Basic Subgroups and a Freeness Criterion for Torsion-Free Abelian Groups

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Abstract

We prove that if an abelian group has a basic subgroup of infinite rank and if every subgroup disjoint from any basic subgroup is free then the group itself is free. We prove several corollaries and related results, including some that do not require the existence of a basic subgroup.

1 Introduction

All groups in this paper are abelian and torsion-free. A subgroup B of a group G is *basic* in G if it is a free, pure subgroup of G and the quotient G/B is divisible. A subgroup H of G is *B-high* if it is maximal among subgroups of G disjoint from B. ("Disjoint" means the intersection is $\{0\}$, not $\emptyset$ which is impossible for subgroups.) Our main result is the following theorem, which confirms a conjecture of the second author.

Theorem 1.1 *Let G be a torsion-free abelian group such that*

1. *G has a basic subgroup of infinite rank, and*

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2. for every basic subgroup B of G, all B-high subgroups of G are free.

Then G is free.

Since subgroups of free groups are free and since Zorn's Lemma allows us to enlarge every subgroup disjoint from B to a B-high subgroup, hypothesis 2 of the theorem can equivalently be stated as "every subgroup of G that is disjoint from a basic subgroup is free."

Part of our motivation for this theorem is to "confine" the non-freeness of groups. Under hypothesis 1, the contrapositive of the theorem tells us that if a group is not free then a "reason" for this can be found disjoint from some basic subgroup.

Another part of our motivation is the desire for a torsion-free analog of a result proved for p-groups in [1]. In the context of p-groups, "direct sum of cyclic groups" plays the role of "free," for example in the definition of "basic." Theorem 1 of [1] says that if G is a p-group without elements of infinite height, if B is basic in G, and if every B-high subgroup of G is a direct sum of cyclic groups, then G is a direct sum of cyclic groups.

The situation is more complicated in the torsion-free case for several reasons. First, unlike p-groups, torsion-free groups — even very well-behaved ones, e.g., separable ones — need not have basic subgroups [3]. So the existence of a basic subgroup must be assumed. Second, we must assume that a basic subgroup has infinite rank. Although that assumption can be avoided in many interesting cases, e.g., if G is $\aleph_1$-free, it cannot be dropped altogether. An example from [4] is used in Section 3 to show what can go wrong. Third, we need to assume freeness of B-high subgroups for all (or at least for many) basic B's, not just for one. Whether this third difference can be eliminated remains an open problem.

In addition to the theorem mentioned above, we shall present several corollaries of the theorem or of its proof. One of these is similar to the theorem but does not require the existence of a basic subgroup; the price for this is that hypothesis 2 must be strengthened.

2 Proof of Main Theorem

The proof of Theorem 1.1 is in two parts, the first of which is a construction carried out in an arbitrary free abelian group of infinite rank. This part has essentially nothing to do with the general group G of the theorem. The second part applies this construction inside a basic subgroup of G. Since the first part may be of some interest in its own right, we formulate as a proposition the result of this construction.

Proposition 2.1 *Let B be a free abelian group of infinite rank. There is a subset S of B and there is a partition of S into countably many pieces S_k ($k \in \mathbb{N}$) with the following properties.*

1. *S freely generates a subgroup $\langle S \rangle$ of B.*
2. *The purification $\langle S \rangle_*$ of $\langle S \rangle$ in B is all of B.*
3. *For each n, let $S(\geq n) = \bigcup_{k \geq n} S_k$. Then $B/\langle S(\geq n) \rangle$ is divisible.*

Proof. Fix a basis W of the free group B, and let κ be the cardinality of W; so κ is infinite by hypothesis. Let $\preceq$ be a well-ordering of W such that each element has strictly fewer than κ predecessors. In other words, the order type of $(W, \preceq)$ is the initial ordinal of cardinality κ.

Assign to each $w \in W$ countably many later (in $\preceq$) members of W, which we call $f(w,k)$ for $k \in \mathbb{N}$, such that all the $f(w,k)$ are distinct. Thus, we have

- $f : W \times \mathbb{N} \to W$ is one-to-one, and
- $f(w,k) \succ w$ for all $w \in W$ and all $k \in \mathbb{N}$.

