\aleph_0}$ obtained from the discrete topology on $\mathbb{Z}$. (Since the product of countably many complete metric spaces always admits a complete metric, we didn't really need to exhibit ρ.) A basic neighborhood (a metric ball) about 0 is a subgroup of the form

$$V_k = \{x \in \Pi \mid x(n) = 0 \text{ for all } n < k\},$$

and a basic neighborhood of any other point in Π is a coset of such a subgroup V_k.

To connect these topological aspects of Π with its algebraic aspects, we need a theorem of Specker ensuring that homomorphisms are continuous.

Specker's Theorem [16, Satz III]. *Every homomorphism $h : \Pi \to \mathbb{Z}$ has the form $h(x) = \sum_{i=0}^{n} a_i x(i)$ for some finite n and some coefficients $a_i \in \mathbb{Z}$.*

The finiteness of n in this theorem ensures that h is continuous with respect to the product topology on Π and the discrete topology (or any other topology) on $\mathbb{Z}$. Since a function into a product space is continuous if and only if all its components are, the following corollary is immediate.

Corollary. *Every homomorphism $h : \Pi \to \Pi$ is continuous.*

We shall also need the following result about subgroups of Π.

Baer's Theorem [1, Thm. 4.7 and Cor. 12.8], [16, Satz I]. *Every countable subgroup of Π is free.*

We finish this introduction by defining and discussing two particular subgroups of Π that will play a role in our results.

For each natural number i, let e_i be the element of Π whose ith component is 1 and whose other components are all 0; these elements e_i are sometimes called the *standard unit vectors*. The subgroup Σ that they generate consists of those elements of Π that have only finitely many non-zero components. It is the free abelian group on the countably many generators e_i.

The group $D \subseteq \Pi$ is defined to consist of those $x \in \Pi$ such that, for each positive integer q, all but finitely many terms of x are divisible by q. Equivalently, D/Σ is the divisible part of Π/Σ. Equivalently again, D is the $\mathbb{Z}$-adic closure of Σ in Π. We record for future reference that D is a Borel set in Π:

$$D = \bigcap_q \bigcup_k \bigcap_{n \geq k} \{x \in \Pi \mid q \text{ divides } x(n)\},$$

where q and k range over positive integers and the sets $\{x \in \Pi \mid q \text{ divides } x(n)\}$ are both open and closed.

There is an analog of Specker's theorem for D and for many other groups "near" Π and D. The result for D is on page 163 of [8], attributed to T. Yen. We quote here a more general form from [3].

Continuity Theorem. *Let G be a pure subgroup of index $< \mathfrak{c}$ in either Π or D, and assume that $\Sigma \subseteq G$. Then every homomorphism $h : G \to \mathbb{Z}$ has the form $h(x) = \sum_{i=0}^{n} a_i x(i)$ for some finite n and some coefficients $a_i \in \mathbb{Z}$. Therefore, every homomorphism $G \to \Pi$ is continuous.*

Here G is taken to be topologized as a subspace of Π.

2. Unions of Chains

In this section, we shall apply the results quoted in Section 1 to obtain restrictions on possible representations of Π, D and similar groups as unions of chains of subgroups.

Theorem 2.1. *The group Π is not the union of a countable chain of proper subgroups, each isomorphic to Π.*

Proof. Suppose we had a countable chain of subgroups $P_0 \subseteq P_1 \subseteq \dots$ of Π, each isomorphic to Π, with union Π. We shall show that some P_n is all of Π.

By the Baire Category Theorem, fix an m such that P_m is not meager. This P_m, being isomorphic to Π, is the image of a homomorphism $h : \Pi \to \Pi$. By the corollary to Specker's Theorem, h is continuous, so P_m is analytic and therefore has the Baire property. (In fact, because h is one-to-one, P_m is a Borel set, by [14, 2E7].) Since it isn't meager, it is comeager in some basic open set, say $a + V_k$ for some $a \in \Pi$. (Recall that $a + V_k$ is the coset consisting of those $x \in \Pi$ such that $x(n) = a(n)$ for all $n < k$.)

