SOME ABELIAN GROUPS WITH FREE DUALS

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Abstract. Let Π be the direct product and Σ the direct sum of countably many copies of the additive group $\mathbb{Z}$ of integers. Specker proved that the dual $\mathrm{Hom}(\Pi, \mathbb{Z})$ of Π is isomorphic to Σ. We extend this result to groups G that are "near" Π. If G is a pure subgroup of Π of index smaller than $2^{\aleph_0}$, then the dual of G is isomorphic to Σ. If Π is a subgroup of countable index in a group H and if H has no direct summand isomorphic to Σ, then the dual of H is free (of finite or countable rank). We also obtain similar results for certain other subgroups of Π.

Introduction

We write Π for the direct product of countably many copies of the additive group of integers, that is, the group of all sequences $x = (x(n))_{n\in\mathbb{N}}$ of integers with componentwise addition. We write Σ for the subgroup of Π consisting of sequences with only finitely many non-zero entries. So Σ is the free abelian group generated by the "unit vectors" e_i, where the sequence e_i has i'th term 1 and all other terms 0. If $x, y \in \Pi$ and at least one of x and y is in Σ, then the sum $\sum_n x(n)y(n)$ makes sense since only finitely many terms are non-zero. We write this sum as $\langle x, y\rangle$. This bilinear (over $\mathbb{Z}$) pairing clearly identifies Π with the dual group $\mathrm{Hom}(\Sigma, \mathbb{Z})$ of Σ; that is, every homomorphism $\phi : \Sigma \to \mathbb{Z}$ is given by $\phi(x) = \langle x, y\rangle$ for a unique $y \in \Pi$, namely $y(i) = \phi(e_i)$.

Specker proved ([6], Satz III) that the same pairing also identifies Σ with the dual of Π. We shall extend this result to groups "near" Π. Specifically, we show that if G is a pure subgroup of Π of index smaller than the cardinality $\mathfrak{c} = 2^{\aleph_0}$ of the continuum then the dual of G is isomorphic to Σ. If, furthermore, G includes Σ then its dual is identified with Σ by the same pairing as above; that is, every homomorphism $\phi : G \to \mathbb{Z}$ is of the form $\phi(x) = \langle x, y\rangle$ for a unique $y \in \Sigma$. We also show that, if Π is a subgroup of countable index in a group H and if H has no direct summand isomorphic to Σ, then the dual of H is free.

The group Π/Σ has a divisible part D/Σ, where D consists of those sequences $x \in \Pi$ such that, for each positive integer r, all but finitely many terms $x(n)$ are

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divisible by r. Like any abelian group, Π/Σ can be written as the direct sum of its divisible part D/Σ and a reduced part A/Σ. The reduced complement A/Σ is uniquely determined up to isomorphism (being isomorphic to the quotient Π/D), and a result of Balcerzyk [1] describes it as the direct product, over all primes p, of the p-adic completions of direct sums of $\mathfrak{c}$ copies of the group J_p of p-adic integers. Equivalently, A/Σ is isomorphic to the product over all primes p of the countable powers ${J_p}^{\aleph_0}$. On the other hand, A/Σ is not at all unique as a subgroup of Π/Σ, for D/Σ has many complements there. In fact, we shall show that, unlike the A/Σ's, their pre-images $A \subseteq \Pi$ are not all isomorphic.

It is known that the dual of D is isomorphic to Σ, via the same pairing $\langle -, - \rangle$ as above. We shall extend this result by showing that, if $\Sigma \subseteq D' \subseteq D$ and D'/Σ is a pure subgroup of index $< \mathfrak{c}$ in D/Σ, then the dual of D' is isomorphic to Σ via the same pairing. Finally, we shall show that, of the various possible A's as described in the preceding paragraph, some have dual Σ via the same pairing; we do not know whether they all do.

Large Subgroups of Π

It will be useful for future reference to isolate explicitly what needs to be proved in order to conclude that the dual of a group G is identified with Σ by the standard pairing $\langle -, - \rangle$. Specifically, suppose that G is a group with $\Sigma \subseteq G \subseteq \Pi$. Then in order that $\langle -, - \rangle$ identify $\mathrm{Hom}(G, \mathbb{Z})$ with Σ, i.e., that every homomorphism $G \to \mathbb{Z}$ be of the form $\langle -, y \rangle$ for a unique $y \in \Sigma$, it is (necessary and) sufficient that G have the following two properties. (Recall that e_i is the i'th unit vector in Π.)

