

Reductions Between Cardinal Characteristics of the Continuum

ANDREAS BLASS

ABSTRACT. We discuss two general aspects of the theory of cardinal characteristics of the continuum, especially of proofs of inequalities between such characteristics. The first aspect is to express the essential content of these proofs in a way that makes sense even in models where the inequalities hold trivially (e.g., because the continuum hypothesis holds). For this purpose, we use a Borel version of Vojtáš's theory of generalized Galois-Tukey connections. The second aspect is to analyze a sequential structure often found in proofs of inequalities relating one characteristic to the minimum (or maximum) of two others. Vojtáš's max-min diagram, abstracted from such situations, can be described in terms of a new, higher-type object in the category of generalized Galois-Tukey connections. It turns out to occur also in other proofs of inequalities where no minimum (or maximum) is mentioned.

1. Introduction

Cardinal characteristics of the continuum are certain cardinal numbers describing combinatorial, topological, or analytic properties of the real line \mathbb{R} and related spaces like ${}^\omega\omega$ and $\mathcal{P}(\omega)$. Several examples are described below, and many more can be found in [4, 14]. Most such characteristics, and all those under consideration in this paper, lie between \aleph_1 and the cardinality $\mathfrak{c} = 2^{\aleph_0}$ of the continuum, inclusive. So, if the continuum hypothesis (CH) holds, they are equal to \aleph_1 . The theory of such characteristics is therefore of interest only when CH fails.

1991 *Mathematics Subject Classification.* 03E05, 03E40, 03E75.

Partially supported by NSF grant DMS-9204276 and NATO grant LG 921395.

The content of my talks at BEST had already been submitted for publication elsewhere. So this paper corresponds rather to the talk I gave at Oberwolfach in October, 1993.

This paper is in final form, and no version of it will be submitted for publication elsewhere.

That theory consists mainly of two sorts of results. First, there are equations and (non-strict) inequalities between pairs of characteristics or sometimes between one characteristic and the maximum or minimum of two others. Second, there are independence results showing that other equations and inequalities are not provable in Zermelo-Fraenkel set theory (ZFC). As examples of results of the first sort, we mention in particular the work of Rothberger [10] and Bartoszyński [1] relating the characteristics associated to Lebesgue measure and Baire category; as examples of the second sort we mention [2] and the earlier work cited there. Many more examples can be found in [4, 14] and the references there.

A curious aspect of the proofs of inequalities (and of equations, which we regard as pairs of inequalities) in this theory is that they contain significant information whether or not CH holds, even though CH makes the inequalities themselves trivial. In other words, the proofs establish additional information beyond the inequalities. Vojtáš [15] introduced a framework in which one can attempt to formulate such additional information. He associates cardinal characteristics to binary relations on the reals and shows that proofs of inequalities between characteristics usually amount to the construction of a suitable pair of functions between the domains and the ranges of the corresponding relations. He calls these pairs of functions generalized Galois-Tukey connections, but for brevity we shall call them morphisms. The existence of such morphisms seems a plausible candidate for the “additional information” established by typical proofs of inequalities between cardinal characteristics.

It turns out, however, by a result of Yiparaki [16], that this additional information is still trivial in the presence of CH. We shall show in Section 2 how to modify Vojtáš’s framework so as to produce non-trivial results even in the presence of CH. The key idea is to add a definability (or absoluteness) requirement on the functions that constitute a morphism.

Section 3 contains some information about Baire category that will be used in subsequent examples.

Section 4 is concerned with a structure often found in proofs of inequalities of the form $x \geq \min(y, z)$, a structure that Vojtáš described with his max-min diagram. We show that this diagram can be neatly interpreted in terms of a construction, which we call *sequential composition*, on the relations associated to y and z . This construction allows us to analyze the “flow of control” in proofs of such three-cardinal inequalities.

Section 5 is devoted to showing that sequential composition is necessary for the results in Section 4. In particular, certain simpler compositions proposed by Vojtáš are not adequate for these results.

Finally, in Section 6, we discuss a situation where sequential composition arises in the natural proof of an inequality involving just two cardinal characteristics, not a maximum or minimum.

I thank Peter Vojtáš and Janusz Pawlikowski for useful discussions of the

topic of this paper.

2. Borel Galois-Tukey Connections

To motivate Vojtáš's framework, we define a few cardinal characteristics of the continuum and discuss the proofs of some inequalities relating them. The characteristics to be used in this example are the *bounding number* \mathfrak{b} , the *dominating number* \mathfrak{d} , the *unsplitting number* \mathfrak{r} , and the *splitting number* \mathfrak{s} , defined as follows. (See [4, 14] for more information about these cardinals.) For functions f and g from ω to ω , we write $f \leq^* g$ to mean that $f(n) \leq g(n)$ for all but finitely many n . A family $\mathcal{B} \subseteq {}^\omega\omega$ of such functions is *unbounded* if there is no $g \in {}^\omega\omega$ such that all elements of \mathcal{B} are $\leq^* g$. The smallest possible cardinality for an unbounded family is \mathfrak{b} . A family $\mathcal{D} \subseteq {}^\omega\omega$ is *dominating* if every $g \in {}^\omega\omega$ is \leq^* some $f \in \mathcal{D}$. The smallest possible cardinality for a dominating family is \mathfrak{d} . A subset X of ω *splits* another such set Y if both $Y \cap X$ and $Y - X$ are infinite. A family $\mathcal{S} \subseteq \mathcal{P}(\omega)$ is a *splitting family* if every infinite $Y \subseteq \omega$ is split by some member of \mathcal{S} . The smallest possible cardinality for a splitting family is \mathfrak{s} . A family \mathcal{R} of infinite subsets of ω is an *unsplit family* (sometimes called *refining* or *reaping*) if no single X splits all the members of \mathcal{R} . The smallest possible cardinality for an unsplit family is \mathfrak{r} .

It is easy to see that all four of the cardinals just defined are between \aleph_1 and \mathfrak{c} inclusive and that $\mathfrak{b} \leq \mathfrak{d}$. As an example for future analysis, we give the proof of the less trivial yet well-known inequality $\mathfrak{s} \leq \mathfrak{d}$. Given a dominating family \mathcal{D} , we shall assign to each $f \in \mathcal{D}$ a set $X_f \subseteq \omega$ in such a way that all these X_f 's constitute a splitting family. This will clearly suffice to prove $\mathfrak{s} \leq \mathfrak{d}$. The construction is as follows. Given f , partition ω into finite intervals $[a_0, a_1), [a_1, a_2), \dots$ each satisfying $f(a_n) < a_{n+1}$. To be specific, let $a_0 = 0$ and $a_{n+1} = 1 + \max\{a_n, f(a_n)\}$. Then let X_f be the union of the even-numbered intervals $[a_{2n}, a_{2n+1})$. To see that the X_f 's constitute a splitting family, let an arbitrary infinite $Y \subseteq \omega$ be given, and let $g : \omega \rightarrow \omega$ be the function sending each natural number n to the next larger element of Y . As \mathcal{D} is dominating, it contains an $f \geq^* g$. In the construction of X_f we have, for all sufficiently large n , that the next element of Y after a_n is $g(a_n) \leq f(a_n) < a_{n+1}$ and therefore lies in the interval $[a_n, a_{n+1})$. So Y meets all but finitely many of these intervals and therefore contains infinitely many members of X_f and infinitely many members of $\omega - X_f$. That is, X_f splits Y , as required.