We claim that $P_m \supseteq V_k$. To see this, fix any $z \in V_k$. Then the map $x \mapsto x + z$ is a homeomorphism of Π to itself, and therefore, since $(a + V_k) \setminus P_m$ is meager, so is its translate $z + ((a + V_k) \setminus P_m)$. By the Baire Category Theorem again, these two meager sets cannot cover $a + V_k$, so fix $x \in a + V_k$ belonging to neither of them. Then $x \in P_m$ and $x - z \in P_m$ (where we used that $x - z \in a + V_k$ since $z \in V_k$ and V_k is a subgroup). As P_m is a subgroup, we conclude that $z \in P_m$, as required.

Since the chain of subgroups P_i covers Π, some P_n must contain the finitely many unit vectors $e_0, \dots, e_{k-1}$. Then $P_{\max\{m,n\}}$ contains these unit vectors and all members of V_k. But that's enough to generate Π, so $P_{\max\{m,n\}} = \Pi$. $\square$

In the statement of Theorem 2.1, both "chain" and "each isomorphic to Π" are essential. Π is the union of countably many subgroups, each isomorphic to Π, for example the subgroup $V_1 = \{x \in \Pi \mid x(0) = 0\}$ and the subgroups $\{x \in \Pi \mid x(1) = rx(0)\}$ for all rational numbers r. Also, Π is the union of a countable chain of proper subgroups (not isomorphic to Π). One way to see this is to represent Π/Σ as the direct sum of its divisible part D/Σ and a reduced part R; as a rational vector space, D/Σ can easily be expressed as a countable increasing union of proper, pure subgroups, and, taking direct sums with R and then pre-images in Π, we get a similar representation for Π.

We do not know whether "countable" is essential in Theorem 2.1. It might be provable that Π is not the union of a chain (of any length) of proper subgroups isomorphic to Π. On the other hand, it may be that the continuum hypothesis implies that Π is expressible as such a union, for a chain of length $\aleph_1$.

The proof of Theorem 2.1 actually establishes considerably more than the theorem asserts. In the first place, the assumption that the subgroups in the chain are isomorphic to Π was used only to show that they are analytic. Thus, the following corollary is already proved.

Corollary 2.2. *The group Π is not the union of a countable chain of analytic, proper subgroups.*

If G is an analytic subgroup of index $< \mathfrak{c}$ in Π or in D and if $\Sigma \subseteq G$, then, by the Continuity Theorem, every homomorphic image of G in Π is also analytic. Thus, Π cannot be covered by a countable chain of such homomorphic images. In particular, Π cannot be covered by a countable chain of subgroups isomorphic to D. We digress for a moment to improve this last special case with a totally different method.

Proposition 2.3. *The group Π is not the sum (hence a fortiori not the union) of fewer than $\mathfrak{c}$ subgroups, each isomorphic to D.*

Proof. Suppose it were such a sum. Each of the summands has a countable subgroup for which the quotient is divisible (because D has such a subgroup, namely Σ). Let H be the sum of those fewer than $\mathfrak{c}$ countable subgroups. Thus, $|H| < \mathfrak{c}$ and Π/H is divisible. This leads to a contradiction by the following argument, essentially the proof of Theorem 2 in [6]. Since Π/H is divisible, we have $\Pi = H + 2\Pi$. Therefore $\Pi/2\Pi = (H + 2\Pi)/2\Pi \cong$

$H/(H \cap 2\Pi)$. Here the group on the left has cardinality $\mathfrak{c}$ while the group on the right, a quotient of H, has smaller cardinality. This contradiction completes the proof. $\square$

Returning to Theorem 2.1, we need a definition in order to state our next improvement of it. For ease of reference, we combine this definition with another that we shall need later.

Definition. The cardinal $\mathbf{cov}(B)$ is the smallest κ such that some κ meager sets cover the real line. The cardinal $\mathbf{add}(B)$ is the smallest κ such that some κ meager sets of reals have a non-meager union.