Finiteness Property. *For each homomorphism $\phi : G \to \mathbb{Z}$, there are only finitely many $i \in \mathbb{N}$ with $\phi(e_i) \neq 0$.*

Uniqueness Property. *The only homomorphism $\phi : G \to \mathbb{Z}$ sending all e_i to 0 is the identically zero homomorphism.*

Indeed, if the finiteness property holds, then to any homomorphism $\phi : G \to \mathbb{Z}$ we can associate an element $y \in \Sigma$ by defining $y(i) = \phi(e_i)$, and it is clear that ϕ and $\langle -, y \rangle$ agree at all the e_i. If, in addition, the uniqueness property holds, then by applying it to the difference between ϕ and $\langle -, y \rangle$ we find that these two homomorphisms agree everywhere. That no other y' can satisfy $\phi(x) = \langle x, y' \rangle$ for all x follows immediately from $\Sigma \subseteq G$, since we can take $x = e_i$ for various i.

The finiteness and uniqueness properties go back to Specker [6]; although they are not named there, they are the two parts of the proof of Satz III. We shall refer to them in all our proofs that various groups have duals isomorphic to Σ via $\langle -, - \rangle$. Notice that, since the unit vectors e_i generate Σ, the uniqueness property for G amounts to saying that zero is the only homomorphism from G/Σ to $\mathbb{Z}$.

Theorem 1. *If G is a pure subgroup of Π of index smaller than $\mathfrak{c}$, then the dual of G is isomorphic to Σ. If, in addition, $\Sigma \subseteq G$, then $\langle -, - \rangle$ identifies the dual of G with Σ.*

It follows that a pure subgroup of Π of index smaller than $\mathfrak{c}$ admits no homomorphism onto Σ. Otherwise, the group would have a direct summand isomorphic to

Σ, and so its dual would have a direct summand isomorphic to Π, which is absurd since the dual is countable.

Proof of theorem. We consider first the case that $\Sigma \subseteq G$, and we obtain the desired result by establishing the finiteness and uniqueness properties for G. There are at least two ways to prove each of these properties, but to save space we give only the shorter proof of each.

Proof of finiteness property. Suppose $\phi : G \to \mathbb{Z}$ were a homomorphism such that the set $S = \{i \in \mathbb{N} \mid \phi(e_i) \neq 0\}$ is infinite. Let $g : S \to \mathbb{N} - \{0\}$ be a function such that, if $m < n$ are any two elements of S, then $g(m)|\phi(e_m)| + 1$ divides $g(n)$. It is trivial to define such a g inductively. For each subset X of S, let $g_X : \mathbb{N} \to \mathbb{N}$ be the function that agrees with g on X and is identically zero on $\mathbb{N} - X$.

Since $\mathbb{Z}$ is countable, the kernel K of ϕ has countable index in G, which has index $< \mathfrak{c}$ in Π by hypothesis. So K has fewer than $\mathfrak{c}$ cosets in Π. But there are $\mathfrak{c}$ functions of the form g_X, so two of these functions lie in the same coset of K. The difference x between these two is therefore a non-zero element of G and satisfies $\phi(x) = 0$. Furthermore, each component $x(n)$ is either 0 or $\pm g(n)$, and non-zero components occur only at positions $n \in S$.

Let m be the first position such that $x(m) \neq 0$. So $x = \pm g(m)e_m + y$, where all the non-zero entries of y agree, up to sign, with the corresponding entries of g and occur at positions later than m in S. By our choice of g, all entries of y are therefore divisible by $g(m)|\phi(e_m)| + 1$. Since $x \in G$ and $g(m)e_m \in \Sigma \subseteq G$, we know that $y \in G$. Since G is pure in Π, y is divisible by $g(m)|\phi(e_m)|+1$ in G, and therefore $\phi(y)$ is divisible by $g(m)|\phi(e_m)| + 1$ in $\mathbb{Z}$. Applying ϕ to the equation $x = \pm g(m)e_m + y$ and remembering how x was chosen, we find that $0 = \pm g(m)\phi(e_m) + \phi(y)$. Since $\phi(y)$ is divisible by $g(m)|\phi(e_m)| + 1$, so is $g(m)\phi(e_m)$, but this is absurd. □

