BAIRE CATEGORY FOR MONOTONE SETS

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Abstract. We study Baire category for downward-closed subsets of 2^ω, showing that it behaves better in this context than for general subsets of 2^ω. We show that, in the downward-closed context, the ideal of meager sets is prime and $\mathfrak{b}$-complete, while the complementary filter is $\mathfrak{g}$-complete. We also discuss other cardinal characteristics of this ideal and this filter, and we show that analogous results for measure in place of category are not provable in ZFC.

1. Introduction

We shall work with the space 2^ω of infinite sequences of zeros and ones, topologized as a product of two-point discrete spaces. Notions of Baire category — meager (= first category), non-meager (= second category), and comeager (= residual) — will always be with respect to this topology. We write B for the σ-ideal of meager sets in the Boolean algebra $\mathcal{P}(2^\omega)$ of subsets of 2^ω, and B^+ for the complement of this ideal.

We identify subsets of the set ω of natural numbers with their characteristic functions. Thus, we often speak of elements of 2^ω as though they were subsets of ω. We write $[\omega]^\omega$ for the subspace of 2^ω consisting of the (characteristic functions of) infinite sets. Restricting attention from 2^ω to $[\omega]^\omega$ does not affect Baire category notions since the difference between the two spaces is countable.

We (weakly) order 2^ω componentwise modulo finite sets, so $x \leq y$ means that, for all sufficiently large n, $x(n) \leq y(n)$. Under the identification of subsets of ω with their characteristic functions, $\leq$ is the relation of almost-inclusion ($A - B$ finite) between subsets.

Let $\mathcal{M}([\omega]^\omega)$ be the lattice of downward-closed (with respect to this ordering) subsets X of $[\omega]^\omega$. $\mathcal{M}([\omega]^\omega)$ is ordered by inclusion. We write $B_\mathcal{M}$ for the σ-ideal $B \cap \mathcal{M}([\omega]^\omega)$ of meager sets in $\mathcal{M}([\omega]^\omega)$, and we write ${B_\mathcal{M}}^+$ for its complement in $\mathcal{M}([\omega]^\omega)$. (We use $[\omega]^\omega$ instead of 2^ω only to avoid having a non-zero intersection (namely the collection of finite subsets of ω) of all the non-zero elements of the lattice.) The purpose of this paper is to exhibit some pleasant properties of $B_\mathcal{M}$ not enjoyed by B.

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In Section 2, we show that, for sets in $\mathcal{M}([\omega]^\omega)$, non-meagerness coincides with groupwise density as defined in [5]. It follows that $B_{\mathcal{M}}{}^+$ is a filter in $\mathcal{M}([\omega]^\omega)$ (so $B_{\mathcal{M}}$ is a prime ideal) and in fact a $(< \mathfrak{g})$-complete filter, where $\mathfrak{g}$ is the groupwise density number introduced in [5].

In Section 3, we study the cardinal characteristics of the ideal $B_{\mathcal{M}}$ and the filter $B_{\mathcal{M}}{}^+$. In particular, we find that it is consistent for the additivity of $B_{\mathcal{M}}$ to be strictly greater than that of B.

Finally, in Section 4, we show that what we did for Baire category in Section 2 cannot be carried over to Lebesgue measure, at least not in ZFC alone. Assuming the continuum hypothesis, we prove that the complement in $\mathcal{M}([\omega]^\omega)$ of the ideal of measure-zero sets is not a filter.

2. Non-Meager Sets

This section is devoted to a combinatorial characterization of non-meager downward-closed subsets of 2^ω and some immediate consequences of this characterization. The characterization is essentially the same as Talagrand's characterization [9] of meager filters (or ideals) on ω. Talagrand's proof does not really use that filters are closed under finite intersections (and ideals under finite unions). So our proof is practically a repetition of Talagrand's; we give it for the sake of completeness.