Clearly, $\aleph_1 \leq \mathbf{add}(B) \leq \mathbf{cov}(B) \leq \mathfrak{c}$. It is known that each of these three inequalities is strict in some but not all models of set theory [7]. It is also known that $\mathbf{cov}(B)$ and $\mathbf{add}(B)$ are unchanged if we replace the real line in their definition by Π. Indeed, Π is homeomorphic to a comeager subset of $\mathbb{R}$, namely the set of irrational numbers; see [9, Thm. 7.7].

The only use, in the proof of Theorem 2.1, of the countability hypothesis was to ensure that the subgroups P_n cannot all be meager. For this purpose, that hypothesis can obviously be weakened to say only that the number of P_n's is smaller than $\mathbf{cov}(B)$. (Whether this is a genuine weakening, i.e., whether $\mathbf{cov}(B) > \aleph_1$, depends, as mentioned above, on what set-theoretic universe one is in.)

Finally, the hypothesis that the P_n form a chain can be weakened to say only that they form a directed family, i.e., that every two of them (and therefore every finitely many of them) are contained in a single one. This weakening would have made no difference if we were still requiring countability, since every countable directed set contains a cofinal chain, but, now that we have weakened the countability assumption, this additional weakening makes sense. The following corollary summarizes all the preceding improvements of Theorem 2.1.

Corollary 2.4. *The group Π is not the union of a directed family of fewer than $\mathbf{cov}(B)$ proper, analytic subgroups.*

We turn next to analogous results for D in place of Π. The proof of Theorem 2.1 cannot be applied to D because D, unlike Π, is not a complete metric space. Not only is D not complete with respect to the metric of Π (because it's not closed in Π), it is not complete with respect to any metric inducing the same topology (because it's not a G_δ set in Π; see [9 Thm. 3.11]).

In fact, D is the union of a countable chain of proper subgroups, each isomorphic to D; consider for example, the subgroups

$$D_k = \{x \in D \mid x(n) \text{ is even for all } n \geq k\}.$$

Nevertheless, we can get an analog of Theorem 2.1 by adding the hypothesis that the subgroups in the chain are pure.

Theorem 2.5. *The group D is not the union of a countable chain of pure subgroups, each isomorphic to D.*

Before starting the proof, we give a definition that will be used both in this proof and in some later results.

Definition. For any $f \in \Pi$, let

$$f\Pi = \{x \in \Pi \mid f(n) \text{ divides } x(n) \text{ for all } n\}.$$

Clearly, $f\Pi$ is a Borel (in fact, closed) subgroup of Π. Notice that, if no component of f is 0, then componentwise multiplication by f is an isomorphism from Π onto $f\Pi$. We shall write $f \cdot -$ for this isomorphism and $-/f$ for its inverse. Notice also that, if $f \in D$ then $f\Pi \subseteq D$.

Proof of Theorem 2.5. Suppose D were the union of a countable chain of pure subgroups $D_0 \subseteq D_1 \subseteq \dots$. Temporarily fix some $f \in D$ with no zero components, and use the isomorphism $f \cdot -$ mentioned above to pull back the subgroups $D_n \cap f\Pi$ to subgroups $P_n = (D_n \cap f\Pi)/f$ of Π. These P_n form a countable chain with union Π.

Recall that D is a Borel set in Π. Each D_n, being isomorphic to D, is then an analytic set, because of the continuity theorem. Since $f\Pi$ is Borel, it follows that $D_n \cap f\Pi$ and P_n are analytic; see [14, Thm. 1E2].

By Corollary 2.2, we deduce that the P_n cannot all be proper subgroups of Π. So there is n with $P_n = \Pi$, which means that $D_n \cap f\Pi = f\Pi$ and thus $f\Pi \subseteq D_n$.

Un-fix f. We have seen that, for every $f \in D$ with no zero components, there is an n such that $f\Pi \subseteq D_n$. (Of course, n can depend on f; otherwise, we could immediately conclude that $D \subseteq D_n$ and the proof would be finished.)