We now present Vojtáš's framework for describing cardinal characteristics and proofs of inequalities, using the preceding proof as an example. First, each characteristic was defined as the smallest possible cardinality for a set \mathcal{Z} of reals such that every real is related in a certain way to one in \mathcal{Z} . More precisely, in each case we had a triple $\mathbf{A} = (A_-, A_+, A)$ of two sets A_\pm and a binary relation

A between them such that the characteristic is

$$\|(A_-, A_+, A)\| = \min\{|\mathcal{Z}| : \mathcal{Z} \subseteq A_+ \text{ and } \forall x \in A_- \exists z \in \mathcal{Z} A(x, z)\},$$

which we call the *norm* of \mathbf{A} . For example, \mathfrak{d} is the norm of $({}^\omega\omega, {}^\omega\omega, \leq^*)$, and \mathfrak{s} is the norm of $(\mathcal{P}_\infty(\omega), \mathcal{P}(\omega), \text{is split by})$. (Here \mathcal{P}_∞ means the family of infinite subsets.) To describe \mathfrak{b} and \mathfrak{r} as norms, it suffices to take the descriptions for \mathfrak{d} and \mathfrak{s} and *dualize* them in the following sense: interchange A_- with A_+ , and replace A with the complement of the converse relation. In general, we write

$$(A_-, A_+, A)^\perp = (A_+, A_-, \{(z, x) : \text{not } A(x, z)\}).$$

(Vojtáš calls the norms of \mathbf{A} and of its dual the dominating and bounding numbers of \mathbf{A} , respectively, by analogy with the example of \mathfrak{d} and \mathfrak{b} above.) We sometimes call triples (A_-, A_+, A) relations, although strictly speaking it is only the third component A that is a relation. To avoid trivialities, we shall tacitly assume that our relations have $A_\pm \neq \emptyset$, that each element of A_- is A -related to some element of A_+ (so $\|\mathbf{A}\|$ is defined), and that not all elements of A_- are A -related to any single element of A_+ (so $\|\mathbf{A}^\perp\|$ is defined). These assumptions amount to requiring the norms of both \mathbf{A} and \mathbf{A}^\perp to be at least 2.

The proof of $\mathfrak{s} \leq \mathfrak{d}$ presented above consists of three parts. First, there was a construction $\xi_+ : {}^\omega\omega \rightarrow \mathcal{P}(\omega)$ sending each f to X_f . Second, there was a (simpler) construction $\xi_- : \mathcal{P}_\infty(\omega) \rightarrow {}^\omega\omega$ sending each Y to the function $\xi_-(Y) = g$ that maps n to the next larger element of Y . Finally, there was the verification that if $g \leq^* f$ then Y is split by X_f . Abstracting this structure, we obtain Vojtáš's notion of a generalized Galois-Tukey connection; we call it simply a morphism and write it in the opposite direction to Vojtáš's.

DEFINITION. A *morphism* ξ from (A_-, A_+, A) to (B_-, B_+, B) is a pair of functions $\xi_- : B_- \rightarrow A_-$ and $\xi_+ : A_+ \rightarrow B_+$ such that, for all $b \in B_-$ and all $a \in A_+$,

$$A(\xi_-(b), a) \implies B(b, \xi_+(a)).$$

Our convention for the direction of morphisms was chosen partly to work well with other uses of the same category [8, 3] and partly so that the direction of a morphism agrees with the direction of the implication displayed in the definition.

The existence of a morphism ξ from (A_-, A_+, A) to (B_-, B_+, B) immediately implies the norm inequality $\|(A_-, A_+, A)\| \geq \|(B_-, B_+, B)\|$. Indeed, if \mathcal{Z} is as in the definition of norm for (A_-, A_+, A) , then $\xi_+(\mathcal{Z})$ has no greater cardinality and serves the same purpose in the definition of the norm of (B_-, B_+, B) . Because of this, we write $\mathbf{A} \geq \mathbf{B}$ to indicate the existence of such a morphism. (Our inequalities, unlike our morphisms, go in the same direction as Vojtáš's.)

A morphism from \mathbf{A} to \mathbf{B} becomes, just by interchanging its two components, a morphism in the opposite direction between the dual objects. Thus, for example, the proof above of $\mathfrak{s} \leq \mathfrak{d}$, exhibiting a morphism from $({}^\omega\omega, {}^\omega\omega, \leq^*)$ to

($\mathcal{P}_\infty(\omega), \mathcal{P}(\omega)$, is split by), also exhibits a morphism from $(\mathcal{P}(\omega), \mathcal{P}_\infty(\omega)$, does not split) to $({}^\omega\omega, {}^\omega\omega, \not\leq^*)$ and thus proves that $\mathfrak{b} \leq \mathfrak{r}$. (In this form, the inequality is essentially due to Solomon [11].)

One can similarly exhibit morphisms that capture the combinatorial content of the proofs of a great many other inequalities between cardinal characteristics. This applies both to trivial inequalities like $\mathfrak{b} \leq \mathfrak{d}$ and deep results like Bartoszyński's theorem that the additivity of category is at least the additivity of measure. See [5] for a presentation of Bartoszyński's theorem that makes the morphism explicit, and see [15] for more examples.

For future reference, we mention that the converse, $\mathfrak{d} \leq \mathfrak{s}$, of the inequality proved above is known not to be provable in ZFC (though of course it holds in some models of ZFC, for example models of CH). It fails, for example, in the model obtained from a model of CH by adding a set $C \subseteq {}^\omega\omega$ of \aleph_2 Cohen reals. In this model, $\mathfrak{d} = \aleph_2$ because any \aleph_1 reals lie in a submodel generated by \aleph_1 members of C and therefore cannot dominate the other members of C . On the other hand, $\mathfrak{s} = \aleph_1$ because the set of ground model reals is non-meager (i.e., of second Baire category) in the extension and any non-meager subset of $\mathcal{P}(\omega)$ is a splitting family (because the reals that fail to split any particular Y form a meager set).

We shall call an equation or (non-strict) inequality between cardinal characteristics “correct” if it is provable in ZFC and “incorrect” if it is independent of ZFC. (It cannot be refutable in ZFC, because it holds in models of CH; recall that we deal only with characteristics that lie between \aleph_1 and \mathfrak{c} .) In models of CH, the difference between correct and incorrect inequalities is hidden, because all the inequalities are true there. Nevertheless, it seems reasonable to say, even in such models, that correct inequalities, like $\mathfrak{s} \leq \mathfrak{d}$, hold for understandable, combinatorial reasons while incorrect inequalities, like $\mathfrak{d} \leq \mathfrak{s}$, hold only because CH “happens” to be true. Can one make mathematical sense of such statements?

Vojtáš's theory, in particular the fact that proofs of inequalities between characteristics usually exhibit morphisms between the corresponding relations, suggests an affirmative answer to this question. The understandable, combinatorial reason for a correct inequality is given by the morphism. So one might hope that there are, even in the presence of CH, no morphisms corresponding to incorrect inequalities. Then, when incorrect inequalities hold in a model, this would not be because of good reasons (i.e., morphisms) but because of other properties of the model (like CH).

A theorem of Yiparaki [16], Chapter 5, dashes this hope. She shows that, if

$$\|\mathbf{A}\| = |A_+| = \|\mathbf{B}^\perp\| = |B_-|,$$

then there is a morphism from \mathbf{A} to \mathbf{B} . In particular, in all our examples there is such a morphism if CH holds, because all the cardinals in the displayed equation

are then equal to \aleph_1 . So models of CH not only satisfy all inequalities, correct or incorrect, they also contain morphisms to justify all these inequalities.

These morphisms, however, are highly non-constructive; their definition involves a multitude of arbitrary choices. We propose therefore to eliminate them by working not with arbitrary morphisms but with well-behaved ones. To be specific, we restrict our attention to objects (A_-, A_+, A) in which A_\pm are Borel sets of reals and A is a Borel relation between them, and we consider only morphisms ξ both of whose components ξ_\pm are Borel functions. The objects and morphisms considered above, in connection with the definitions of \mathfrak{d} and \mathfrak{s} (and their duals) and the proof of $\mathfrak{s} \leq \mathfrak{d}$ (and its dual) are all Borel in this sense. Yiparaki's proof, on the other hand, involves non-Borel morphisms.

As an example of what the restriction to Borel morphisms accomplishes, we show that there is no Borel morphism corresponding to the incorrect inequality $\mathfrak{d} \leq \mathfrak{s}$.