One way to produce such an f is to partition W into κ subsets A_w (one for each $w \in W$), each of size κ and therefore each cofinal in $(W \preceq)$. Then let $f(w,k)$ be the kth element of A_w that is $\succ w$.

Also fix an enumeration of the prime numbers $\langle p_k : k \in \mathbb{N} \rangle$, such that each prime is listed infinitely often.

For each $k \geq 1$, we define

$$S_k = \{w - p_k \cdot f(w,k) \mid w \in W\},$$

and for $k = 0$ we define

$$S_0 = \{w - p_0 \cdot f(w,0) \mid w \in W\} \cup R$$

where $R = W - \mathrm{range}(f)$. Finally, we set $S = \bigcup_{k \in \mathbb{N}} S_k$, and we claim that S and the S_k are as required in the proposition.

Because W is an independent set and f is one-to-one, it is clear by inspection that the various S_k are disjoint, so they form a partition of $S \subseteq B$. It remains to verify statements (1), (2), and (3) of the proposition.

Verification of (1): Suppose, toward a contradiction, that we had a non-trivial $\mathbb{Z}$-linear relation, say $\sum_i c_i s_i = 0$, where the s_i are finitely many distinct elements of S and the c_i are non-zero integers. Thinking of each s_i as a linear combination of members of W, let w^* be the last (with respect to $\preceq$) element of W involved in any of the s_i. Consider all s_i in which w^* occurs, and consider what these occurrences could look like. *A priori*, there are three possibilities.

- $s_i = w^* \in R$.
- $s_i = w^* - p_k \cdot f(w^*, k)$ for some k.
- $s_i = w - p_k \cdot w^*$ for some $w \in W$ and $k \in \mathbb{N}$ such that $f(w, k) = w^*$.

The second of these possibilities, however, contradicts our choice of w^*, because $f(w^*, k) \succ w^*$, so we need only consider the first and third. Only one of these can arise, since the first requires $w^* \in R = W - \text{range}(f)$ while the third requires $w^* = f(w, k) \in \text{range}(f)$. Furthermore, in the third case, w and k are uniquely determined, because f is one-to-one. Therefore, w^* can occur in only one s_i in our alleged linear relation. But then this linear relation cannot hold in B, because the coefficient of the basis element w^* cannot be cancelled. This contradiction completes the proof of (1).

Verification of (2): It suffices to prove that $\langle S \rangle_*$ contains every element w of W, and we do this by transfinite induction along the well-ordering $\preceq$. So let $w \in W$ and assume $w' \in \langle S \rangle_*$ for all $w' \prec w$. If $w \notin \text{range}(f)$, then $w \in R \subseteq S_0 \subseteq S \subseteq \langle S \rangle_*$, and we are done. So we may assume $w = f(w', k)$ for some w' and k. By our choice of f, we know $w' \prec w$ and so $w' \in \langle S \rangle_*$. We also have $w' - p_k \cdot w \in S_k \subseteq S \subseteq \langle S \rangle_*$. Being a subgroup, $\langle S \rangle_*$ must contain $p_k \cdot w$. Being pure, it must contain w.

Verification of (3): Fix n and let $[x]$ denote the image in $B/\langle S(\geq n) \rangle$ of $x \in B$. Since W generates B, it suffices to show that $[w]$ is divisible by p in $B/\langle S(\geq n) \rangle$, for all $w \in W$ and all primes p. Consider a specific $[w]$ and a specific p. Our enumeration of primes p_k listed every prime infinitely often, so we can fix some $k \geq n$ such that $p_k = p$. Then

$$w - p \cdot f(w, k) = w - p_k \cdot f(w, k) \in S_k \subseteq S(\geq n) \subseteq \langle S(\geq n) \rangle.$$

Thus in $B/\langle S(\geq n) \rangle$ we have $[w] = p \cdot [f(w, k)]$. This completes the proof of the proposition. □

Using the proposition, we can now prove Theorem 1.1.

Proof. Let G be as in the hypothesis of Theorem 1.1, let B be any basic subgroup of it of infinite rank, and let H be any B-high subgroup of G. Because B is free and has infinite rank, we can apply Proposition 2.1 to it; let S and S_k be as in that proposition. In addition to the notation $S(\geq n)$ used there, we shall use the analogous notation $S(< n) = \bigcup_{k<n} S_k$. As before, we shall use a subscript $*$ to indicate purification. For subgroups of B, it makes no difference whether we purify in B or in G because B is pure in G.