In [5], a family $G \subseteq [\omega]^\omega$ was defined to be *groupwise dense* if it is downward-closed (i.e., $G \in \mathcal{M}([\omega]^\omega)$) and, for every partition of ω into finite intervals, the union of some infinitely many of the intervals belongs to G. (Actually, the definition in [5] used, instead of a partition of ω into intervals, an arbitrary family of disjoint, finite subsets of ω, but it was shown in [5] that no generality is lost by considering only partitions into intervals.)

Theorem 1. *A set $X \in \mathcal{M}([\omega]^\omega)$ is meager if and only if it is not groupwise dense.*

Proof. Suppose first that X is not groupwise dense. Let $\{I_0, I_1, \dots\}$ be a partition of ω into finite intervals, no infinite union of which is in X. Since X is downward closed, no element $x \in X$ includes infinitely many I_n. In other words, X is covered by the sets

$$N_k = \{x \in \mathcal{M}([\omega]^\omega) \mid (\forall n \geq k)(\exists m \in I_n)\, x(m) = 0\}.$$

Each N_k is closed and nowhere dense, so X is meager.

For the converse, suppose $X \in \mathcal{M}([\omega]^\omega)$ is meager, and let it be covered by an increasing sequence of closed, nowhere dense sets F_n. We shall partition ω into finite intervals I_n and we shall define functions $s_n : I_n \to 2$ in such a way that, if $y \in 2^\omega$ and the restriction of y to some I_n is s_n, then $y \notin F_n$ for that n. Once we do this, the intervals I_n will witness that X is not groupwise dense. Indeed, suppose $x \in X$ is (the characteristic function of) a set that includes I_n for infinitely many n. Then there is $y \leq x$ whose restriction to each of these infinitely many I_n is s_n. But then $y \notin F_n$ for infinitely many n. As all the F_n form an increasing sequence, y is not in their union and hence not in X. This is a contradiction, as $x \in X$ and X is downward-closed.

So it remains to construct the I_n and s_n. We proceed by induction on n and let q be the smallest natural number larger than all elements of all the previously defined $I_0, \dots, I_{n-1}$. (If $n = 0$, take $q = 0$.) We shall obtain s_n as the union of 2^q functions $t_0, t_1, \dots, t_r$, where $r = 2^q - 1$ and the domains of the t_i are adjacent intervals $[q, e_0)$, $[e_0, e_1)$, $\dots$, $[e_{r-1}, e_r)$. So $I_n = [q, e_r)$. We construct t_i by induction on i.

Fix a list $u_0, u_1, \dots, u_r$ of all the 2^q functions $[0, q) \to 2$. Then inductively choose t_i so that no $y \in 2^\omega$ extending $u_i \cup t_0 \cup \dots \cup t_{i-1} \cup t_i$ is in F_n. Such a choice is always possible because F_n is nowhere dense and hence is disjoint from some basic open subset of the (basic open) set of extensions of $u_i \cup t_0 \cup \dots \cup t_{i-1}$.

Now set $s_n = t_0 \cup \dots \cup t_r$. If $y \in 2^\omega$ extends s_n, then, since it also extends u_i for some i, it extends $u_i \cup t_0 \cup \dots \cup t_{i-1} \cup t_i$ and is therefore not in F_n. So s_n has the required property and the proof is complete. $\square$

The cardinal $\mathfrak{g}$ was defined in [5] as the smallest number of groupwise dense families whose intersection is empty. This cardinal is easily seen to be uncountable; in fact it is no smaller than the distributivity number $\mathfrak{h}$. (We refer to [10] for general information on cardinal characteristics of the continuum and to [3, 4] for more details about $\mathfrak{g}$.) Thus, the intersection of any fewer than $\mathfrak{g}$ non-meager sets in $\mathcal{M}([\omega]^\omega)$ is non-empty. Contrast this with what happens in $\mathcal{P}(2^\omega)$ where there are pairwise disjoint, non-meager sets (in fact $2^{\aleph_0}$ of them). The following result slightly improves these observations by changing the conclusion "non-empty" to "non-meager."