PROPOSITION 1. *There is no Borel morphism*

$$(\xi_-, \xi_+) : (\mathcal{P}_\infty(\omega), \mathcal{P}(\omega), \text{is split by}) \rightarrow (\omega^\omega, \omega^\omega, \leq^*).$$

PROOF. Suppose we had such a morphism (ξ_-, ξ_+) consisting of two Borel maps. Let $c \in \omega^\omega$ be Cohen-generic over the universe V (in some Boolean extension), and by abuse of notation write (ξ_-, ξ_+) also for the pair of Borel maps in $V[c]$ having the same codes as the original (ξ_-, ξ_+) had in V . Since the ground model reals remain a non-meager and hence splitting set in the extension, $\xi_-(c)$ is split by some X in the ground model. Because ξ is a morphism, it follows that $c \leq^* \xi_+(X)$. But the ground model contains X and the code for ξ_+ and therefore also $\xi_+(X)$. This is absurd, as no real from the ground model can dominate a Cohen real. \square

More generally, we can show that there are no Borel morphisms between Borel relations when the corresponding inequality of characteristics can be violated by forcing. To express this precisely, suppose we have Borel \mathbf{A} and \mathbf{B} such that some notion of forcing P forces $\|\mathbf{A}\| < \|\mathbf{B}\|$, where we have, as above, abused notation by writing \mathbf{A} and \mathbf{B} for the objects in the forcing extension having the same Borel codes as the original \mathbf{A} and \mathbf{B} in the ground model. Then there is no Borel morphism (in the ground model) $\xi : \mathbf{A} \rightarrow \mathbf{B}$. Indeed, for ξ to be such a morphism would be a Π_1^1 assertion about the Borel codes of ξ , \mathbf{A} , and \mathbf{B} . Such assertions are preserved by forcing extensions. But P forces that there is no such morphism, because it forces the opposite ordering of the norms. (Note that we proved Proposition 1 by adding a single Cohen real, whereas the general argument just given would involve adding at least \aleph_2 Cohen reals.)

We emphasize that, in the situation of the preceding paragraph, the inequality $\|\mathbf{A}\| \geq \|\mathbf{B}\|$ may well be true in the ground model, but there cannot be a Borel morphism causing it.

It should be noted that independence proofs for inequalities between cardinal characteristics (e.g., [2, 14]) typically take the form considered above, i.e., they produce a P forcing a strict inequality in the opposite direction. In many cases, the proofs in the literature involve not only a forcing construction but some requirements on the ground model over which the forcing is done. But these requirements, usually CH or Martin's axiom, can themselves be forced, so there is a P as in the discussion above.

Summarizing, we have that incorrect inequalities between cardinal characteristics, though they may be true in some models and may be given by morphisms in some models, are not given by Borel morphisms in any model, provided their incorrectness can be established by a forcing argument.

3. Baire Category

In Section 4, we shall use some cardinal characteristics related to Baire category as an example to motivate and illustrate the operation of sequential composition of relations. In preparation for this, we devote the present section to introducing the notation and preliminary results needed to make the later discussion proceed smoothly. Along the way, we shall see a few more examples of the structures discussed in Section 2.

In what follows, we shall use the Cantor space ${}^\omega 2$, with its usual (product) topology, as the underlying space in all our discussions of Baire category. One of the characteristics that we shall need is the additivity of Baire category, $\mathbf{add}(B)$, the smallest number of meager (= first category) sets whose union is not meager. Like the characteristics discussed in the preceding section, $\mathbf{add}(B)$ lies between \aleph_1 and \mathfrak{c} inclusive. It can be described as the norm of $(B, B, \not\subseteq)$, where B is the collection of meager subsets of ${}^\omega 2$. This description is not amenable, as it stands, to the Borel considerations of the preceding section, because B is not a set of reals but a set of sets of reals. One can, however, replace B with the collection of meager F_σ sets, i.e., countable unions of nowhere-dense closed sets; this does not affect $\mathbf{add}(B)$, since every meager set is included in a meager F_σ set. It is easy to code meager F_σ sets by reals in such a way that the set of codes is Borel, and with some additional work one can arrange that the relations we need, like $\not\subseteq$, are also Borel in the codes. We omit the details of this since we shall soon introduce a different way of viewing $\mathbf{add}(B)$ for which these matters are easier to handle.

The other Baire category characteristic that we shall need is the covering number, $\mathbf{cov}(B)$, the smallest number of meager sets needed to cover ${}^\omega 2$. This is the norm of $({}^\omega 2, B, \subseteq)$, and again we can replace B by the collection of meager F_σ sets or by the collection of their codes. It is obvious that $\mathbf{add}(B) \leq \mathbf{cov}(B)$. As expected, this trivial inequality corresponds to a trivial morphism, with $\xi_- : B \rightarrow {}^\omega 2$ sending any meager set to some real not in it and with $\xi_+ : B \rightarrow B$ being the identity map. (The map ξ_- involves an arbitrary choice, but from a

code for a meager F_σ set one can obtain, in a Borel fashion, a specific real not in that set, just by following the proof of the Baire category theorem.)

It will be convenient to use a particular, easily coded basis for the ideal of meager sets. To introduce this basis, we first define a *chopped real* to be a pair (x, Π) , where $x \in {}^\omega 2$ and where Π is a partition of ω into finite intervals $I_0 = [a_0, a_1)$, $I_1 = [a_1, a_2)$, \dots , with $0 = a_0 < a_1 < a_2 < \dots$. The idea is that a real $x : \omega \rightarrow 2$ has been chopped into finite pieces by the partition Π of its domain. We say that a real $y \in {}^\omega 2$ *matches* the chopped real (x, Π) if y agrees with x on infinitely many of the intervals of Π , i.e., if there are infinitely many $n \in \omega$ such that $y \upharpoonright I_n = x \upharpoonright I_n$. We write $\text{Match}(x, \Pi)$ for the set of all y that match (x, Π) . It is easy to verify that $\text{Match}(x, \Pi)$ is a dense G_δ set. Talagrand [12], in obtaining a combinatorial description of meager filters, proved that every dense G_δ set in ${}^\omega 2$ has a subset of the form $\text{Match}(x, \Pi)$; equivalently, every meager set is included in the complement of $\text{Match}(x, \Pi)$ for some chopped real.

Thus, we can use chopped reals and the corresponding matching sets instead of arbitrary meager sets or meager F_σ sets in describing the cardinal characteristics of Baire category. Specifically, let us define CR to be the set of chopped reals and define

$$\mathbf{U} = (CR, {}^\omega 2, \text{matches}^\smile),$$

where \smile means to take the converse of a relation. Then the norm of the dual of \mathbf{U} is the smallest size for a family of chopped reals such that no single real matches them all. Equivalently, by Talagrand's result, it is the smallest number of dense G_δ sets with empty intersection. Taking the complements of those sets, we find that this is precisely the covering number. So we have $\text{cov}(B) = \|\mathbf{U}^\perp\|$. (The norm of \mathbf{U} itself is the smallest cardinality of a non-meager set of reals, called the *uniformity* of category; it is important in its own right, but we shall not need it here. The reason for defining \mathbf{U} as we did, rather than dually, is to avoid excessive negations and to avoid some dualizations in the next section.)

For a similar description of the additivity of category, we need to describe, in terms of chopped reals, the inclusion relation between the corresponding dense G_δ sets. We say that one chopped real (x, Π) *engulfs* another (x', Π') if all but finitely many intervals of Π include intervals of Π' on which x and x' agree.

LEMMA. $\text{Match}(x, \Pi) \subseteq \text{Match}(x', \Pi)'$ if and only if (x, Π) engulfs (x', Π') .

PROOF. First, suppose (x, Π) engulfs (x', Π') and y matches (x, Π) . So there are infinitely many intervals I of Π on which y agrees with x . Except for finitely many, each such I includes an interval J of Π' on which x and x' agree. Thus we get infinitely many intervals J of Π' on which y and x' agree, so y matches (x', Π') .

