CARDINAL CHARACTERISTICS AND THE PRODUCT OF COUNTABLY MANY INFINITE CYCLIC GROUPS

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Abstract. We study, from a set-theoretic point of view, those subgroups of the infinite direct product $\mathbb{Z}^{\aleph_0}$ for which all homomorphisms to $\mathbb{Z}$ annihilate all but finitely many of the standard unit vectors. Specifically, we relate the smallest possible size of such a subgroup to several of the standard cardinal characteristics of the continuum. We also study some related properties and cardinals, both group-theoretic and set-theoretic. One of the set-theoretic properties and the associated cardinal are combinatorially natural, independently of any connection with algebra.

Introduction

Let $\Pi = \mathbb{Z}^{\aleph_0}$ be the direct product of a countable infinity of copies of the infinite cyclic group $\mathbb{Z}$. Specker [23] proved that Π as well as many of its subgroups G have the following property, in which e_n is the element of Π whose nth component is 1 and whose other components are all zero (the nth standard unit vector). If h is a homomorphism from Π (or G) to $\mathbb{Z}$, then $h(e_n) = 0$ for all but finitely many n. The subgroups G for which Specker established this property all have, like Π itself, the cardinality of the continuum, $\mathfrak{c} = 2^{\aleph_0}$, and the question naturally arises whether any smaller subgroups G of Π also have this property. Eda [9] showed that this question is undecidable on the basis of the usual (Zermelo-Fraenkel) axioms of set theory (ZFC). Specifically, he proved that the answer is negative in models of Martin's Axiom but positive in models obtained by adjoining many random reals. In fact, his proofs give somewhat more precise information about the minimum cardinality κ of a subgroup of Π satisfying Specker's theorem. In terms of the cardinal characteristics of the continuum introduced in [8] and described in Section 1 below, Eda's proofs establish that $\mathfrak{p} \le \kappa \le \mathfrak{d}$.

One purpose of this paper is to improve these estimates; we shall show that the additivity of Lebesgue measure $\mathbf{add}(L)$ is another lower bound for κ and that the bounding number $\mathfrak{b}$ is an upper bound. (Definitions of these cardinals are recalled in Section 1.) The new upper bound subsumes Eda's, since $\mathfrak{b} \le \mathfrak{d}$; the new lower

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bound is incomparable with Eda's, since neither of $\mathfrak{p}$ and $\mathbf{add}(L)$ is provably larger than the other.

These proofs suggest some additional group-theoretic questions concerning homomorphisms into free groups. We say that an abelian group G "binds" a subgroup H if every homomorphism of G into a free group maps H to a group of finite rank. We consider questions about the smallest cardinalities in which various sorts of binding (a group binding an infinite rank subgroup, a group binding itself, etc.) can occur non-trivially. A rather surprising result is that one of these cardinalities is $\geq \mathbf{add}(L)$ but $\leq$ the additivity of Baire category, $\mathbf{add}(B)$. These questions are connected to the Specker phenomenon described above and also to questions considered by Eklof and Shelah [11] about groups that satisfy $G \cong G \oplus F$ with F free of finite rank.

Finally, we consider some purely set-theoretic questions arising out of these problems. These questions seem quite natural from a combinatorial point of view, but do not seem to have been previously considered.

This paper is organized as follows. Section 1 presents the definitions of the cardinal characteristics of the continuum that we will need later and some relevant known theorems about them. Section 2 is devoted to the Specker phenomenon and the proof that it occurs in cardinality $\mathfrak{b}$, i.e., the upper bound mentioned above. The lower bound is established in Section 4 as a consequence of a stronger result about binding. Section 3 begins with a general discussion of binding, describes its connection with the Specker phenomenon and with the cardinal $\mathfrak{p}$, and ends with the connection with the Eklof-Shelah work mentioned above. Section 4 contains the purely set-theoretic notions of "predicting" and "evading," their connection with cardinal characteristics and applications to the group-theoretic topics of the previous sections. Finally, Section 5 adapts some of the arguments from the earlier sections to give information about the slender property of complete Boolean algebras, defined by Eda [10].

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Terminology and Notation

By "group" we mean abelian group, and we write the group operation as addition. $\mathbb{Z}$ is the group of integers. For any set I, $\mathbb{Z}^I$ is the group of functions $I \to \mathbb{Z}$ with addition of corresponding values as the operation; it is the direct product of $|I|$ copies of $\mathbb{Z}$, where $|I|$ means the cardinality of I. $\mathbb{Z}^{(I)}$ is the subgroup of $\mathbb{Z}^I$ consisting of functions whose values are 0 at all but finitely many elements of I; it is the direct sum of $|I|$ copies of $\mathbb{Z}$. When I is the set ω of natural numbers, we write Π and Σ for $\mathbb{Z}^\omega$ and $\mathbb{Z}^{(\omega)}$, respectively. If x and y are in $\mathbb{Z}^I$ and at least one of them is in $\mathbb{Z}^{(I)}$, then the sum $\sum_{i\in I} x(i)y(i)$ is finite, and we denote it by the inner product notation $\langle x, y\rangle$.