Theorem 2. *The intersection of fewer than $\mathfrak{g}$ groupwise dense sets is groupwise dense. Thus, ${B_\mathcal{M}}^+$ is a $(< \mathfrak{g})$-complete filter in $\mathcal{M}([\omega]^\omega)$.*

Proof. Let fewer than $\mathfrak{g}$ groupwise dense sets G_i be given, and let G be their intersection. To show G is groupwise dense, let ω be partitioned into intervals I_n (listed in their natural order). For each i, let H_i be the collection of those infinite subsets x of ω such that $\bigcup_{n \in x} I_n \in G_i$.

We check that each H_i is groupwise dense. Let ω be partitioned into intervals J_k. Then the sets $J'_k = \bigcup_{n \in J_k} I_n$ are intervals and constitute a partition of ω. So the union of some infinitely many of them is in the groupwise dense family G_i. The union of the infinitely many corresponding J_k is then in H_i.

Since the H_i are groupwise dense and there are fewer than $\mathfrak{g}$ of them, they have a common member x. Then $\bigcup_{n \in x} I_n$ is in all G_i. So we have found a member of G that is the union of infinitely many I_n. $\square$

Corollary. *$B_\mathcal{M}$ is a prime ideal in $\mathcal{M}([\omega]^\omega)$.*

Proof. In any lattice, an ideal whose complement is a filter is prime. $\square$

3. Cardinal Characteristics

Associated to any ideal I of sets are four cardinal characteristics, defined as follows. Let X be the union of all the sets in I.

add(I) is the smallest cardinality of a subfamily of I whose union is not a member of I.

cov(I) is the smallest cardinality of a subfamily of I whose union is X.

unif(I) is the smallest cardinality of a subset of X that is not in I.

cof(I) is the smallest cardinality of a subfamily C of I such that every member of I is a subset of some member of C.

For more information about such characteristics, see [2, 6, 10].

An ideal in $\mathcal{M}([\omega]^\omega)$, like $B_\mathcal{M}$, is not an ideal of sets, as it is not closed under arbitrary subsets. Nevertheless, three of the four cardinal characteristics make good sense in this more general context. The exception is **unif**, which would get the trivial value 1, since singletons are not in $B_\mathcal{M}$ simply because they are not closed downward. To reasonably extend **unif** to ideals in $\mathcal{M}([\omega]^\omega)$ we define **unif**(I) to be the smallest cardinality of a subset of X not included in any member of I. With this correction, all four characteristics can also be described as the corresponding characteristics of the ideal of sets obtained by closing I downward in $\mathcal{P}(2^\omega)$.

Our first goal in this section is to compute the cardinal characteristics of $B_\mathcal{M}$. Afterward, we shall also consider characteristics of the complementary filter ${B_\mathcal{M}}^+$.

Recall [10] that the bounding number $\mathfrak{b}$ is defined to be the smallest possible cardinality of a family F of functions $\omega \to \omega$ such that no single function $\omega \to \omega$ eventually majorizes each member of F. Similarly, the dominating number $\mathfrak{d}$ is defined as the smallest cardinality of any family F of functions $\omega \to \omega$ such that every function $\omega \to \omega$ is eventually majorized by one from F.

Theorem 3. $\mathbf{add}(B_\mathcal{M}) = \mathbf{unif}(B_\mathcal{M}) = \mathfrak{b}$, *and* $\mathbf{cov}(B_\mathcal{M}) = \mathbf{cof}(B_\mathcal{M}) = \mathfrak{d}$.

Proof. We record for future reference that the X in the definition of **cov** and **unif**, for the ideal $B_\mathcal{M}$, is the family of all infinite, co-infinite subsets of ω.

To each partition $\Pi = \{I_0, I_1, \dots\}$ of ω into finite intervals, associate the set $M(\Pi)$ of all infinite subsets of ω that include only finitely many of the I_n. By Theorem 1, each $M(\Pi)$ is in $B_\mathcal{M}$ and each member of $B_\mathcal{M}$ is a subset of $M(\Pi)$ for some Π.