Conversely, suppose (x, Π) does not engulf (x', Π') , so there are infinitely many intervals I of Π that contain no interval of Π' on which x and x' agree.

Discarding some of these intervals I , we can arrange that no interval of Π' meets more than one of them. Define $y \in {}^\omega 2$ by making it agree with x on the union of these intervals I and making it disagree with x' everywhere else. Then y matches (x, Π) but not (x', Π') . \square

Define

$$\mathbf{V} = (CR, CR, \text{is engulfed by}).$$

Then the norm of the dual of V is the minimum number of chopped reals such that no single real engulfs them all. Equivalently, by Talagrand's result and the lemma, it is the minimum number of dense G_δ sets such that no single dense G_δ set is included in them all. Taking complements, we find that this is just the additivity of category. So $\mathbf{add}(B) = \|\mathbf{V}^\perp\|$. (The norm of \mathbf{V} itself is the *cofinality* characteristic of Baire category.)

The trivial inequality $\mathbf{add}(B) \leq \mathbf{cov}(B)$ is given by a trivial morphism $\mathbf{U}^\perp \rightarrow \mathbf{V}^\perp$ or equivalently $\mathbf{V} \rightarrow \mathbf{U}$ in which one component is the identity map $CR \rightarrow CR$ while the other component $CR \rightarrow {}^\omega 2$ sends a chopped real to its first component (forgetting the partition and keeping only the real).

A non-trivial inequality [7], namely $\mathbf{add}(B) \leq \mathfrak{b}$, is also easy to see from this point of view. Notice that \mathfrak{b} is, as discussed in Section 2, the norm of \mathbf{W}^\perp , where

$$\mathbf{W} = ({}^\omega \omega, {}^\omega \omega, \leq^*).$$

So to prove the inequality in question, it suffices to exhibit a morphism $\mathbf{W}^\perp \rightarrow \mathbf{V}^\perp$ or equivalently $\xi : \mathbf{V} \rightarrow \mathbf{W}$. Define ξ_- to map any $f \in {}^\omega \omega$ to the chopped real $(0, \Pi)$ where 0 is the identically zero function $\omega \rightarrow 2$ and where Π is chosen so that, for any $n \in \omega$, $f(n)$ is no more than one interval past n (i.e., if $n \in I_k$ then $f(n) \in I_l$ for some $l \leq k + 1$). (To get a Borel map ξ_- , the intervals should be chosen in a canonical manner, for example by choosing each endpoint a_n as small as possible subject to the constraints that $f(n)$ be at most one interval past n and that all the intervals be nonempty.) Define ξ_+ to map any chopped real (x, Π) to the function in ${}^\omega \omega$ sending each $n \in \omega$ to the right endpoint of the interval after the one that contains n . It is straightforward to verify that, if $\xi_-(f)$ is engulfed by (x, Π) , then $f \leq^* \xi_+(x, \Pi)$, so ξ serves as the required morphism.

The inequalities above combine to give $\mathbf{add}(B) \leq \min\{\mathbf{cov}(B), \mathfrak{b}\}$. In fact, equality holds here [13, 7], but the converse inequality cannot be separated into two simpler inequalities each proved by exhibiting a morphism. On the contrary, this converse inequality, like any inequality of the form $x \geq \min\{y, z\}$ (or $x \leq \max\{y, z\}$) involves an interaction of all three cardinals. In the next section, we shall review the proof of $\mathbf{add}(B) \geq \min\{\mathbf{cov}(B), \mathfrak{b}\}$ and use it to motivate a construction that allows such proofs to be presented as morphisms between suitable relations.

4. Sequential Composition

To treat inequalities involving the maximum or minimum of two cardinal characteristics, it is natural to seek a construction which, given two relations \mathbf{A} and \mathbf{B} , produces another relation \mathbf{C} with $\|\mathbf{C}\| = \max\{\|\mathbf{A}\|, \|\mathbf{B}\|\}$ and $\|\mathbf{C}^\perp\| = \min\{\|\mathbf{A}^\perp\|, \|\mathbf{B}^\perp\|\}$. Such a construction is the product $\mathbf{A} \times \mathbf{B}$ given by letting C_- be the disjoint union $A_- \sqcup B_-$, letting C_+ be the product $A_+ \times B_+$, and defining $C(x, (a, b))$ to hold when either $x \in A_-$ and $A(x, a)$ or $x \in B_-$ and $B(x, b)$. This is the product in the category-theoretic sense with respect to our definition of morphisms. (With Vojtáš's convention it is, as he points out, the coproduct.) It would be pleasant if proofs of three-cardinal inequalities, like the $\mathbf{add}(B) \geq \min\{\mathbf{cov}(B), \mathfrak{b}\}$ mentioned at the end of the preceding section, could be presented as constructions of morphisms from a product, and Vojtáš asks ([15]) whether this can be done, after noting that the usual proof has a more complicated structure (described below).

An earlier (preprint) version of [15] contained a different sort of product, which we shall call the *old product* to distinguish it from the (categorical) product in the preceding paragraph. The old product \mathbf{C} of \mathbf{A} and \mathbf{B} has $C_- = A_- \times B_-$, $C_+ = A_+ \times B_+$, and $C((x, y), (a, b))$ if and only if both $A(x, a)$ and $B(y, b)$. The norm of \mathbf{C} and its dual are the maximum and minimum of the norms of the factors and their duals, respectively, as long as the norms are infinite. (More precisely, $\|\mathbf{C}^\perp\| = \min\{\|\mathbf{A}^\perp\|, \|\mathbf{B}^\perp\|\}$ and $\max\{\|\mathbf{A}\|, \|\mathbf{B}\|\} \leq \|\mathbf{C}\| \leq \|\mathbf{A}\| \cdot \|\mathbf{B}\|$. The last two inequalities can both be strict, but of course only when the norms involved are finite. For example, $(3, 3, \neq)$ has norm 2 but its old product with itself has norm 3.) In the preprint version of [15], Vojtáš asked whether proofs of certain three-cardinal inequalities could be presented as constructions of morphisms from an old product.

We shall show that the answer to his question is negative for both the product and the old product if we require morphisms to consist of Borel maps. (Without this requirement, Yiparaki's result applies and shows that an affirmative answer is consistent, being true in models of CH.) Thus the more complicated structure, described by Vojtáš in his max-min diagram, is essential. But we shall see that this more complicated structure can also be described in terms of a construction of a suitable (more complicated) \mathbf{C} from \mathbf{A} and \mathbf{B} .

To motivate this construction and set the stage for the proof of its necessity, we review the proof that $\mathbf{add}(B) \geq \min\{\mathbf{cov}(B), \mathfrak{b}\}$, using the notation and machinery of the preceding section.

Let (x_α, Π_α) for $\alpha < \kappa$ be $\kappa < \min\{\mathbf{cov}(B), \mathfrak{b}\}$ chopped reals. We must produce a single chopped real (y, Θ) engulfing them all. Since $\kappa < \mathbf{cov}(B)$, fix a real y matching all the (x_α, Π_α) . This will be the first component of the chopped real we seek; it remains to produce Θ such that, for each $\alpha < \kappa$, for all but finitely many blocks J of Θ , there is a block I of Π_α such that $I \subseteq J$ and

$x_\alpha \upharpoonright I = y \upharpoonright I$.

For each α , define $f_\alpha : \omega \rightarrow \omega$ by letting $f_\alpha(n)$ be the right endpoint of the next interval of Π_α , after the one containing n , on which y agrees with x_α . (Such an interval exists because y matches (x_α, Π_α) .) As $\kappa < \mathfrak{b}$, fix some $g : \omega \rightarrow \omega$ eventually majorizing every f_α . Then choose Θ so that, for each of its intervals, g of the left endpoint is smaller than the right endpoint. For each $\alpha < \kappa$, for all but finitely many intervals $J = [a, b]$ of Θ , we have $f_\alpha(a) \leq g(a) < b$ and therefore, by definition of f_α , y agrees with x_α on some interval of Π_α that starts after a and ends before b and thus is contained in J . This shows that (y, Θ) is as required.