Proof of uniqueness property. Suppose $\phi : G \to \mathbb{Z}$ annihilates all the e_i and therefore all elements of Σ. Let x be an arbitrary element of G; we must show $\phi(x) = 0$. Define a homomorphism $\rho : \Pi \to \Pi$ by sending each $z \in \Pi$ to the sequence $\rho(z)$ whose n'th component is

$$\rho(z)(n) = x(n) \sum_{k \leq n} z(k),$$

and let $G' = \rho^{-1}(G)$. Since G is pure of index $< \mathfrak{c}$ in Π, so is G'. Furthermore, for each $i \in \mathbb{N}$, $\rho(e_i)$ agrees with x from the i'th component on (and has zeros in all earlier components), so $\rho(e_i)$ differs from $x \in G$ by an element of $\Sigma \subseteq G$ and is therefore also in G. Thus G' contains every e_i and so $\Sigma \subseteq G'$.

Thus our proof of the finiteness property for G is equally applicable to G', and we apply it to the homomorphism $\phi \circ \rho$. We obtain that, for all sufficiently large $i \in \mathbb{N}$, $\phi(\rho(e_i)) = 0$. But, as we saw above, x is the sum of $\rho(e_i)$ and an element of Σ. So ϕ, annihilating $\rho(e_i)$ and all elements of Σ, also annihilates x, as required. □

We have proved the part of the theorem referring to subgroups that include Σ. The general case can be reduced to the case $G \supseteq \Sigma$ as follows. By Theorems 3 and

4 of Nunke [5], the closure (in the product topology of Π obtained from the discrete topology on $\mathbb{Z}$) of any $G \subset \Pi$ of infinite rank is isomorphic to Π by an isomorphism $\gamma : \Pi \to \overline{G}$ that sends Σ into G. If G is pure and of index $< \mathfrak{c}$ in Π, hence certainly in its closure $\overline{G}$, then $\gamma^{-1}(G)$ is pure and of index $< \mathfrak{c}$ in Π, and it contains Σ. So, by what we have already proved, the dual of $\gamma^{-1}(G)$ is isomorphic to Σ. Since $\gamma^{-1}(G)$ is isomorphic to G, the proof is complete. (Note that the duality for G is given not by $\langle -, - \rangle$ but by $\langle \gamma^{-1}(-), - \rangle$.) $\square$

The hypothesis of purity in Theorem 1 is needed; Π has (non-pure) subgroups of countable index whose duals are not free. For example, consider the decomposition $\Pi/\Sigma = D/\Sigma \oplus A/\Sigma = X \oplus Y \oplus (A/\Sigma)$ where X is a countable-dimensional rational linear subspace of D/Σ and Y is a complementary subspace. Let X_0 be the subgroup of X generated (as a subgroup) by a basis (over the rationals) of X, and let G be the pre-image in Π of $X_0 \oplus Y \oplus (A/\Sigma)$. This G has countable index in Π because X/X_0 is countable. But G has an obvious projection onto the infinite-rank free group X_0, i.e., it has a direct summand isomorphic to Σ, and so its dual has a direct summand isomorphic to Π.

Of course, the hypothesis of small index is also needed in Theorem 1. If we allowed index $\mathfrak{c}$, then G could be Σ whose dual is Π, or G could be the group of bounded sequences, which Nöbeling [4] proved to be free and whose dual is therefore isomorphic to the product of $\mathfrak{c}$ copies of $\mathbb{Z}$.

Subgroups of D

Theorem 2. *If G is a pure subgroup of index $< \mathfrak{c}$ in D and if $\Sigma \subseteq G$, then the dual of G is isomorphic to Σ via $\langle -, - \rangle$.*

Proof. The proof of the finiteness property in Theorem 1 can be used equally well here. One merely needs to make sure that the functions g_X are in D, i.e., are eventually divisible by each positive integer, and this is easy to arrange during the inductive definition of g.