To connect the notions involved in the definitions of the characteristics of $B_\mathcal{M}$ with those involved in the definitions of $\mathfrak{b}$ and $\mathfrak{d}$, we use the following three constructions relating partitions Π as above, co-infinite subsets of ω, and functions $\omega \to \omega$.

To any Π as above, we assign a function $F_\Pi : \omega \to \omega$ as follows. For any $k \in \omega$, let n be the number such that $k \in I_n$, and let $F_\Pi(k)$ be the largest element of I_{n+2}.

For any $f : \omega \to \omega$, let G_f be an infinite, co-infinite subset of ω such that, for each $k \in \omega$, the second element of $\omega - G_f$ after k is larger than $f(k)$. To obtain such a G_f, inductively choose the (infinitely many) elements $a_0 < a_1 < \dots$ of its complement so that each a_{n+1} is greater than $f(k)$ for all $k \le a_n$.

Finally, for any $f : \omega \to \omega$, let H_f be some partition of ω into finite intervals $[a, b]$ each of which satisfies $f(a) \le b$. It is clear that such a partition exists; just define the intervals one at a time by induction.

The essential properties of these constructions are given by the following two lemmas.

Lemma 1. *Let* Π *be a partition of* ω *into finite intervals, and suppose* $g : \omega \to \omega$ *eventually majorizes* F_Π. *Then* $M(\Pi) \subseteq M(H_g)$.

Proof. Consider any block $[a, b]$ of the partition H_g; by definition it satisfies $b \geq g(a)$. If a is large enough, then by hypothesis $g(a) \geq F_\Pi(a)$ and so $b \geq F_\Pi(a)$. By definition of F_Π, this means that $[a, b]$ includes an entire block of the partition Π (actually two entire blocks, but we don't need that here). Therefore, a set cannot include infinitely many blocks of H_g without also including infinitely many blocks of Π. By definition of $M(\Pi)$ and $M(H_g)$, this completes the proof. $\square$

Corollary. $\mathfrak{b} \leq \mathbf{add}(B_\mathcal{M})$.

Proof. Let $\kappa = \mathbf{add}(B_\mathcal{M})$. So there are κ sets in $B_\mathcal{M}$ whose union is not in $B_\mathcal{M}$. Enlarging these sets if necessary, we assume without loss of generality that they are $M(\Pi_i)$ for some κ partitions Π_i of ω into finite intervals. We shall prove that $\mathfrak{b} \leq \kappa$ by showing that the functions F_{Π_i} are not all eventually majorized by any single function g. Indeed, if g eventually majorized all the F_{Π_i}, then by the lemma the union of all the $M(\Pi_i)$ would be included in $M(H_g)$ which is in $B_\mathcal{M}$; this would contradict the fact that this union is not in $B_\mathcal{M}$. $\square$

Corollary. $\mathbf{cof}(B_\mathcal{M}) \leq \mathfrak{d}$.

Proof. Let a family of $\mathfrak{d}$ functions $f_i : \omega \to \omega$ be such that every function $g : \omega \to \omega$ is eventually majorized by some f_i. Taking g to be F_Π for an arbitrary partition Π of ω into finite intervals, and applying the lemma, we find that all sets of the form $M(\Pi)$ and therefore all sets in $B_\mathcal{M}$ are included in sets of the form $M(H_{f_i})$. Therefore, the $\mathfrak{d}$ sets of the latter form are as required in the definition of $\mathbf{cof}(B_\mathcal{M})$. $\square$

Lemma 2. *Let Π be a partition of ω into finite intervals, and suppose f is such that $G_f \in M(\Pi)$. Then F_Π eventually majorizes f.*

Proof. As $G_f \in M(\Pi)$, every interval in Π, except for finitely many, must meet the complement of G_f. So, for sufficiently large k, as there are two intervals of Π between k and $F_\Pi(k)$ (by definition of F_Π), there must also be at least two elements of $\omega - G_f$ between k and $F_\Pi(k)$. In particular, the second element of $\omega - G_f$ after k is at most $F_\Pi(k)$. But it is also at least $f(k)$ (by definition of G_f). $\square$

Corollary. $\mathfrak{d} \leq \mathbf{cov}(B_\mathcal{M})$.