Let us describe this proof in terms of the relations \mathbf{U} , \mathbf{V} , and \mathbf{W} from the preceding section. Recall that

$$\begin{aligned}\mathbf{U} &= (CR, {}^\omega 2, \text{matches}^\smile), \\ \mathbf{V} &= (CR, CR, \text{is engulfed by}), \\ \mathbf{W} &= ({}^\omega \omega, {}^\omega \omega, \leq^*),\end{aligned}$$

so $\|\mathbf{U}^\perp\| = \mathbf{cov}(B)$, $\|\mathbf{V}^\perp\| = \mathbf{add}(B)$, and $\|\mathbf{W}^\perp\| = \mathfrak{b}$. (Thus the inequality $\mathbf{add}(B) \geq \min\{\mathbf{cov}(B), \mathfrak{b}\}$ should correspond to a morphism from some sort of “product” of \mathbf{U} and \mathbf{W} to \mathbf{V} ; the problem is to define an appropriate sort of product.)

The proof of $\mathbf{add}(B) \geq \min\{\mathbf{cov}(B), \mathfrak{b}\}$ above began by regarding the given elements (x_α, Π_α) of V_- (for which we needed to find a V -related element of V_+) as elements of U_- and finding a $y \in U_+$ that is U -related to them all. Thus, the proof implicitly used the identity map to convert elements of V_- into elements of U_- . (In other applications, a non-trivial map will occur here.) The next step was to define, from y and (x_α, Π_α) , the element $f_\alpha \in W_-$; so the proof uses a map $V_- \times U_+ \rightarrow W_-$. (Our definition of f_α presupposed that y matches (x_α, Π_α) ; to get a total map, let f_α be identically zero if y fails to match (x_α, Π_α) .) Finally, from y and an element $g \in W_+$ that is W -related to each f_α , we produced the required $(y, \Theta) \in V_+$ that is V -related to each (x_α, Π_α) . The constructions in the proof can be summarized as three maps (where we have omitted the subscripts α from x , Π , and f)

$$\begin{aligned}\alpha : V_- &\rightarrow U_- : (x, \Pi) \mapsto (x, \Pi), \\ \beta : V_- \times U_+ &\rightarrow W_- : ((x, \Pi), y) \mapsto f, \text{ and} \\ \gamma : U_+ \times W_+ &\rightarrow V_+ : (y, g) \mapsto (y, \Theta).\end{aligned}$$

Their key property is that from $U(\alpha(x, \Pi), y)$ and $W(\beta((x, \Pi), y), g)$ we were able to infer $V((x, \Pi), \gamma(y, g))$.

Notice that if β were a function of only (x, Π) rather than both (x, Π) and y , then the situation above would precisely describe a morphism ξ from the

old product of \mathbf{U} and \mathbf{W} to \mathbf{V} . Indeed, α and β could then serve as the two components of $\xi_- : V_- \rightarrow U_- \times W_-$ and γ could serve as ξ_+ .

But β in the proof definitely needs y as an argument, so this proof is not described by a morphism from an old product. It is even less describable in terms of the (new, categorical) product; that would require each element of V_- to have an image in U_- or in W_- , not both.

The awkward U_+ in the domain of β can be moved to the codomain by considering what category theorists call the exponential adjoint and computer scientists call currying:

$$\hat{\beta} : V_- \rightarrow U_+ W_- : (x, \Pi) \mapsto (y \mapsto \beta((x, \Pi), y)).$$

The key property of α , β , and γ can, of course, be trivially rewritten in terms of α , $\hat{\beta}$, and γ . The result is that the first two of these are the components of ξ_- and the third is ξ_+ for a morphism ξ to \mathbf{V} from the following compound of \mathbf{U} and \mathbf{W} , which we call their *sequential composition* (cf. also [3]):

$$\mathbf{U}; \mathbf{W} = (U_- \times U_+ W_-, U_+ \times W_+, \{(a, \rho), (u, w) \mid U(a, u) \text{ and } W(\rho(u), w)\}).$$

Notice that, quite generally, a morphism from a sequential composition to another relation can be regarded (via exponential adjointness) as consisting of three functions that enjoy the key property of α , β , and γ discussed above. Such a triple of functions is precisely what Vojtáš [15] describes with his max-min diagram.

The following proposition records the connection between sequential composition and maxima and minima of cardinal characteristics. Note that this proposition has a product of cardinals where the product of relations has a maximum of cardinals and the old product has (as discussed above) something between the maximum and the product of cardinals. Of course, for infinite cardinals (the case we're interested in) all these cardinals coincide.

PROPOSITION 2. *For any relations \mathbf{A} and \mathbf{B} , we have*

$$\|\mathbf{A}; \mathbf{B}\| = \|\mathbf{A}\| \cdot \|\mathbf{B}\| \quad \text{and} \quad \|(\mathbf{A}; \mathbf{B})^\perp\| = \min\{\|\mathbf{A}^\perp\|, \|\mathbf{B}^\perp\|\}.$$

PROOF. We prove only the least trivial part, namely the \geq half of the first equation. By definition, $\|\mathbf{A}; \mathbf{B}\|$ is the smallest possible cardinality for a set $S \subseteq A_+ \times B_+$ such that, for every element $x \in A_-$ and every function $\rho : A_+ \rightarrow B_-$, some $(a, b) \in S$ satisfies $A(x, a)$ and $B(\rho(a), b)$. Fix such an S of minimum cardinality; we must show that this cardinality is at least the product of the norms of \mathbf{A} and \mathbf{B} .

For each $a \in A_+$, let $S_a = \{b \in B_+ \mid (a, b) \in S\}$. Let $X = \{a \in A_+ \mid |S_a| \geq \|\mathbf{B}\|\}$. It will suffice to prove that X has cardinality at least $\|\mathbf{A}\|$, for then we have at least $\|\mathbf{A}\|$ elements a each S -related to at least $\|\mathbf{B}\|$ elements b and so we have at least $\|\mathbf{A}\| \cdot \|\mathbf{B}\|$ pairs $(a, b) \in S$.

So suppose toward a contradiction that $|X| < \|\mathbf{A}\|$. By definition of norm, we can fix some $x \in A_-$ that is not A -related to any element of X . Using the definition of norm again along with the definition of X , we can choose, for each $a \in A_+ - X$, an element $\rho(a) \in B_-$ that is not B -related to any $b \in S_a$. Extend ρ arbitrarily to a function $\rho : A_+ \rightarrow B_-$. By our choice of S , there is some $(a, b) \in S$ with $A(x, a)$ and $B(\rho(a), b)$. From $A(x, a)$ we infer, by our choice of x , that $a \notin X$. But then from $B(\rho(a), b)$ and our choice of ρ it follows that $b \notin S_a$. That contradicts $(a, b) \in S$. \square

By virtue of this proposition and the general properties of morphisms, we see that the existence of a morphism $\mathbf{A}; \mathbf{B} \rightarrow \mathbf{C}$ implies $\|\mathbf{C}\| \leq \|\mathbf{A}\| \cdot \|\mathbf{B}\|$ ($= \max\{\|\mathbf{A}\|, \|\mathbf{B}\|\}$ if the norms are infinite) and $\|\mathbf{C}^\perp\| \geq \min\{\|\mathbf{A}^\perp\|, \|\mathbf{B}^\perp\|\}$. In other words, morphisms from sequential compositions provide a way to present proofs of three-cardinal inequalities relating (in the non-trivial direction) one cardinal to the maximum or minimum of two others. The example of $\mathbf{add}(B) \geq \min\{\mathbf{cov}(B), \mathfrak{b}\}$ shows that this sort of presentation occurs naturally in one example; other examples are given in Vojtáš's discussion of the max-min diagram [15], and another example (involving a sequential composition one of whose factors is an old product) is discussed in [3].