The uniqueness property is easier here than in Theorem 1, because G/Σ, being a pure subgroup of D/Σ, is divisible and therefore cannot have a non-trivial homomorphism to $\mathbb{Z}$. $\square$

Corollary. *Let G be as in Theorem 2 and not equal to D. Then G is not isomorphic to D.*

Proof. For any G as in Theorem 2, its dual is Σ and its double dual is therefore Π, both dualities being given by $\langle -, - \rangle$. It follows that the canonical embedding of G in its double dual is just the inclusion of G in Π. The $\mathbb{Z}$-adic closure of G in its double dual is therefore D, since D is the $\mathbb{Z}$-adic closure of Σ and $\Sigma \subseteq G \subseteq D$. So if $G \neq D$ then G, unlike D, is not $\mathbb{Z}$-adically closed in its double dual. Therefore, G cannot be isomorphic to D. $\square$

Theorem 3. *D has $2^{\mathfrak{c}}$ pairwise non-isomorphic subgroups that have duals isomorphic to Σ via $\langle -, - \rangle$.*

Proof. Let U be any non-trivial ultrafilter on $\mathbb{N}$. (See [2], Section 7 for background information on ultrafilters.) Define a subgroup D_U of D to consist of those $x \in D$

such that $\{n \in \mathbb{N} \mid x(n) = 0\} \in U$. Then D_U is a pure subgroup of D and a supergroup of Σ. As in the proof of Theorem 2, the uniqueness property is trivial, so we need only verify the finiteness property.

Let $\phi : D_U \to \mathbb{Z}$ and suppose S were an infinite subset of $\mathbb{N}$ all of whose elements i satisfy $\phi(e_i) \neq 0$. By shrinking S if necessary, we may assume that the complement of S is in U. (In more detail: Split S into two disjoint infinite pieces. The pieces cannot both be in U; replace S by a piece not in U. Then, as U is an ultrafilter, the complement is in U.) Let π be the homomorphism $D \to D_U$ which, when applied to an $x \in D$, leaves the components $x(i)$ for $i \in S$ unchanged but changes all the other components to 0. Then the homomorphism $\phi \circ \pi : D \to \mathbb{Z}$ sends each e_i for $i \in S$ to a non-zero value (since π doesn't affect such e_i's and ϕ sends them to non-zero values). That contradicts the finiteness property for D.

Since there are $2^{\mathfrak{c}}$ non-trivial ultrafilters on $\mathbb{N}$ (see [2], Corollary 7.15), we have $2^{\mathfrak{c}}$ subgroups D_U which would verify Theorem 3 if they were all pairwise non-isomorphic. In fact, not all of them but enough of them are non-isomorphic.

Any isomorphism between two of the D_U's would induce an isomorphism between their double duals, i.e., an automorphism of Π sending one D_U to the other. (Here we use the fact that, as in the proof of the corollary above, the canonical embedding of D_U in its double dual is just its inclusion into Π, since all the dualities are given by the same pairing $\langle -, - \rangle$.) But Π has only $\mathfrak{c}$ automorphisms, since $\mathrm{Hom}(\Pi, \Pi) = \mathrm{Hom}(\Pi, \mathbb{Z}^{\aleph_0}) \cong (\mathrm{Hom}(\Pi, \mathbb{Z}))^{\aleph_0} \cong \Sigma^{\aleph_0}$. So each isomorphism class of D_U's has size only $\mathfrak{c}$. Since there are $2^{\mathfrak{c}}$ D_U's, there must be $2^{\mathfrak{c}}$ isomorphism classes. □

The Reduced Part of Π/Σ

We show that the reduced complement A/Σ of the divisible part D/Σ in Π/Σ can be chosen so that its pre-image in Π has free dual.

Theorem 4. *There is a group A such that $\Sigma \subseteq A \subseteq \Pi$, and $\Pi/\Sigma = (D/\Sigma) \oplus (A/\Sigma)$, and the dual of A is isomorphic to Σ via $\langle -, - \rangle$.*

Proof. There are only countably many finite sequences s of integers. So we can associate to each such s an infinite set $C_s \subseteq \mathbb{N}$ in such a way that, first, distinct s and s' are assigned disjoint C_s and $C_{s'}$, and, second, all elements of C_s are larger than the length of s. Define X to be the subset of Π consisting of those sequences $x \in \Pi$ such that

$$\forall s \, \forall n \in C_s \, [\text{If } s \text{ is an initial segment of } x, \text{ then } x(n) \text{ is odd, and otherwise } x(n) \text{ is even.}]$$

Let $\langle X \rangle$ be the subgroup of Π generated by X.