Since each D_n is a proper subgroup of D, fix some $g_n \in D \setminus D_n$. For each natural number z, define $h_n(z)$ to be an integer so large that, for all $k \geq h_n(z)$, $g_n(k)$ is divisible by $z!$; such an integer exists because $g_n \in D$. There is an increasing function h, from natural numbers to natural numbers, that eventually majorizes each h_n. For example, we can define $h(z)$ to be z plus the maximum of $h_0(z), h_1(z), \dots, h_{z-1}(z)$, so that h majorizes h_n beyond n. It will be convenient to have $h(0) = 0$.

Now define $f \in \Pi$ by letting $f(k) = z!$ for the largest z such that $h(z) \leq k$. Such a largest z exists, because h is increasing and $h(0) = 0$.

Notice that, for any fixed positive integer q, all sufficiently large k satisfy $h(q) \leq k$, which means that the z in the definition of $f(k)$ is $\geq q$ and therefore q divides $f(k)$. This proves that $f \in D$. Since no component of f is 0, the first part of this proof provides an n such that $f\Pi \subseteq P_n$. Fix this n for the rest of the proof.

Our choice of h ensures that $h(z) \geq h_n(z)$ for all sufficiently large z. By taking k large enough, we can ensure that $f(k) = z!$ for a sufficiently large z, where $k \geq h(z) \geq h_n(z)$. Then, by definition of h_n, we know that $g_n(k)$ is divisible by $z! = f(k)$.

We thus have $f(k)$ dividing $g_n(k)$ for all sufficiently large k. It follows that, for a suitable positive integer c, $f(k)$ divides $cg_n(k)$ for all k (not just all sufficiently large k). For example, we could take c to be the product of those finitely many $f(k)$ that do not divide the corresponding $g_n(k)$. Thus, we have $cg_n \in f\Pi$. By our choice of n, we have $cg_n \in D_n$. Using the assumption that D_n is pure, we infer that $g_n \in D_n$, which contradicts the definition of g_n. $\square$

Theorem 2.5 admits improvements similar (but not quite identical) to those of Theorem 2.1. As before, we can replace the assumption that the subgroups are isomorphic to D by the assumption that they are analytic, and we can weaken "chain" to "directed family."

But a difference arises when we try to weaken the countability hypothesis, since this hypothesis was used twice in the proof of Theorem 2.5. The first use occurred when we invoked Corollary 2.2, and we have already seen that "countable" can be replaced with "$< \mathbf{cov}(B)$" there. The second use occurred when we defined the function h to eventually majorize all the (countably many) h_n. Can uncountably many functions be similarly eventually majorized by a single one? The answer depends on one's set-theoretic universe. More precisely, define $\mathfrak{b}$ to be the smallest cardinality of any family of functions $\mathbb{N} \to \mathbb{N}$ not eventually majorized by a single function. Clearly, $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{c}$; any combination of equalities and strict inequalities here is consistent with the usual axioms of set theory. The second use of countability in the proof of Theorem 2.5 still works as long as the number of h_n's is smaller than $\mathfrak{b}$.

Thus, for the proof of Theorem 2.5 to apply to an uncountable directed family of D_n's, we need the cardinality of this family to be smaller than both $\mathbf{cov}(B)$ and $\mathfrak{b}$. This requirement can be simplified by using the known fact [7, 13] that the smaller of $\mathbf{cov}(B)$ and $\mathfrak{b}$ is exactly $\mathbf{add}(B)$.

Corollary 2.6. *The group D is not the union of a directed family of fewer than $\mathbf{add}(B)$ pure, proper, analytic subgroups.*

We close this section with some results about isomorphic copies of Π inside D. We considered a family of such copies, namely those of the form $f\Pi$, in the proof of Theorem 2.5. It turns out that every copy of Π in D is included in one of this special form.

Theorem 2.7. *If $h : \Pi \to D$ is any homomorphism, then there is an $f \in D$ (with no zero components) such that $h(\Pi) \subseteq f\Pi$.*

Proof. Temporarily fix a positive integer q. The group D is covered by the countable chain of Borel subgroups

$$D_k = \{x \in D \mid q \text{ divides } x(n) \text{ for all } n \geq k\}.$$

Since h is continuous, the groups $h^{-1}(D_k)$ are Borel sets in Π. By Corollary 2.2, there must be a k such that $h^{-1}(D_k) = \Pi$, which means $h(\Pi) \subseteq D_k$.