Proof. Let $\kappa = \mathbf{cov}(B_\mathcal{M})$, and let κ sets in $B_\mathcal{M}$ be given whose union contains all infinite, co-infinite subsets of ω. Enlarging these κ sets if necessary, we may assume that they have the form $M(\Pi_i)$. We shall show that the corresponding F_{Π_i} constitute a dominating family. So let any $f : \omega \to \omega$ be given. Since G_f is an infinite, co-infinite subset of ω, it lies in some $M(\Pi_i)$, and by the lemma F_{Π_i} eventually majorizes f. $\square$

Corollary. $\mathbf{unif}(B_\mathcal{M}) \leq \mathfrak{b}$.

Proof. Let $\mathfrak{b}$ functions f_i be given, not all eventually majorized by any single function $\omega \to \omega$. In particular, they are not all eventually majorized by F_Π for any single Π. By the lemma, the infinite, co-infinite sets G_{f_i} do not all lie in any single $M(\Pi)$ and therefore do not all lie in any single set in $B_\mathcal{M}$. $\square$

The corollaries above, together with the general facts that **add** $\leq$ **unif** and **cov** $\leq$ **cof** for any ideal, clearly complete the proof of the theorem. $\square$

Corollary. *It is consistent, relative to ZFC, that the additivity number for $B_{\mathcal{M}}$ strictly exceeds the additivity number for B.*

Proof. In view of the theorem, this corollary merely asserts the consistency of $\mathbf{add}(B) < \mathfrak{b}$, which is well known; see for example [2, 6]. Among the models satisfying this strict inequality are those obtained from a model of the generalized continuum hypothesis by adding $\aleph_2$ Laver or Mathias reals in a countable-support iteration and the model obtained from a model of Martin's axiom and $2^{\aleph_0} \geq \aleph_2$ by adding at least $\aleph_1$ random reals. $\square$

We remark that the cardinal characteristics computed for $B_{\mathcal{M}}$ in Theorem 3 are the same as the characteristics of the ideal of K_σ sets (countable unions of compact sets) in ω^ω.

The rest of this section is devoted to the cardinal characteristics of the filter ${B_{\mathcal{M}}}^+$ of non-meager sets in $\mathcal{M}([\omega]^\omega)$. Cardinal characteristics of a filter F on a set X are defined to be the corresponding characteristics of the ideal $\{A \subseteq X | X - A \in F\}$. So in the case at hand we are concerned with the ideal of non-comeager, upward-closed subsets of $[\omega]^\omega$. If one wants to work in the lattice $\mathcal{M}([\omega]^\omega)$ of downward-closed (rather than upward-closed) subsets of ω, one can simply replace all subsets of ω by their complements, so the ideal becomes the ideal of non-comeager sets in $\mathcal{M}([\omega]^\omega)$.

Theorem 4. $\mathbf{add}({B_{\mathcal{M}}}^+) = \mathbf{cov}({B_{\mathcal{M}}}^+) = \mathfrak{g}$ *and* $\mathbf{unif}({B_{\mathcal{M}}}^+) = 2^{\aleph_0}$.

Proof. Untangling the definitions, we find that $\mathbf{add}({B_{\mathcal{M}}}^+)$ is the minimum number of sets in the filter ${B_{\mathcal{M}}}^+$ whose intersection is not in ${B_{\mathcal{M}}}^+$ and that $\mathbf{cov}({B_{\mathcal{M}}}^+)$ is the minimum number of sets in the filter ${B_{\mathcal{M}}}^+$ whose intersection is empty. By Theorem 2 and the definition of $\mathfrak{g}$, both of these cardinals equal $\mathfrak{g}$.