Lemma. $\langle X \rangle \cap D = 0$.

Proof of lemma. Consider an arbitrary element u of $\langle X \rangle \cap D$. Being in $\langle X \rangle$, it has the form $c_1 x_1 + \cdots + c_k x_k$ for some $x_1, \dots, x_k \in X$ and $c_1, \dots, c_k \in \mathbb{Z}$. We shall show that all c_i are 0, which proves the lemma. So suppose some $c_i \neq 0$. Factoring out the largest possible power of 2, we write u in the form $(b_1 x_1 + \cdots + b_k x_k) 2^p$, where at least one b_i is odd. Being in D, u has all but finitely many components divisible by 2^{p+1}, so $b_1x_1 + \cdots + b_kx_k$ has all but finitely many components even.

Reducing all coefficients b_i in $b_1x_1 + \cdots + b_kx_k$ modulo 2, and deleting all terms with even (now 0) coefficients, we obtain a non-empty sum $y_1 + \cdots + y_l$ of distinct elements of X, such that all but finitely many of its components are even. ("Non-empty" is because at least one b_i was odd.) Choose $q \in \mathbb{N}$ so large that all of $y_1, \dots, y_l$ have distinct initial segments $s_1, \dots, s_l$ of length q. Consider any $n \in C_{s_1}$. As all of $y_1, \dots, y_k$ are in X and among these only y_1 has s_1 as an initial segment, we have, by definition of X, that $y_1(n)$ is odd and all the other $y_i(n)$ are even. So the n'th component of $y_1 + \cdots + y_l$ is odd. This happens for infinitely many n (as C_{s_1} is infinite), contrary to the fact that all but finitely many components of $y_1 + \cdots + y_l$ are even. This contradiction completes the proof of the lemma. □

By the lemma (and the fact that $\Sigma \subseteq D$), the canonical projection $\Pi \to \Pi/\Sigma$ sends $\langle X \rangle$ to a subgroup of Π/Σ whose intersection with D/Σ is zero. As D/Σ is divisible, we can enlarge this image of $\langle X \rangle$ to a complement of D/Σ in Π/Σ, and this complement has the form A/Σ for some subgroup A of Π that includes Σ and $\langle X \rangle$. We note for future reference that $\Pi/A \cong D/\Sigma$ is torsion-free, so A is pure in Π.

To complete the proof of the theorem, we need only show that the dual of A is isomorphic to Σ via $\langle -, - \rangle$. For this purpose, it suffices to establish the finiteness and uniqueness properties for A. The uniqueness property is very easy. A/Σ has trivial dual because it is a direct summand of Π/Σ which has trivial dual.

To establish the finiteness property, let $\phi : A \to \mathbb{Z}$ be a homomorphism, and suppose $S = \{i \in \mathbb{N} \mid \phi(e_i) \neq 0\}$ is infinite. Fix an increasing sequence $(p_i)_{i \in S}$ of odd prime numbers, indexed by S, with the additional property that p_i does not divide $\phi(e_i)$ for any $i \in S$. We shall define a sequence x of integers with the following properties.

(1) $x \in X$.
(2) If $m < n$ and $m \in S$ then p_m divides $x(n)$.
(3) If $n \in S$ then

$$\sum_{k=0}^{n} x(k)\phi(e_k) \equiv \frac{p_n - 1}{2} \pmod{p_n}.$$

To produce such an x we define its terms $x(n)$ inductively. If all $x(m)$ for $m < n$ have already been defined, then (3) specifies the congruence class of $x(n)$ modulo p_n (because p_n does not divide the coefficient $\phi(e_n)$ of $x(n)$ in (3)), and (2) specifies the congruence class of $x(n)$ modulo the earlier odd primes p_m in our sequence. Finally, if $n \in C_s$ for some s, then (1) specifies the congruence class of $x(n)$ modulo 2 as follows. First, the s such that $n \in C_s$ is unique, since the C_s's are pairwise disjoint. Second, since our choice of C_s ensures that n exceeds the length of s, the values $x(m)$ that are already available suffice to determine whether s is an initial segment of x and thus to determine what parity (1) requires $x(n)$ to have. Thus, the desired properties of $x(n)$ are finitely many congruences with distinct, prime moduli. So the Chinese remainder theorem guarantees the existence of such an $x(n)$, and the inductive construction of x succeeds.