Now un-fix q. Of course, the k we found in the preceding discussion may depend on q, so we call it $k(q)$. It has the property that, for all $x \in \Pi$ and all $n \geq k(q)$, we have $h(x)(n)$ divisible by q. We may assume that $k(1) = 0$.

Define f by letting $f(m) = q!$ for the largest $q \leq m$ such that $k(q!) \leq m$. Then $f \in D$ since $f(m)$ is divisible by $q!$ for all $m \geq \max\{q, k(q!)\}$. Finally, we check that $h(\Pi) \subseteq f\Pi$. Consider any $x \in \Pi$ and any m; we must show that $f(m)$ divides $h(x)(m)$. But $f(m) = q!$ for a certain q satisfying $k(q!) \leq m$, which means, by definition of $k(q!)$, that $h(x)(m)$ is divisible by $q!$, as required. □

In the following corollaries, we deal with isomorphic copies of Π although Theorem 2.7 would allow us to deal with homomorphic images of Π just as easily. The extra generality is illusory, since Nunke [15, Thm. 5] showed that all homomorphic images of Π that are subgroups of Π are isomorphic to either Π or $\mathbb{Z}^n$ for some finite n.

Corollary 2.8. *No isomorphic copy of Π in D includes Σ.*

Proof. If an isomorphic copy G of Π in D included Σ, then Theorem 2.7 would give an $f \in D$ such that $\Sigma \subseteq G \subseteq f\Pi$. But the only functions f for which $\Sigma \subseteq f\Pi$ are those that take only the values ± 1, and these are not in D. □

Corollary 2.9. *The group D is not the union of countably many isomorphic copies of* Π.

Proof. If D were such a union, then by Theorem 2.7 we could assume that the countably many isomorphic copies of Π involved in the union are of the form $f_n\Pi$ for certain $f_n \in D$ with no zero components. As in the proof of Theorem 2.5, we can find a single $f \in D$ such that, for each n, all sufficiently large k have $f(k)$ dividing $f_n(k)/2$. In particular, for each n there exists some k (in fact any sufficiently large k will do) such that $f_n(k)$ does not divide $f(k)$. But then f is not in any of the $f_n\Pi$, contrary to assumption. $\square$

Corollary 2.10. *The group D is not the union of a chain (of any length) of isomorphic copies of* Π.

Proof. Suppose it were such a union. For each of the countably many members s of Σ, fix a group in the chain that contains s. The union of the countably many chosen groups is either all of D (if these groups are cofinal in the chain) or included in a larger subgroup from the chain. The first possibility contradicts Corollary 2.9, and the second contradicts Corollary 2.8. $\square$

3. Quotients of D

In this section, we prove two theorems saying that, under certain circumstances, a small quotient of D is necessarily free of finite rank. The "certain circumstances" are that the quotient is separable or that the kernel is analytic; the necessary "smallness" of the quotient depends on which of these two circumstances is assumed.

Recall that a group G is called *separable* if it can be embedded as a pure subgroup in $\mathbb{Z}^\kappa$ for some cardinal κ.

Theorem 3.1. *Every separable quotient of D of cardinality $< \mathfrak{c}$ is free of finite rank.*

Proof. Let G be such a quotient, with epimorphism $h : D \to G$. Let B be the purification in G of $h(\Sigma)$. Since D/Σ is divisible, so is G/B; furthermore, B is pure in G and, as a countable subgroup of some $\mathbb{Z}^\kappa$, B is free. Therefore, the hypotheses of Theorem 1 of [5] are satisfied, and we conclude that G may be identified with a pure subgroup of Π (rather than some larger $\mathbb{Z}^\kappa$).