One unpleasant aspect of sequential composition needs to be discussed. Because of the function-space construction in the definition of sequential composition, even if \mathbf{A} and \mathbf{B} are, as we advocated in Section 2, Borel relations between Borel sets of reals, $\mathbf{A}; \mathbf{B}$ will not be of this form, simply because its domain is not a set of reals at all but rather a set one type higher. So it does not yet make sense to talk about Borel morphisms involving sequential compositions.

Fortunately, when we deal (as we did above) with morphisms ξ from a sequential composition $\mathbf{A}; \mathbf{B}$ to some \mathbf{C} , then the troublesome (higher type) part is $\xi_- : C_- \rightarrow A_- \times {}^{A_+}B_-$ or more precisely its second component $C_- \rightarrow {}^{A_+}B_-$. We can declare such a function to be Borel if and only if its exponential adjoint $C_- \times A_+ \rightarrow B_-$ is Borel. In this sense, the morphism involved in our proof of $\mathbf{add}(B) \geq \min\{\mathbf{cov}(B), \mathfrak{b}\}$ is Borel, and so are the morphisms involved in the other proofs to which we alluded two paragraphs ago.

The exponential adjoint trick does not work for morphisms to a sequential composition, for then the function space ${}^{A_+}B_-$ occurs in the domain rather than the codomain of a function. The only sensible way to talk about such functions being Borel seems to be to work not with the space of all functions $A_+ \rightarrow B_-$ but with the subspace of Borel functions, or rather with the set of Borel codes for such functions. We postpone further discussion of this until after we see, in the next section, an occurrence “in nature” of something that ought to be a Borel morphism to a sequential composition.

5. Non-existence of Borel morphisms

The purpose of this section is to show that, in the proof of the inequality

$\mathbf{add}(B) \geq \min\{\mathbf{cov}(B), \mathfrak{b}\}$, one cannot replace the sequential composition $\mathbf{U}; \mathbf{W}$ with the product or the old product. In fact, we shall see that \mathbf{V} is in some sense equivalent to $\mathbf{U}; \mathbf{W}$. We begin with a proposition showing that products (old or new) do not suffice. It also shows that the order of the components in a sequential composition is essential.

PROPOSITION 3. *Let \mathbf{U} , \mathbf{V} , and \mathbf{W} be as in the preceding two sections. There is no Borel morphism to \mathbf{V} from the product of \mathbf{U} and \mathbf{W} , nor from their old product, nor from $\mathbf{W}; \mathbf{U}$.*

PROOF. Notice first that there are morphisms from $\mathbf{W}; \mathbf{U}$ to the old product and from the old product to the product. Both morphisms are the identity on the $+$ -components $W_+ \times U_+$. On the $-$ -components, we have $W_- \sqcup U_- \rightarrow W_- \times U_- \rightarrow W_- \times {}^{W_+}U_-$, where the first map is defined by fixing elements $w_0 \in W_-$ and $u_0 \in U_-$ and sending any $w \in W_-$ to (w, u_0) and any $u \in U_-$ to (w_0, u) , and the second map is the identity on the first component W_- while on the second component it sends any $u \in U_-$ to the constant function in ${}^{W_+}U_-$ with value u . It is trivial to check that these functions define Borel morphisms. So it suffices to show that there is no Borel morphism $\xi : \mathbf{W}; \mathbf{U} \rightarrow \mathbf{V}$.

Suppose there were such a ξ . Form a forcing extension of the universe M by first adding a Mathias real m and then, over the resulting model, adding a Cohen real c . Extend ξ to a Borel morphism in $M[m, c]$, still called ξ , with the same Borel code. More precisely, the two functions that constitute ξ correspond (via exponential adjointness in one component) to three Borel maps

$$\alpha : V_- \rightarrow W_-, \quad \beta : V_- \times W_+ \rightarrow U_-, \quad \gamma : W_+ \times U_+ \rightarrow V_+$$

with the key property that, for all $x \in V_-$, $w \in W_+$, and $u \in U_+$, if $W(\alpha(x), w)$ and $U(\beta(x, w), u)$, then $V(x, \gamma(w, u))$. We use the same symbols $U_\pm, V_\pm, W_\pm, W, \alpha, \beta$, and γ for the objects in $M[m, c]$ having the same Borel codes (in M). The key property is a Π_1^1 sentence, so it remains true in the forcing extension.

We apply the key property with an arbitrary $x \in V_- \cap M$, with $w = m$ and with $u = c$. Then, since α is coded in M and since a Mathias real dominates all ground model reals, $\alpha(x) \leq^* m$, i.e., $W(\alpha(x), m)$. Similarly, since β is coded in the ground model M , the chopped real $\beta(x, m)$ is in $M[m]$ and is therefore matched by c (since matching $\beta(x, m)$ is a comeager requirement coded in $M[m]$ and c is a Cohen real over $M[m]$). That is, $U(\beta(x, m), c)$.

By the key property, it follows that $V(x, \gamma(m, c))$. In other words, we have a chopped real $\gamma(m, c) \in M[l, c]$ that engulfs every chopped real x from the ground model M . Recalling the connection between chopped reals (related by engulfing) and meager sets (related by inclusion) from Section 3, we find that all the meager Borel sets coded in M are subsets of a single meager set in $M[m, c]$. But Pawlikowski has shown that $M[m, c]$ has no meager set that includes all

the meager Borel sets coded in M ; see [9], Proposition 1.2 and the discussion following Corollary 1.3. This contradiction shows that no such morphism ξ can exist. \square

The preceding proposition tells us that, of the various combinations of \mathbf{U} and \mathbf{W} considered so far, $\mathbf{U}; \mathbf{W}$ is the only one admitting a Borel morphism to \mathbf{V} and therefore the only one that can be used to prove $\mathbf{add}(B) \geq \min\{\mathbf{cov}(B), \mathfrak{b}\}$. In fact, Corollary 1.3 of [9] strongly suggests that $\mathbf{U}; \mathbf{W}$ is in some sense equivalent to \mathbf{V} . We state this corollary as the next proposition and give a proof somewhat different from that in [9] (in that we use more forcing but do not use, e.g., the Kuratowski-Ulam theorem) in order to suggest what the proper sense of equivalence between $\mathbf{U}; \mathbf{W}$ and \mathbf{V} should be.

PROPOSITION 4 (PAWLIKOWSKI). *Let M be any inner model. The union of all the meager Borel sets coded in M is meager if and only if there is a Cohen real c over M and there is a real d dominating all the reals of $M[c]$.*

PROOF. The “if” half is immediate from the existence of a Borel morphism $\xi : \mathbf{U}; \mathbf{W} \rightarrow \mathbf{V}$ with code in M . Indeed, if c and d are as in the statement of the proposition, if x is any chopped real in M , and if $\alpha : V_- \rightarrow U_-$, $\beta : V_- \times U_+ \rightarrow W_-$, and $\gamma : U_+ \times W_+ \rightarrow V_+$ are the parts of ξ (as above), then we have $U(\alpha(x), c)$ (because c is a Cohen real over M which contains $\alpha(x)$), and $W(\beta(x, c), d)$ (because $\beta(x, c)$ is in $M[c]$ which d dominates), and therefore $V(x, \gamma(c, d))$ (because ξ is a morphism). Thus, $\gamma(c, d)$ engulfs all chopped reals $x \in M$, and therefore it codes a meager set that includes all meager Borel sets coded in M .

For the “only if” direction, suppose we have a meager set that includes all the meager Borel sets coded in M . Then, by Section 3, we have a chopped real (y, Θ) that engulfs all the chopped reals of M . Then clearly y matches every chopped real from M and is therefore Cohen generic over M . It remains to produce a real g dominating all the reals of $M[y]$. We claim that such a g is obtained by letting $g(n)$ be the right endpoint of the next interval in Θ after the one that contains n .