Since $x \in X \subseteq A$, $\phi(x)$ is defined.

Consider an arbitrary $n \in S$. All components of x past position n are divisible by p_n, because of (2). So we can write

$$x = \sum_{k=0}^{n} x(k)e_k + p_n z$$

for some $z \in \Pi$. Since A contains both x and $\sum_{k=0}^{n} x(k)e_k$ (the latter because it is in $\Sigma \subseteq A$), it contains $p_n z$ and, being pure, also contains z. So we obtain

$$\begin{aligned}\phi(x) &= \sum_{k=0}^{n} x(k)\phi(e_k) + p_n\phi(z) \\ &\equiv \frac{p_n - 1}{2} \pmod{p_n},\end{aligned}$$

by (3). This congruence clearly implies $|\phi(x)| \geq \frac{p_n-1}{2}$, i.e., $p_n \leq 2|\phi(x)| + 1$.

This inequality, holding for each of the infinitely many $n \in S$, gives us infinitely many distinct p_n all below $2|\phi(x)| + 1$, obviously a contradiction. □

Corollary. *The groups A such that $\Sigma \subseteq A \subseteq \Pi$ and $\Pi/\Sigma = (D/\Sigma) \oplus (A/\Sigma)$ are not all isomorphic.*

Proof. We begin by counting how many such A's there are. Clearly, each such A is the pre-image in Π of a complement A/Σ of D/Σ in Π/Σ. So our task is to count these complements. Fix one of them, and call it A_0/Σ. Then the others are in one-to-one correspondence with the homomorphisms $A_0/\Sigma \to D/\Sigma$; indeed, quite generally, the complements of Y in a direct sum $X \oplus Y$ are in one-to-one correspondence with the homomorphisms $X \to Y$. (To see this, regard $X \oplus Y$ as a set of ordered pairs, regard functions — in particular homomorphisms — as sets of ordered pairs, and verify that the complements of Y are literally the same sets of ordered pairs as the homomorphisms $X \to Y$.) Since A_0/Σ is torsion-free and has the cardinality $\mathfrak{c}$ of the continuum, it has a free subgroup F of rank $\mathfrak{c}$. A homomorphism from F to D/Σ is specified by arbitrarily specifying the images in D/Σ of the generators of F; as D/Σ has cardinality $\mathfrak{c}$, the number of such specifications is $\mathfrak{c}^\mathfrak{c} = 2^\mathfrak{c}$. Each homomorphism $F \to D/\Sigma$ extends to A_0/Σ because D/Σ is divisible. So there are at least $2^\mathfrak{c}$ homomorphisms $A_0/\Sigma \to D/\Sigma$ and therefore at least $2^\mathfrak{c}$ complements A/Σ for D/Σ in Π/Σ and at least $2^\mathfrak{c}$ groups A as in the corollary. (Of course, we can improve "at least" to "exactly" because Π, having cardinality $\mathfrak{c}$, has only $2^\mathfrak{c}$ subsets.)

Now assume that A_0 was chosen as in Theorem 4, so its dual is countable. Since a homomorphism $\gamma : A_0 \to \Pi$ is specified by giving its countably many component homomorphisms $\gamma_n : A_0 \to \mathbb{Z}$, the number of such γ's is only $\aleph_0^{\aleph_0} = \mathfrak{c}$. In particular, there are at most $\mathfrak{c}$ isomorphisms from A_0 to other A's as in the corollary.

Thus, of the $2^{\mathfrak{c}}$ such A's, at most $\mathfrak{c}$ are isomorphic to A_0. Since $2^{\mathfrak{c}} > \mathfrak{c}$, the proof of the corollary is complete. □

We do not know whether every complement A/Σ of D/Σ in Π/Σ has the property $\mathrm{Hom}(A,\mathbb{Z}) = \Sigma$ that Theorem 4 asserts of one such complement. In general, the connection between the various complements A/Σ and between their pre-images $A \subseteq \Pi$ is rather mysterious. We digress briefly to illustrate another aspect of this.