$\mathbf{unif}({B_{\mathcal{M}}}^+)$ is the minimum number of elements of $[\omega]^\omega$ needed to meet every set in ${B_{\mathcal{M}}}^+$, so it is obviously at most $2^{\aleph_0}$. To prove the reverse inequality, consider any fewer than $2^{\aleph_0}$ elements $a_i \in [\omega]^\omega$; we must find a groupwise dense $X \in \mathcal{M}([\omega]^\omega)$ that contains none of the a_i. There is an obvious choice of X, namely

$$X = \{x \in [\omega]^\omega | (\forall i)\, a_i \not\subseteq^* x\},$$

which contains no a_i and is downward-closed. To see that X is groupwise dense, which will complete the proof, consider an arbitrary partition of ω into finite intervals I_n. Fix a family of $2^{\aleph_0}$ pairwise almost disjoint infinite subsets d_ξ of ω. Each a_i is almost included in at most one of the sets

$$D_\xi = \bigcup_{n \in d_\xi} I_n,$$

as the a_i are infinite and the D_ξ, like the d_ξ, are almost disjoint. As there are more d_ξ's then a_i's, there must be a D_ξ that includes no a_i and is therefore in X. As D_ξ is a union of infinitely many I_n, the proof that X is groupwise dense is complete. $\square$

We do not know the value of $\mathbf{cof}({B_{\mathcal{M}}}^+)$, but we have the following partial information.

Theorem 5. $\mathbf{cof}(B_{\mathcal{M}}{}^{+}) \geq 2^{\aleph_0}$ *and* $\mathbf{cof}(B_{\mathcal{M}}{}^{+}) > \mathfrak{b}$.

Proof. The first inequality follows from Theorem 4, since $\mathbf{cof} \geq \mathbf{unif}$ for any proper ideal. The second inequality follows from thie first if $\mathfrak{b} < 2^{\aleph_0}$, so we assume from now on that $\mathfrak{b} = 2^{\aleph_0}$. To show that $\mathbf{cof}(B_{\mathcal{M}}{}^{+}) > 2^{\aleph_0}$, let $2^{\aleph_0}$ groupwise dense families X_α ($\alpha < 2^{\aleph_0}$) be given; we shall construct a groupwise dense Y such that no $X_\alpha \subseteq Y$.

List all the partitions of ω into finite intervals as Π_α ($\alpha < 2^{\aleph_0}$). We construct Y by an induction of length $2^{\aleph_0}$; at each step we declare one set $y_\alpha \in [\omega]^\omega$ to be in Y and one set $x_\alpha \in [\omega]^\omega$ to be out of Y. Since Y is to be monotone, we ensure that no y_α is $\geq$ any x_β.

At stage α, we proceed as follows. As in the proof of Theorem 4, form $2^{\aleph_0}$ almost disjoint sets y, each of which is a union of infinitely many intervals from Π_α. Each of the x_β's defined at earlier stages, being infinite, is $\leq$ at most one of these y's. As there are fewer than $2^{\aleph_0}$ such x_β's, we can choose one of our y's that is $\geq$ none of them; take this y as y_α.

Next, notice that, since $\{y_\beta \mid \beta \leq \alpha\}$ has cardinality $< 2^{\aleph_0} = \mathfrak{b} = \mathbf{unif}(B_{\mathcal{M}})$, its downward closure cannot be groupwise dense and therefore cannot include X_α. So we can define x_α to be some element of X_α that is not $\leq y_\beta$ for any $\beta \leq \alpha$. This completes stage α of our construction.

After all $2^{\aleph_0}$ stages, let Y be the downward closure of $\{y_\alpha \mid \alpha < 2^{\aleph_0}\}$. Y is groupwise dense because it contains, for each Π_α, an infinite union y_α of its intervals. Y includes no X_α since $x_\alpha \in X_\alpha$ and $x_\alpha \notin Y$. $\square$

4. Variants

The two concepts connected by Theorem 1, Baire category and groupwise density, have close relatives, to which one might reasonably try to extend Theorem 1. Groupwise density is, as its name suggests, a variant of the more familiar notion of density, and Baire category is in many respects similar to Lebesgue measure [7]. In this section, we show that neither of the variants of Theorem 1 suggested by these analogies is provable. One fails outright by a trivial argument. The other is at least consistently false; we do not know whether it is consistently true.