To verify that this g works, we consider any $f : \omega \rightarrow \omega$ in $M[y]$ and show that $f \leq^* g$. We may suppose without loss of generality that f is non-decreasing. Since f belongs to the Cohen extension $M[y]$, it is the denotation with respect to y of some name $\dot{f} \in M$. (Here and in the rest of this proof, names and forcing are with respect to the usual notion of forcing for adding a Cohen real, ${}^{<\omega}2$ ordered by reverse inclusion.) We assume without loss of generality that all conditions force “ \dot{f} is a function from ω to ω .”

We define a chopped real $(x, \Pi) \in M$ with the property that, if $[a, b]$ is any interval in Π and if p is any Cohen condition that (has length at least b and) agrees with x on $[a, b]$, then p forces “ $\dot{f}(a) \leq b$.” We proceed by induction. After n intervals of Π and the restrictions of x to those intervals have been defined, we

produce the next interval $[a, b]$ and the restriction of x to it as follows. Of course a is the first number not in the intervals already defined. Fix a list u_0, u_1, \dots, u_{r-1} of all the functions $a \rightarrow 2$. We inductively define functions $x_i : [a, l_i] \rightarrow 2$ with $a = l_0 \leq l_1 \leq \dots \leq l_r$ by starting with the empty function as x_0 and obtaining x_{i+1} as an extension of x_i such that $u_i \cup x_{i+1}$ forces a particular value v_i for $\dot{f}(a)$. Such an extension exists because \dot{f} is forced to be a total function on ω . After x_r and l_r have been reached, let b be the largest of l_r and all the v_i , and extend x_r arbitrarily to $[a, b]$; this defines the restriction of the desired x to the next interval $[a, b]$ of Π . If a condition p agrees with x on $[a, b]$ then it agrees with some u_i on a and therefore extends $u_i \cup x_{i+1}$ and forces “ $\dot{f}(a) = v_i \leq b$ ” as required.

As (x, Π) is in M , it is engulfed by (y, Θ) . That is, each interval $[m, n]$ of Θ , with only finitely many exceptions, includes an interval $[a, b]$ of Π on which x and y agree. Then the initial segment $y \upharpoonright b$ of y is a condition of the sort considered in defining x , so it forces “ $\dot{f}(a) \leq b$.” Since y is Cohen generic, this forced statement is true in $M[y]$, i.e., $f(a) \leq b$. But then, as f is non-decreasing, we have, for all elements k of the interval of Θ immediately preceding $[m, n]$, that

$$f(k) \leq f(a) \leq b \leq n = g(k).$$

This applies to all sufficiently large $k \in \omega$ because $[m, n]$ can be any interval of Θ with only finitely many exceptions. Therefore, $f \leq^* g$, as required. \square

We would like to regard the “only if” half of the preceding proof as presenting a morphism $\eta : \mathbf{V} \rightarrow \mathbf{U}; \mathbf{W}$ in the direction opposite to the ξ involved in the “if” half. This would mean that the proof involves (1) a construction η_- whose input consists of a chopped real (z, Ψ) (from U_-) and a function $\Phi : U_+ \rightarrow W_-$ and whose output is a chopped real (x, Π) and (2) a construction η_+ whose input consists of a chopped real (y, Θ) and whose output consists of a real c and an element g of W_+ . The required key property for a morphism is that (with the notation of (1) and (2)), if (y, Θ) engulfs (x, Π) , then c matches (z, Ψ) and g dominates $\Phi(c)$.

Our proof looks vaguely but not exactly like this. Notice that a Cohen-forcing name \dot{f} of a real, as in our proof of the proposition, defines a function Φ into ${}^\omega\omega$ from a comeager subset D of ${}^\omega 2$, namely, $\Phi(u)(n)$ is the unique number forced to be the value of $\dot{f}(n)$ by some initial segment of u . (D consists of those u whose initial segments force values for all $\dot{f}(n)$; this D is comeager because \dot{f} is forced to name a real.) For a Φ of this special sort (or, more precisely, for any extension of such a Φ to all of ${}^\omega 2 = U_+$), our proof produced, from \dot{f} , a certain (x, Π) . For the output of η_- , when the input consists of (z, Ψ) and such a special Φ , we take a chopped real that engulfs (z, Ψ) , the (x, Π) constructed from \dot{f} in the proof, and an (x', Π') such that $\text{Match}(x', \Pi')$ is included in the proper domain D of Φ . (It is trivial to produce a chopped real engulfing any finitely (or even countably) many given chopped reals.) For the output of η_+ on input (y, Θ) , we

take the pair whose first component is y and whose second component is the g constructed from Θ during the proof (the “end of the next interval” function). Then the key property that η needs is just what is established by the proof.

All this, however, was done only for the very special Φ 's that correspond to names \dot{f} . The proof gives no hint what η_- should do if its second argument Φ is not of this form. This difficulty is, of course, connected with the difficulty mentioned earlier that sets like U_+W_- are one type higher than the (Borel) sets of reals that our theory is equipped to handle.

Both difficulties can be attacked by working not with all functions $U_+ \rightarrow W_-$ but with some subfamily that can be coded by reals (or with the codes rather than the functions). In the case at hand, it is tempting to take this family to be just those Φ 's that are given by Cohen names \dot{f} ; these names, which are essentially reals, can then serve as the codes. But this choice of a subfamily is obviously tailored to just this one example. We can do much better by noticing that any Borel function $\Phi : U_+ \rightarrow W_-$ agrees on some comeager set A with the function given by some \dot{f} . Thus, we can allow Borel functions as inputs to η_- at the cost of redefining the output of η_- so that it also engulfs a chopped real whose matching set is included in A .

The restriction to Borel functions is one that can be sensibly imposed in general. That is, we can modify the general definition of sequential composition $\mathbf{A}; \mathbf{B}$, when all the components of \mathbf{A} and \mathbf{B} are Borel sets and relations, by replacing the set A_+B_- of functions with the subset of Borel functions or, better, with the set of Borel codes for such functions.

To be specific about the coding, we note that Borel functions are precisely the functions recursive in a real and the type 2 object 2E (see for example [6], Theorem VI.1.8). So as a code for such a function we can use the pair of the Kleene index (a natural number) for this recursive function and the real parameter.

This framework covers the standard examples, including the particular ones we have discussed. It should, however, probably be extended a bit because the set of codes of Borel maps from a Borel set to a Borel set is in general not itself a Borel set; it is only $\mathbf{\Pi}_1^1$. So it seems reasonable to allow the domains and codomains of relations to be $\mathbf{\Pi}_1^1$ (i.e., semi-recursive in 2E and a real) sets of reals; the relations should probably still be required to be Borel in the weak sense that they and their complements relative to the domain and codomain are $\mathbf{\Pi}_1^1$ (so that dualization works). Morphisms should be functions with $\mathbf{\Pi}_1^1$ graphs, i.e., partial recursive in 2E and reals but total on the $\mathbf{\Pi}_1^1$ sets in question.

Rather than speculate further on the basis of very limited examples, let me just list what I would like to see in an ideal framework. The sets, relations, and morphisms should come from a class that is broad enough to encompass the known examples but narrow enough to have absoluteness properties so that non-existence of incorrect morphisms can be proved. Furthermore, the framework

should be closed under the naturally occurring constructions, including a suitable version of sequential composition. I believe that the framework described in the preceding paragraph may do all this, but this has not yet been fully checked.

I should perhaps mention that Pawlikowski told me that, with suitable coding, the components of morphisms that occur in classical proofs of cardinal characteristic inequalities can be taken to be not merely Borel but continuous. It is not clear, however, that this helps in the present situation, since the (natural) codes for continuous functions form only a Π_1^1 set.

6. Unsplitting

In this section, we briefly discuss a situation where sequential composition arises naturally in connection with an inequality involving just two cardinal characteristics, and a rather trivial inequality at that. Recall that the unsplitting number \mathfrak{r} is the smallest number of infinite subsets of ω such that no single set splits them all into two infinite pieces. Equivalently, it is the norm of $\mathbf{R} = (\omega 2, \mathcal{P}_\infty(\omega), R)$ where $R(f, Y)$ means that f is almost constant on Y , i.e., is constant on $Y - F$ for some finite F .