Making this identification and applying the continuity theorem to h, we conclude that G is analytic. By Suslin's Perfect Set Theorem, the assumption that $|G| < \mathfrak{c}$ implies that G is countable. By Baer's Theorem, G is free and is therefore a direct summand of D. But it follows immediately from the continuity theorem that D has no free direct summands of infinite rank. (The continuity theorem says that D has only countably many homomorphisms to $\mathbb{Z}$, but such a summand would have at least $\mathfrak{c}$ such homomorphisms.) Therefore, G has finite rank. $\square$

Theorem 3.2. *Suppose E is a pure, analytic subgroup of D of countable index. Then D/E is free of finite rank.*

Proof. Temporarily fix some $f \in D$ with no zero components. Recall that we have $f\Pi \subseteq D$ with $f\Pi$ isomorphic to Π via $-/f$ and $f \cdot -$. By hypothesis, E is analytic, and so are all its cosets $y + E$ because addition is continuous. Since $f \cdot -$ is clearly continuous, the inverse images in Π of these cosets, namely $((y + E) \cap f\Pi)/f$, are also analytic. They can't all be meager, as there are only countably many of them and they cover Π. By the Lusin-Sierpiński Theorem, one of them almost (i.e., except for a meager set) covers a basic neighborhood $x + V_k$. By a Baire category and subtraction argument, as in the proof of Theorem 2.1, it follows that $(E \cap f\Pi)/f$ includes V_k. This means that $V_k \cap f\Pi \subseteq E$.

Consider now some $z \in V_k$ such that $f(n)$ divides $z(n)$ for all but finitely many n. We say that such a z is *almost in* $f\Pi$. Then multiplying z by a suitable positive integer (for example the product of all the finitely many $f(n)$ that don't divide the corresponding $z(n)$) we obtain an element in $V_k \cap f\Pi$, hence in E. But E is a pure subgroup of D, so $z \in E$.

Now un-fix f. We have shown that, for every $f \in D$ with no zero components, there exists k such that E contains all elements of V_k that are almost in $f\Pi$. Let $k(f)$ be the smallest such k.

We claim that, as f varies, $k(f)$ remains bounded. To prove this claim, suppose it were false, and fix a sequence of f's, say f_m, such that the corresponding $k(f_m)$ increase without bound. As in the proof of Theorem 2.5, obtain a single $f \in D$ (with no zero components) such that, for each m, all sufficiently large n have $f(n)$ dividing $f_m(n)$. It follows that everything that is almost in $f_m\Pi$ is also almost in $f\Pi$. Therefore, $k(f_m) \leq k(f)$ for all m. This contradicts the choice of the f_m, so the claim is established.

By virtue of the claim just verified, we fix a single k so large that E contains all elements of $V_k \cap f\Pi$, for all $f \in D$ without zero components. In other words, $E \supseteq D \cap V_k$. Therefore, D/E is a quotient of $D/(D \cap V_k) \cong \mathbb{Z}^k$. As E is pure in D, D/E is torsion-free. But the only torsion-free quotients of $\mathbb{Z}^k$ are free abelian groups of finite rank, as required. $\square$

In Theorem 3.2, the hypothesis that the index of E is countable was used only to apply the Baire category theorem to see that the countably many sets $((y + E) \cap f\Pi)/f$ cannot all be meager. So it suffices to assume that the index of E in D is $< \mathbf{cov}(B)$. The following corollary summarizes this improvement of Theorem 3.2 along with Theorem 3.1 and some known results.

Corollary 3.3. *For any subgroup E of D, the following are equivalent.*

(1) *D/E is a free abelian group of finite rank.*
(2) *E is a pure subgroup of D of countable index and $E \cong D$.*
(3) *E is a pure, analytic subgroup of D of index $< \mathbf{cov}(B)$.*
(4) *E is a subgroup of D of index $< \mathfrak{c}$ and D/E is separable.*

Proof. Theorem 3.1 says that (4) implies (1), and that (1) implies (4) is obvious. Theorem 3.2 and the discussion following its proof give that (3) implies (1). The only non-obvious part of the implication from (1) to (2) is that $E \cong D$, but this follows since the freeness of the quotient implies that E is a summand of D and it is shown in [4] that all infinite rank direct summands of D are isomorphic to D.

Finally, for the implication from (2) to (3), it suffices to recall that D is a Borel set and so, by the Continuity Theorem, every isomorphic copy of D in Π is analytic. $\square$

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