Temporarily fix an arbitrary $n \in \mathbb{N}$.

Claim 1: $\langle S(\geq n) \rangle_*$ is a basic subgroup of G.

Proof. $\langle S(\geq n)\rangle_*$ is certainly pure, being the purification of something, and free, being a subgroup of the free group B. We know that G/B is divisible, as B is basic in G, and that $B/\langle S(\geq n)\rangle_*$ is divisible, as it is a quotient of $B/\langle S(\geq n)\rangle$, which is divisible, according to (3) of Proposition 2.1. Therefore $G/\langle S(\geq n)\rangle_*$ is divisible. □

Claim 2: $H + \langle S(< n)\rangle$ is disjoint from $\langle S(\geq n)\rangle$.

Proof. Suppose $h + s = s'$ with $h \in H$, $s \in \langle S(< n)\rangle$, and $s' \in \langle S(\geq n)\rangle$. Then $h = s' - s \in \langle S\rangle \subseteq B$. But H and B are disjoint, so $h = s' - s = 0$. Because S is linearly independent, by (1) of Proposition 2.1, we conclude $s' = 0 = s$. □

Claim 3: $(H + \langle S(< n)\rangle)_*$ is disjoint from $\langle S(\geq n)\rangle_*$.

Proof. In any torsion-free group, disjoint subgroups have disjoint purifications. □

Claim 4: $(H + \langle S(< n)\rangle)_*$ is free.

Proof. Being disjoint from $\langle S(\geq n)\rangle_*$ by Claim 3, $(H + \langle S(<n)\rangle)_*$ is included in some $\langle S(\geq n)\rangle_*$-high subgroup by Zorn's Lemma. That high subgroup is free, by hypothesis (2) and Claim 1. So its subgroup $(H + \langle S(<n)\rangle)_*$ is also free. □

Now un-fix n.

Claim 5: $\bigcup_{n\in\mathbb{N}} (H + \langle S(< n)\rangle)_* = G$.

Proof. The left side of the claim, call it L, is the union of an increasing chain of pure subgroups of G, so it is itself a pure subgroup of G. It obviously includes H. It also includes

$$\bigcup_{n\in\mathbb{N}} \langle S(< n)\rangle_* = \langle S\rangle_* = B,$$

where we used part (2) of Proposition 2.1 and the fact that purification commutes with union of chains.

To finish the proof of the claim, we check that the only pure subgroup L of G that includes both B and H is all of G. Consider any $g \in G$; we want to show $g \in L$. If $g \in H$, we're done, so assume $g \notin H$. As H is B-high, $H + \langle g \rangle$ is not disjoint from B. Say $h + mg = b \neq 0$ with $h \in H$, $m \in \mathbb{Z}$, $b \in B$, and $m \neq 0$ (as H and B are disjoint). Then $mg = b - h \in L$ and $g \in L$ because L is pure. □

Claim 5 exhibits G as the union of a countable, increasing sequence of pure subgroups, each free by Claim 4. By a well-known result of Hill [5, Theorem 2], it follows that G is free. □

Every free group of infinite rank obviously satisfies the hypotheses of Theorem 1.1. From this and the theorem, it follows that those hypotheses exactly characterize free groups of infinite rank. There are several immediate consequences of this — for example that the hypotheses of Theorem 1.1 are preserved by infinite-rank subgroups and by direct sums — which seem to have no easy direct proofs, i.e., no proofs avoiding Theorem 1.1.

3 Variations

In this section, we present several corollaries of the main theorem or of its proof. The first eliminates the hypothesis that the basic subgroup must have infinite rank, at the cost of assuming that G is $\aleph_1$-free, i.e., that all countable subgroups of G are free.

Unfortunately some cost is necessary; the main theorem becomes false if one simply drops the assumption that a basic subgroup has infinite rank. This follows from an example constructed in [4], a torsion-free abelian group of rank 2 such that all subgroups of rank 1 are cyclic and all torsion-free quotients of rank 1 are divisible. In such a group, every pure subgroup B of rank 1 is basic, and every B-high subgroup is free, yet the whole group clearly cannot be free.