Theorem 5. *Every complement A/Σ of D/Σ in Π/Σ has a pure subgroup P isomorphic to Π. For some choices of A and P, there is a lifting of P, i.e., a pure subgroup $P' \subseteq A$ mapped isomorphically to P by the projection $\Pi \to \Pi/\Sigma$.*

Proof. $\mathbb{Z}$ is a pure subgroup of its $\mathbb{Z}$-adic completion, which is isomorphic to the product, over all primes p, of the groups J_p of p-adic integers. Taking the $\aleph_0$ power of this situation, we find that Π is isomorphic to a pure subgroup of the product, over all p, of ${J_p}^{\aleph_0}$. But, as mentioned in the introduction, this last product is isomorphic to any complement A/Σ of D/Σ in Π/Σ. This proves the first assertion of the theorem.

For the second, we first construct P', then P, and then A. Partition $\mathbb{N}$ into infinitely many infinite sets C_n indexed by $n \in \mathbb{N}$, and let P' consist of those functions $x : \mathbb{N} \to \mathbb{Z}$ (i.e., $x \in \Pi$) that are constant on each C_n. Clearly, P' is isomorphic to Π by the map sending each such x to the sequence y given by $y(n) =$ the constant value of x on C_n. It is also clear that both P' and $P' + \Sigma$ are pure in Π. Also, $P' \cap D = 0$ because a function constant on an infinite set C_n cannot be eventually divisible by each positive integer unless it is zero on C_n. So the projection from Π to Π/Σ sends P' isomorphically onto a pure subgroup P of Π/Σ that intersects D/Σ only at 0. As D/Σ is divisible, P can be enlarged to a complement A/Σ of D/Σ, and the proof is complete. □

We do not know whether every complement A/Σ of D/Σ has a pure subgroup, isomorphic to Π, that admits a lifting.

Small Supergroups of Π

We now turn from subgroups of Π to supergroups. As in Theorem 1, we find that such groups have free duals if they are not too far from Π.

Theorem 6. *Suppose Π is a subgroup of countable index in a group H, and suppose H has no direct summand isomorphic to Σ. Then the dual of H is free.*

Proof. Let H be as in the hypothesis of the theorem, and let C be the (countable) quotient group H/Π. So we have an exact sequence

$$0 \longrightarrow \Pi \longrightarrow H \longrightarrow C \longrightarrow 0$$

and thus an exact sequence of duals

$$\mathrm{Hom}(\Pi,\mathbb{Z}) \longleftarrow \mathrm{Hom}(H,\mathbb{Z}) \longleftarrow \mathrm{Hom}(C,\mathbb{Z}) \longleftarrow 0.$$

If $\mathrm{Hom}(C, \mathbb{Z}) = 0$, then this exact sequence shows that the dual of H is a subgroup of the dual Σ of Π, so it is free and the proof is complete in this case.

It remains to consider the case that C has a non-trivial homomorphism to $\mathbb{Z}$ and therefore can be written as $C = C_1 \oplus \mathbb{Z}$ (where we write simply $\mathbb{Z}$ for an isomorphic copy of $\mathbb{Z}$ in C). The pre-image in H of the $\mathbb{Z}$ summand of C is the direct sum of Π and a copy of $\mathbb{Z}$, so it is isomorphic to Π and we therefore have an exact sequence

$$0 \longrightarrow \Pi \longrightarrow H \longrightarrow C_1 \longrightarrow 0$$

(with a new embedding $\Pi \to H$). As before, if the dual of C_1 is trivial, we are done. Otherwise, we can split off another $\mathbb{Z}$ and add it to Π obtaining $C_1 = C_2 \oplus \mathbb{Z}$ and an exact sequence

$$0 \longrightarrow \Pi \longrightarrow H \longrightarrow C_2 \longrightarrow 0.$$

Repeat this process as long as the duals of the C_n's are non-zero. If the process ever terminates (with some C_n having trivial dual), then, as before, the dual of H is free, as desired.