We treat first the easier situation, the one involving density. Recall that a set $X \in \mathcal{M}([\omega]^\omega)$ is *dense* if every infinite $A \subseteq \omega$ has an infinite subset $B \subseteq A$ with $B \in X$. (This is the usual notion of density for the notion of forcing consisting of the infinite subsets of ω ordered by inclusion.) It is easy to see that groupwise density implies density; just consider a partition of ω into intervals, each of which contains at least one element of A. It is also easy to see that the dense sets form a filter in $\mathcal{M}([\omega]^\omega)$ and in fact a countably complete filter; its additivity number is the cardinal $\mathfrak{h}$ introduced and studied in [1] and usually called the distributivity number (because it measures the distributivity of the complete Boolean algebra associated to the forcing mentioned above).

A density analog of Theorem 1 would say that the filter of dense sets is prime in $\mathcal{M}([\omega]^\omega)$, i.e., that the non-dense sets constitute an ideal (preferably even a σ-ideal). It is easy to see, however, that this analog is false. Let $X_0 \in \mathcal{M}([\omega]^\omega)$

consist of those $a \subseteq \omega$ in which all but finitely many elements are even, and let X_1 be defined similarly with "odd" in place of "even." Then neither X_0 nor X_1 is dense, but their union is dense, so the non-dense sets fail to form an ideal.

Turning to the less trivial case of Lebesgue measure, we note that the sets in $\mathcal{M}([\omega]^\omega)$ of measure zero constitute a σ-ideal and we ask whether this ideal is prime, i.e., whether the sets of positive outer measure constitute a filter (preferably a countably complete filter) in $\mathcal{M}([\omega]^\omega)$. The following theorem gives a consistent negative answer. In the statement and proof of the theorem, "measure" refers to the version of Lebesgue measure appropriate for the space 2^ω, namely the product measure obtained from the uniform measure on 2.

Theorem 6. *Assume the continuum hypothesis. Then there exist two sets in $\mathcal{M}([\omega]^\omega)$, each of positive outer measure, whose intersection has measure zero.*

Proof. Since the continuum hypothesis is assumed, let all the Borel sets of measure 1 be listed in a sequence indexed by the countable ordinals. We define two sequences of elements x_α and y_α of 2^ω, each indexed by the countable ordinals α, subject to the following four requirements:

(1) Both x_α and y_α belong to the αth Borel set of measure 1 (in our fixed list).
(2) x_α has density 1/2 in y_β for all $\beta < \alpha$.
(3) y_α has density 1/2 in x_β for all $\beta \leq \alpha$.
(4) Both x_α and y_α have density 1/2 in ω.

By "a has density 1/2 in b," where a and b are infinite subsets of ω, we mean that the ratio of elements of a among the first n elements of b tends to 1/2 as n increases,

$$\lim_{n\to\infty} \frac{|a \cap b \cap \{0,1,\ldots,n-1\}|}{|b \cap \{0,1,\ldots,n-1\}|} = \frac{1}{2}.$$

We note that, by the strong law of large numbers, for any fixed infinite b, almost all a have density 1/2 in b. If we attempt to define the x_α and y_α by induction on α (defining the x before the y at each stage because of the $<$ in (2) and the $\leq$ in (3)), we find that, at each step, the element of 2^ω that we wish to define is subject to countably many requirements, each of which is satisfied by almost all elements of 2^ω. Since measure is countably additive, the element we need always exists (in fact, almost any element will do), so the inductive definition succeeds.

Having defined the x_α and y_α, we let X be the smallest element of $\mathcal{M}([\omega]^\omega)$ containing all the x_α. So X consists of those $a \in [\omega]^\omega$ that are $\leq x_\alpha$ for some α. Thanks to (1), X intersects every Borel set of measure 1 and therefore has positive outer measure. Similarly, let Y be the smallest set in $\mathcal{M}([\omega]^\omega)$ containing all the y_α; it too has positive outer measure.