Corollary 3.1 *Let G be an $\aleph_1$-free abelian group such that*

1. *G has a basic subgroup and*
2. *for every basic subgroup B of G, all B-high subgroups of G are free.*

Then G is free.

Proof. Let G satisfy the hypotheses of the corollary, and let B be any basic subgroup of G. If B has infinite rank, then we simply invoke Theorem 1.1, which is applicable since $\aleph_1$-free obviously implies torsion-free. We're also done if $B = G$, since a basic subgroup is free by definition. So we may assume that B has finite rank and that $G/B \neq \{0\}$. Since G/B is divisible and torsion-free, it includes a copy of $\mathbb{Q}$. Consider the preimage in G of this copy. It has finite rank, because both B and $\mathbb{Q}$ do, but it is not finitely generated, because $\mathbb{Q}$ isn't. So it is countable but not free, and this contradicts the hypothesis that G is $\aleph_1$-free. □

The next corollary slightly weakens the need to consider *all* basic subgroups in the main theorem's second hypothesis.

Corollary 3.2 *Let G be a torsion-free abelian group such that*

1. *G has a basic subgroup B of infinite rank and*
2. *for every basic subgroup B' of G that is a subgroup of B (and has the same rank as B), all B'-high subgroups of G are free.*

Then G is free.

In this corollary and the next one, there is a parenthesized clause to the effect that certain basic subgroups have equal rank. Our proofs establish the apparently stronger versions of the corollaries that include these clauses. Deletion of these clauses would not, however, really weaken the corollaries, because, as Dugas and Irwin pointed out in [3], all basic subgroups of the same group necessarily have the same rank.

Proof. In the proof of Theorem 1.1, hypothesis 2 was applied only to the basic subgroups $\langle S(\geq n)\rangle_*$, which are subgroups of B. They have the same rank as B because, in the proof of Proposition 2.1, each of the pieces S_k of S has cardinality equal to the cardinality of W, which is the rank of B. Thus, hypothesis 2 of the present corollary covers all the applications actually made of the apparently stronger hypothesis 2 of the main theorem. □

Next we use a result from [2] to infer that, under suitable assumptions, at least one B-high subgroup must be free, so the requirement that they all be free is satisfied if they all are isomorphic.

Corollary 3.3 *Let G be a torsion-free abelian group of uncountable rank κ such that*

1. *G has a basic subgroup of the same rank κ, and*
2. *for each basic subgroup B (of rank κ), all the B-high subgroups of G are isomorphic.*

Then G is free.

Proof. It suffices to show that, if G and a basic subgroup B have the same uncountable rank κ, then there exists a free, B-high subgroup of G. Once this is shown, we can simply invoke the preceding corollary to complete the proof.

Given such G and B, we obtain, from Theorem 2 of [2], a free, pure subgroup F of G such that $G = B + F$. (It is for the purity of F that we must go beyond what Reid proved in [6] and that we need the uncountability of κ. A counterexample in [2] shows that when $\kappa = \aleph_0$ then there may be no suitable F.)

Let H be a $(B \cap F)$-high subgroup of F. Since F is free, so is H, and it remains only to check that H is B-high in G. Obviously, $H \cap B = \{0\}$, so we need only check that, if $g \in G - H$ then $H + \langle g \rangle$ contains a non-zero element of B.

There is a trivial case, namely when $g \in F$. Then, as H is $(B \cap F)$-high in F, $H + \langle g \rangle$ contains a non-zero element of $B \cap F$.

So assume from now on that $g \notin F$. As $G = B + F$, we can write $g = b + f$ with $b \in B$, $f \in F$, and (as $g \notin F$) $b \notin F$. Now if $f \in H$ then $H + \langle g \rangle$ contains $-f + g = b$, which is a non-zero element of B, and we're done.