It remains to consider the possibility that the process never terminates. Each C_n, starting with $C_0 = C$, splits as $C_{n+1} \oplus \mathbb{Z}$. Given any element $c \in C$, we produce a sequence of integers by taking its $\mathbb{Z}$-components in the successive splittings. That is, we let $c_0 = c \in C = C_0$ and, once $c_n \in C_n$ is defined, we use the splitting $C_n = C_{n+1} \oplus \mathbb{Z}$ to write $c_n = (c_{n+1}, z(n))$. This inductively defines a sequence $z = (z(n))_{n \in \mathbb{N}} \in \Pi$. This construction assigning to each $c \in C$ a sequence $z \in \Pi$ is clearly a homomorphism $\rho : C \to \Pi$.

For any $n \in \mathbb{N}$, the decomposition $C = C_n \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \dots \oplus \mathbb{Z}$ with n copies of $\mathbb{Z}$ shows that every n-tuple of integers occurs as the first n components of $\rho(c)$ for some $c \in C$. Thus the image of ρ has infinite rank. Furthermore, since C is countable, the image of ρ is a countable subgroup of Π and is therefore free by [6] Satz I. So the image of ρ is isomorphic to Σ. So, composing this isomorphism with ρ and with the projection $H \to C$, we get a homomorphism from H onto Σ. Since Σ is free, H has a direct summand isomorphic to Σ, contrary to hypothesis. This contradiction shows that the process of splitting off $\mathbb{Z}$'s to produce C_n's cannot continue forever. And we had already seen that, once the process terminates, the proof is complete. □

We remark that the last part of this proof could be replaced with a reference to Stein's result, Corollary 19.3 in [3], vol. I, that every countable group is the direct sum of a free group and a group with no free quotients

Theorem 6 can be generalized by replacing Π with any group that has free dual and that is isomorphic to its direct sum with $\mathbb{Z}$. Simply replace Π by that group throughout the proof except in the last paragraph.

In the conclusion of Theorem 6, we cannot improve "free" to "isomorphic to Σ"; the dual of H may well have finite rank and may even be zero, as the following example shows. Let X_p be the group of rational numbers whose denominators are powers of the prime p, and let $X = X_2 \oplus X_3$. Clearly, each X_p has trivial dual and therefore so does X. Notice also that the pairs (a, a) with $a \in \mathbb{Z}$ form a pure subgroup of X isomorphic to $\mathbb{Z}$. Let Y be the direct sum of countably many copies of X. Then Y also has trivial dual, and it has a pure subgroup S isomorphic to Σ, say by an isomorphism ρ. Let H be the result of first forming the direct sum of Π and Y and then forming a quotient by identifying each element of $\Sigma \subseteq \Pi$ with the corresponding (under ρ) element of $S \subseteq Y$. No elements of Π or of Y are sent to 0 by this identification, so we can regard Π and Y as subgroups of H; together they generate H, and their intersection is precisely the identified $\Sigma = S$. As Y is countable, Π has countable index in H. The quotient H/Π is isomorphic to Y/S; since S is pure in Y, Π is pure in H. So the hypotheses of Theorem 6 are satisfied. Yet every homomorphism $\phi : H \to \mathbb{Z}$ is trivial. Indeed, the restriction of ϕ to Y is trivial because Y has trivial dual. Since Σ has been identified with a subgroup S of Y, ϕ is also trivial on Σ and therefore on Π (by Specker's Satz III [6], specifically by the uniqueness property for Π). But Y and Π generate H, so ϕ is trivial on all of H.

A Question

Problem. *Can a subgroup of* Π *have a free dual of uncountable rank?*

We point out that one simple approach to obtaining such a subgroup does not work. There is no subgroup of Π isomorphic to $\mathbb{Z}^{\aleph_1}$. The reason is that, for any subgroup of Π, there are countably many homomorphisms to $\mathbb{Z}$ (namely the projections of the product Π to its factors) that separate points. But a homomorphism from $\mathbb{Z}^{\aleph_1}$ to $\mathbb{Z}$ depends on only finitely many of the $\aleph_1$ coordinates, so countably many such homomorphisms collectively depend on only countably many coordinates and therefore cannot separate all the points of $\mathbb{Z}^{\aleph_1}$. (A similar argument shows that $\mathbb{Z}^{\kappa}$ cannot be embedded in $\mathbb{Z}^{\lambda}$ if $\kappa > \lambda$, though a slight reformulation of the proof is needed if κ is above a measurable cardinal.)

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