A group G is *torsionless* if for every non-zero $x \in G$ there is a homomorphism $h : G \to \mathbb{Z}$ with $h(x) \neq 0$. This is equivalent to requiring that G be embeddable in $\mathbb{Z}^I$ for some set I. (For one direction of the equivalence, use the projection homomorphisms $\mathbb{Z}^I \to \mathbb{Z}$ that evaluate functions in $\mathbb{Z}^I$ at a specific element of I. For the other direction, take I to be the set of all homomorphisms $G \to \mathbb{Z}$ and embed G in $\mathbb{Z}^I$ by $g \mapsto (i \mapsto i(g))$.) In any group G, the elements that are mapped to 0 by all homomorphisms $G \to \mathbb{Z}$ form a subgroup N, and the quotient G/N is the largest torsionless quotient of G.

We use the standard (among set-theorists) notation ω for the set of natural numbers. When discussing functions on ω or subsets of ω, we often use an asterisk * to indicate that finitely many exceptions are allowed. For example, if A and B are subsets of ω, then $A \subseteq^* B$ means that $A \setminus B$ is finite. Similarly, if f and g are functions from ω into an ordered set, then $f \leq^* g$ means that $f(n) \leq g(n)$ for all but finitely many n.

1. Cardinal Characteristics of the Continuum

For information about cardinal characteristics of the continuum, including the facts stated without proof below, we refer to Vaughan [24] and the earlier works cited there. We shall have occasion to refer to nine of the standard cardinal characteristics of the continuum, three related to Lebesgue measure, three related to Baire category, and three of a more combinatorial nature.

The characteristics related to Lebesgue measure are

(1) $\mathbf{add}(L)$, the additivity of measure, i.e., the smallest number of measure zero sets whose union is not of measure zero,
(2) $\mathbf{cov}(L)$, the covering number for measure, i.e., the smallest number of measure zero sets that can cover the real line, and
(3) $\mathbf{unif}(L)$, the uniformity number for measure, i.e., the smallest cardinality of a set not of measure zero.

To avoid possible confusion, we emphasize that "not of measure zero" is not synonymous with "of non-zero measure," because the former includes the possibility of not being measurable.

The analogous characteristics for Baire category are

(4) $\mathbf{add}(B)$, the smallest number of first category sets whose union is of second category,
(5) $\mathbf{cov}(B)$, the smallest number of first category sets that can cover the real line, and
(6) $\mathbf{unif}(B)$, the smallest cardinality of a set of second category.

These definitions referred to measure and category in the real line, but the cardinals defined here would be the same if we used either of the following two spaces instead. 2^ω is the set of infinite sequences of zeros and ones, with the product topology obtained from the discrete topology on $\{0, 1\}$ and the product measure obtained from the measure on $\{0,1\}$ that gives each of the two elements measure $1/2$. Similarly ω^ω is the set of infinite sequences of non-negative integers, with the product topology obtained from the discrete topology on ω and with the product measure obtained from the measure on ω that gives each point n measure $2^{-(n+1)}$.

To define the three remaining characteristics that we will need, we recall the notations $\subseteq^*$ and $\leq^*$ introduced above, and we also introduce the following terminology. A family $\mathcal{F}$ of sets is said to have the *strong finite intersection property* if the intersection of every finite subfamily of $\mathcal{F}$ is infinite. In terms of these concepts we define

(7) $\mathfrak{p}$ is the smallest cardinality of a family $\mathcal{F}$ of subsets of ω such that $\mathcal{F}$ has the strong finite intersection property but there is no infinite $A \subseteq \omega$ satisfying $A \subseteq^* F$ for all $F \in \mathcal{F}$,
(8) $\mathfrak{d}$ is the smallest cardinality of a family $\mathcal{D}$ of functions from ω to ω such that every function from ω to ω is $\leq^*$ some function in $\mathcal{D}$, and
(9) $\mathfrak{b}$ is the smallest cardinality of a family $\mathcal{B}$ of functions from ω to ω such that no function from ω to ω is $\geq^*$ all the functions in $\mathcal{B}$. (In other words, for every $g : \omega \to \omega$ there is $f \in \mathcal{B}$ such that $g(n) < f(n)$ for infinitely many n.)