We define a similar characteristic \mathfrak{r}_3 using splittings into three pieces rather than two. That is, \mathfrak{r}_3 is the norm of $\mathbf{R}_3 = (\omega 3, \mathcal{P}_\infty(\omega), R_3)$ where again $R_3(f, Y)$ means that f is almost constant on Y .

It is easy to see that $\mathfrak{r}_3 = \mathfrak{r}$. In fact, there are morphisms $\xi : \mathbf{R}_3 \rightarrow \mathbf{R}$ and $\eta : \mathbf{R}; \mathbf{R} \rightarrow \mathbf{R}_3$ defined as follows. ξ_- is the inclusion map $\omega 2 \rightarrow \omega 3$ and ξ_+ is the identity map of $\mathcal{P}_\infty(\omega)$. To define η , we first introduce, for $f : \omega \rightarrow 3$, the notations f' and f'' for the functions $\omega \rightarrow 2$ obtained by identifying 2, whenever it occurs as a value of f , with 0 or 1 respectively. That is,

$$f' : \omega \rightarrow 2 : n \mapsto \begin{cases} 0, & \text{if } f(n) = 0 \\ 1, & \text{otherwise,} \end{cases}$$

and f'' is defined similarly except that the first case is “if $f(n) = 0$ or 2.” Also fix, for each infinite $Y \subseteq \omega$ a bijection $e_Y : \omega \rightarrow Y$, for example the unique increasing bijection. Now we define η as follows. η_- sends $f : \omega \rightarrow 3$ to the pair consisting of f' and the map $G : \mathcal{P}_\infty(\omega) \rightarrow \omega 2$ defined by $G(Y) = f'' \circ e_Y$. (In other words, $G(Y)$ is essentially $f'' \upharpoonright Y$ but with its domain shifted from Y to ω by e_Y .) η_+ sends a pair of infinite sets (Y, Y') to $e_Y(Y')$, the set that “occupies in Y the locations that Y' occupies in ω .” It is easy to check that ξ and η are morphisms, and therefore $\mathfrak{r} \leq \mathfrak{r}_3 \leq \mathfrak{r} \cdot \mathfrak{r} = \mathfrak{r}$.

The idea behind the morphism η is that to get a 3-unsplit family (in the obvious sense), it suffices to start with a 2-unsplit family \mathcal{R} and then, within each of its members Y , form a new 2-unsplit family by transferring \mathcal{R} from ω to Y via e_Y . The union of these new, transferred families is 3-unsplit because, if $f : \omega \rightarrow 3$, then there is some $Y \in \mathcal{R}$ on which f takes at most two values

infinitely often and there is some $Y' \in e_Y[\mathcal{R}]$ on which those two values are reduced to one.

Notice that the “second order” families $e_Y[\mathcal{R}]$ depend in an essential way on the Y 's. In other words, the domain of η in this proof apparently needs to be a sequential composition. There is no evident way to use a product or even an old product instead.

We therefore conjecture that there is no Borel morphism from the old product of two copies of \mathbf{R} to \mathbf{R}_3 . Intuitively, this means that, in the two uses of a 2-unsplit family \mathcal{R} to produce a 3-unsplit family, the second use cannot be made independent of the first.

Although the conjecture seems highly plausible, we do not know how to prove even the weaker conjecture that there is no Borel morphism from \mathbf{R} to \mathbf{R}_3 . The intuitive meaning of this is merely that, to produce a 3-unsplit family, we must use the 2-unsplit family \mathcal{R} twice, not just once.

To see the difficulty in proving these conjectures, recall that our previous proofs of non-existence of Borel morphisms involved the construction of suitable forcing models. To apply this method directly to prove the conjecture, we would want to find a forcing extension that contains an infinite subset Y of ω unsplit by all ground model functions $\omega \rightarrow 2$ but split by some ground model function $f : \omega \rightarrow 3$. This clearly cannot be achieved, since a set unsplit by f' and f'' (in the notation of the definition of η above) is also unsplit by f .

An alternative approach would be concentrate on the dual morphism $\mathbf{R}_3^\perp \rightarrow \mathbf{R}^\perp$. Now the forcing method suggests building a forcing extension in which every function $\omega \rightarrow 2$ is almost constant on an infinite Y from the ground model but some $f : \omega \rightarrow 3$ is not. But this cannot be achieved either. If $f : \omega \rightarrow 3$, then there is an infinite Y in the ground model on which f' is almost constant. Furthermore, $f'' \circ e_Y$ is almost constant on some Y' in the ground model. If e_Y was defined reasonably (e.g., as the increasing bijection), then it too is in the ground model, and so is $e_Y(Y')$ on which f is almost constant.

So the forcing method seems to be of no use in proving even the weak form of the conjecture. Some direct analysis of Borel morphisms seems to be needed.

REFERENCES

1. T. Bartoszyński, *Additivity of measure implies additivity of category*, Trans. Amer. Math. Soc. **281** (1984), 209–213.
2. T. Bartoszyński, H. Judah, and S. Shelah, *The Cichoń diagram*, J. Symbolic Logic **58** (1993), 401–423.
3. A. Blass, *Questions and answers — a category arising in linear logic, complexity theory, and set theory*, Proceedings of a linear logic workshop held at MSI, Cornell, in June, 1993 (J.Y. Girard, Y. Lafont, and L. Regnier, eds.).
4. E. van Douwen, *The integers and topology*, Handbook of Set Theoretic Topology (K. Kunen and J. Vaughan, eds.), North-Holland, 1984, pp. 111–168.
5. D. Fremlin, *Cichoń's diagram*, Séminaire Initiation à l'Analyse (G. Choquet, M. Rogalski, and J. Saint-Raymond, eds.), Univ. Pierre et Marie Curie, 1983/84, pp. (5-01)–(5-13).

6. P. Hinman, *Recursion Theoretic Hierarchies*, Perspectives in Mathematical Logic, Springer-Verlag, 1977.
7. A. Miller, *Some properties of measure and category*, Trans. Amer. Math. Soc. **266** (1981), 93-114.
8. V.C.V. de Paiva, *A Dialectica-like model of linear logic*, Category Theory and Computer Science (D.H. Pitt, D.E. Rydeheard, P. Dybjer, A. Pitts, and A. Poigné, eds.), Lecture Notes in Computer Science 389, Springer-Verlag, 1989, pp. 341-356.
9. J. Pawlikowski, *Why Solovay real produces Cohen real*, J. Symbolic Logic **51** (1986), 957-968.
10. F. Rothberger, *Eine Äquivalenz zwischen der Kontinuumhypothese und der Existenz der Lusinischen und Sierpińskischen Mengen*, Fund. Math. **30** (1938), 215-217.
11. R.C. Solomon, *Families of sets and functions*, Czechoslovak Math. J. **27** (1977), 556-559.
12. M. Talagrand, *Compacts de fonctions mesurables et filtres non mesurables*, Studia Math. **67** (1980), 13-43.
13. J. Truss, *Sets having calibre \aleph_1* , Logic Colloquium 76 (R.O. Gandy and J.M.E. Hyland, eds.), Studies in Logic and Foundations of Math. 87, North-Holland, 1977, pp. 595-612.
14. J. Vaughan, *Small uncountable cardinals and topology*, Open Problems in Topology, ed. by J. van Mill and G.M. Reed, North-Holland, 1990, pp. 195-218.
15. P. Vojtáš, *Generalized Galois-Tukey connections between explicit relations on classical objects of real analysis*, Set Theory of the Reals (H. Judah, ed.), Israel Mathematical Conference Proceedings 6, American Mathematical Society, 1993, pp. 619-643.
16. O. Yiparaki, *On Some Tree Partitions*, Ph.D. thesis, University of Michigan, 1994.

MATHEMATICS DEPT., UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109, U.S.A.

E-mail address: ablass@umich.edu