To complete the proof, we check that $X \cap Y$ has measure zero. Consider an arbitrary element $a \in X \cap Y$. By definition of X and Y, we have $a \leq x_\alpha$ and $a \leq y_\beta$ for some α and β. Consider first the case where $\alpha > \beta$. Since y_β has density 1/2 in ω by (4) and x_α has density 1/2 in y_β by (2), it follows that $x_\alpha \cap y_\beta$ has density 1/4 in ω. The same conclusion follows in the other case, where $\alpha \leq \beta$, by the same argument with x_α and y_β interchanged and with (3) in place of (2). In either case, a, being included in $x_\alpha \cap y_\beta$ modulo a finite set, cannot have density $1/2$ in ω. This shows that $X \cap Y$ is disjoint from the set of elements of 2^ω of density $1/2$. Since the latter set has measure 1, $X \cap Y$ has measure 0. $\square$

The assumption of the continuum hypothesis in Theorem 6 can be weakened to the assumption $\mathbf{cov}(L) = \mathbf{cof}(L)$, where L is the ideal of sets of measure zero. The only changes needed in the proof are that the countable ordinals are replaced by the ordinals below $\mathbf{cof}(L)$ and that instead of enumerating all the Borel sets of measure 1 we enumerate only enough of them to have all the others as supersets.

We conclude this paper with comments on some related work of Plewik [8], dealing with ideals, rather than arbitrary downward-closed families, of subsets of ω. By an ideal, we mean an $X \in \mathcal{M}([\omega]^\omega)$ such that $x \cup y \in X$ for all $x, y \in X$. (Here we view x and y as subsets of ω.) Let $\mathfrak{g}'$ be the smallest number of non-meager ideals with empty intersection. Plewik [8] showed that this definition is unchanged if we replace "empty" with meager, and he proved $\mathfrak{h} \le \mathfrak{g}' \le \mathfrak{d}$. The following result describes the connection between $\mathfrak{g}'$ and $\mathfrak{g}$. It uses the *splitting number* $\mathfrak{s}$ defined (cf. [10]) as the smallest possible cardinality for a family of subsets of ω such that every infinite subset x of ω is split by some y in the family, in the sense that both $x \cap y$ and $x - y$ are infinite.

Theorem 7. $\min(\mathfrak{g}', \mathfrak{s}) \le \mathfrak{g} \le \mathfrak{g}'$.

Proof. That $\mathfrak{g} \le \mathfrak{g}'$ is clear, since ideals are among the members of $\mathcal{M}([\omega]^\omega)$. To prove the other inequality, let $\kappa < \min(\mathfrak{g}', \mathfrak{s})$, and let κ groupwise dense families G_i be given. We must find an element in their intersection. For each i, let H_i be the ideal generated by G_i. Being supersets of the G_i, the H_i are non-meager, and, since there are fewer than $\mathfrak{g}'$ of them, all the H_i have a common member x. Thus, for each i, some finitely many elements y_{ik} of G_i cover x. (Here k ranges from 1 to some finite n_i.) Since the total number of all the y_{ik}, as both i and k vary, is at most $\kappa < \mathfrak{s}$, there must be an infinite $z \subseteq x$ not split by any y_{ik}. That is, for each i and k, either z is almost included in y_{ik} or they are almost disjoint. For any fixed i, the y_{ik} (as k varies) cover x, so they cover z, so (because there are only finitely many of them) they cannot all be almost disjoint from z. So for each i some y_{ik} almost includes z. But G_i is downward-closed and contains y_{ik}. So $z \in G_i$ for all i. $\square$

Note that the second inequality in the theorem and the fact that $\mathfrak{h} \le \mathfrak{g}$, pointed out in [3], imply Plewik's result that $\mathfrak{h} \le \mathfrak{g}'$.

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