Clearly, $\mathfrak{p}$ would be unchanged if ω in its definition were replaced by any countably infinite set. Similarly, $\mathfrak{d}$ and $\mathfrak{b}$ would be unchanged if their definitions referred to functions $A \to \omega$ instead of $\omega \to \omega$, where A is countably infinite. Also, $\mathfrak{d}$ would be unchanged if the relation of "almost everywhere majorization," $\leq^*$, were replaced with "everywhere majorization," for we can adjoin to $\mathcal{D}$, without increasing its cardinality, all those functions that differ only finitely from functions in $\mathcal{D}$.

For orientation, we point out that all nine of these cardinals lie between $\aleph_1$ and the cardinality $\mathfrak{c}$ of the continuum, inclusive. In particular, if the continuum hypothesis holds, then all are equal to $\aleph_1 = \mathfrak{c}$. It follows from results of Martin and Solovay [17] that under Martin's axiom all these cardinals equal $\mathfrak{c}$.

We shall need some information about the relationships between these nine cardinals. Most of this information is summarized in the following diagram, in which an arrow from one cardinal to another indicates that the former is provably less than or equal to the latter; that is $\to$ means $\leq$. (Except for the part involving $\mathfrak{p}$, this is part of what is called Cichoń's diagram; see [3, 12, 24]. The use of arrows instead of inequality signs is due to the custom of working not with the cardinals κ themselves but with the hypotheses $\kappa = \mathfrak{c}$; inequalities between the cardinals obviously yield implications between these hypotheses, and in practice the converse also holds, i.e., the known proofs of implications between these hypotheses also establish the inequalities.)

$$\begin{tikzcd}
\mathbf{cov}(L) \arrow[r] & \mathbf{unif}(B) & & \\
& \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} & \\
\mathbf{add}(L) \arrow[uu] \arrow[r] & \mathbf{add}(B) \arrow[u] \arrow[r] & \mathbf{cov}(B) \arrow[u] \arrow[r] & \mathbf{unif}(L) \\
& \mathfrak{p} \arrow[u] & &
\end{tikzcd}$$

A few of the inequalities are obvious from the definitions, namely $\mathbf{add} \le \mathbf{cov}$ and $\mathbf{add} \le \mathbf{unif}$ for both measure and category, and $\mathfrak{b} \le \mathfrak{d}$. The inequalities $\mathbf{cov}(B) \le \mathfrak{d}$ and $\mathfrak{b} \le \mathbf{unif}(B)$ follow easily from the observation that, for each $g \in {}^\omega\omega$, the set of all $f \le^* g$ is of first category in ${}^\omega\omega$.

Among the non-trivial inequalities, $\mathbf{cov}(L) \le \mathbf{unif}(B)$ and $\mathbf{cov}(B) \le \mathbf{unif}(L)$ are due to Rothberger [21], $\mathbf{add}(L) \le \mathbf{add}(B)$ is due to Bartoszyński [1] and independently to Raisonnier and Stern [20], $\mathfrak{p} \le \mathbf{add}(B)$ is due to Martin and Solovay [17], and $\mathbf{add}(B) \le \mathfrak{b}$ is due to Miller [19]. In fact, Miller proved a stronger result, which we shall need later and therefore display here for emphasis:

$$\mathbf{add}(B) = \min\{\mathbf{cov}(B), \mathfrak{b}\}. \tag{10}$$

The diagram above is complete in the sense that every provable (in ZFC) inequality between our nine cardinals is given by a chain of arrows in the diagram. For inequalities not involving $\mathfrak{p}$, the proofs are summarized in [3]. The consistency of $\mathfrak{p} < \mathbf{add}(L)$ is proved in [15], and the consistency of $\mathfrak{p} > \mathbf{cov}(L)$ is stated in [13], attributed to Miller (unpublished).

To close this section, we retract our earlier statement that $\mathfrak{p}$, $\mathfrak{d}$, and $\mathfrak{b}$ are more combinatorial than the measure and category characteristics. Not that the former aren't combinatorial, but it turns out that the latter have combinatorial descriptions as well. We shall need these descriptions, due to Bartoszyński [1,2], for $\mathbf{add}(L)$ and $\mathbf{cov}(B)$; for related results, see also [18,19, 20].

For the first of these descriptions, we use Bartoszyński's notion of a *slalom*, namely a function s with domain ω, such that $s(n)$ is a set of cardinality $(n+1)^2$ for each n. We say that a function $f : \omega \to \omega$ *goes through* the slalom s if $f(n) \in s(n)$ for all but finitely many n. With this terminology, we have that $\mathbf{add}(L)$ is the smallest possible cardinality of a family $\mathcal{F}$ of functions $\omega \to \omega$ such that, for each slalom s, some $f \in \mathcal{F}$ does not go through s.