So assume from now on that $f \notin H$. As H is $(B \cap F)$-high in F, $H + \langle f \rangle$ contains a non-zero element of $B \cap F$. Say $h + mf \in B - \{0\}$ with $h \in H$ and $m \in \mathbb{Z}$. Notice that $m \neq 0$, for otherwise we would have h in both $B - \{0\}$ and H which is absurd since H is disjoint from B. Now $H + \langle g \rangle$ contains $h + mg = (h + mf) + mb \in B$, so we're done unless this element of B is 0. But then $mb = -h - mf \in F$ and, as F is pure, $b \in F$, a contradiction. □

Finally, we present a version of the main theorem applicable to groups that have no basic subgroup. For this we need the notion of a *pure independent set* in a torsion-free group G. This is an independent set $I \subseteq G$ that generates a pure subgroup $\langle I \rangle$ of G; in other words, it is a basis for a pure, free subgroup of G. We shall have to consider maximal pure independent sets. Notice that a pure independent set $I \subseteq G$ is maximal if and only if the quotient $G/\langle I \rangle$ has no pure subgroup isomorphic to $\mathbb{Z}$. (In particular, a basis for a basic subgroup is maximal pure independent.) Notice also that, if I is a maximal pure independent set in G, it need not follow that $\langle I \rangle$ is a maximal pure free subgroup of G; there may well be pure free subgroups of G properly extending $\langle I \rangle$, but they cannot have bases extending I. Finally notice that, by Zorn's Lemma, every torsion-free abelian group has a maximal pure independent subset, and in fact every pure independent set is included in a maximal one.

Corollary 3.4 *Let G be a torsion-free abelian group with a pure free subgroup of infinite rank. Assume that, for every maximal pure independent $I \subseteq G$, all $\langle I \rangle$-high subgroups of G are free. Then G is free.*

Proof. The hypothesis implies, via Zorn's Lemma, that there is an infinite maximal pure independent set $I \subseteq G$. Fix such an I and write B for $\langle I \rangle$. Then proceed as in the first two paragraphs of the proof of Theorem 1.1, introducing S, S_k, etc. and temporarily fixing n as there. Of course we cannot simply copy Claim 1 from the earlier proof, since B is no longer basic, but we have the following substitute.

Claim 1′: $G/\langle S(\geq n)\rangle_*$ has no pure subgroup isomorphic to $\mathbb{Z}$.

Proof. Because $\langle I \rangle$ is maximal pure independent, G/B contains no pure copy of $\mathbb{Z}$. From part 3 of Proposition 2.1 it follows that $B/\langle S(\geq n)\rangle_*$ is divisible. Thus $G/\langle S(\geq n)\rangle_*$ has a divisible subgroup such that the quotient by it contains no pure copy of $\mathbb{Z}$. It follows easily that $G/\langle S(\geq n)\rangle_*$ itself contains no pure copy of $\mathbb{Z}$. (We point out that this has a direct proof, making no use of the fact that a divisible subgroup is always a direct summand. If a group X has a divisible subgroup Y and X/Y contains no pure $\mathbb{Z}$, then for each $x \in X$, since its image $[x] \in X/Y$ doesn't generate a pure $\mathbb{Z}$, some prime p must divide $[x]$. Say $[x] = p[v]$ in X/Y. Then $x - pv \in Y$ and, as Y is divisible, $x - pv = py$ for some $y \in Y$. Then $x = pv + py$, being divisible by p, cannot generate a pure $\mathbb{Z}$ in X.) □

Since $\langle S(\geq n)\rangle_*$ is free (being a subgroup of B) and pure in G, Claim 1′ implies that any basis for it is a maximal pure independent subset of G. By the hypothesis of the corollary, every $\langle S(\geq n)\rangle_*$-high subgroup of G is free.

Now the rest of the proof of Theorem 1.1, starting with Claim 2, goes through without further changes, showing that G is free. □

The same proof applies more generally.

Corollary 3.5 *Let $\mathcal{C}$ be a class of abelian groups such that, whenever $C \in \mathcal{C}$, D is divisible, and $A \cong C \oplus D$ then $A \in \mathcal{C}$. Let G be a torsion-free abelian group such that*

1. *G has a pure free subgroup B of infinite rank such that $G/B \in \mathcal{C}$, and*

2. *for each such B, all B-high subgroups of G are free.*

Then G is free.

This corollary includes the preceding one as the special case where $\mathcal{C}$ consists of the groups containing no pure copy of $\mathbb{Z}$. It also includes the main theorem as the special case where $\mathcal{C}$ consists of the divisible groups.

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