We say that two functions f and g on ω are infinitely often equal if $f(n) = g(n)$ for infinitely many n. Then $\mathbf{cov}(B)$ is the smallest possible cardinality of a family $\mathcal{F}$ of functions $\omega \to \omega$ such that no single function $g : \omega \to \omega$ is infinitely often equal to each of the functions in $\mathcal{F}$.

2. The Specker Phenomenon

We say that a subgroup G of $\Pi = \mathbb{Z}^\omega$ *exhibits the Specker phenomenon* if it contains a sequence $(a_n)_{n\in\omega}$ of linearly independent elements such that every homomorphism $G \to \mathbb{Z}$ maps a_n to 0 for all but finitely many n. (The concept, but not the name, is from [9].) When we need to be more specific, we shall say that G exhibits the Specker phenomenon *witnessed by* $(a_n)_{n\in\omega}$. Thus, Specker's theorem [23, Satz III] asserts that Π exhibits the Specker phenomenon witnessed by the sequence of standard unit vectors (e_n).

Definition. The smallest cardinal of any subgroup of Π exhibiting the Specker phenomenon is denoted by $\mathfrak{se}$.

The symbol $\mathfrak{se}$ stands for Specker and Eda. Eda studied this cardinal in [9], proving that $\mathfrak{se} = \mathfrak{c}$ follows from Martin's axiom but is false in a model obtained by adding many random reals. In fact, his argument for the random real case shows that $\mathfrak{se} \leq \mathfrak{d}$. (It is well-known that $\mathfrak{d} < \mathfrak{c}$ in the random real model.) His proof that Martin's axiom implies $\mathfrak{se} = \mathfrak{c}$ also establishes somewhat more, though indirectly. The application of Martin's axiom in the proof uses a partially ordered set which does not merely (as Martin's axiom requires) satisfy the countable antichain condition but in fact is σ-centered. Bell [5] showed that Martin's axiom for σ-centered orderings is equivalent to $\mathfrak{p} = \mathfrak{c}$. In conjunction with Eda's argument, this proves that $\mathfrak{p} = \mathfrak{c}$ implies $\mathfrak{se} = \mathfrak{c}$, and a trivial variation of the argument shows that $\mathfrak{p} \leq \mathfrak{se}$. (We shall give a simpler proof of this, not using Bell's theorem but following the lines of Eda's proof, in Section 3.) Thus, we can summarize Eda's results (in the light of Bell's theorem) as $\mathfrak{p} \leq \mathfrak{se} \leq \mathfrak{d}$.

In this section, we shall improve the upper bound on $\mathfrak{se}$ from $\mathfrak{d}$ to $\mathfrak{b}$. Later, in Section 4, we shall give a new lower bound, namely $\mathbf{add}(L)$.

Before proceeding, it seems appropriate to make some remarks about the definition of the Specker phenomenon. On the one hand, the definition might seem too restrictive — why are only subgroups of Π considered? On the other hand, one might be even more restrictive and bring the definition closer to Specker's result by requiring that the witnesses a_n be the standard unit vectors e_n. We wish to point out that, once some trivialities are removed, such variations in the definition do not affect $\mathfrak{se}$. As an example of a triviality to be removed, we note that, if G were not required to be a subgroup of Π, then every divisible group of infinite rank would exhibit the Specker phenomenon simply because it has no non-zero homomorphisms to $\mathbb{Z}$.

Quite generally, if G is not torsionless, then its homomorphisms to $\mathbb{Z}$ are "the same" as those of its largest torsionless quotient G/N, so, when dealing with homomorphisms to $\mathbb{Z}$ (e.g., when discussing the Specker phenomenon), it is reasonable to divide out the irrelevant N and work with the torsionless G/N. Thus, when relaxing the restriction, in the definition of Specker phenomenon, that G be a subgroup of Π, we should still require that it be torsionless.

Theorem 1. *Either of the following two modifications of the definition of* $\mathfrak{se}$ *does not change its value.*
(a) Instead of requiring G *to be a subgroup of* Π, *require only that* G *be torsionless.*
(b) Require the witness sequence to be the sequence (e_n) *of all standard unit vectors.*

Proof. (a) The assumption that G is torsionless is equivalent to requiring that G

be embeddable in $\mathbb{Z}^I$ for some index set I. Suppose now that $G \subseteq \mathbb{Z}^I$ exhibits the Specker phenomenon, witnessed by (a_n). For each non-zero linear combination c of (any finitely many) a_n's, select one coordinate $i \in I$ such that the ith component of c is not zero. Let J be the set of i's so selected; note that J is a countable subset of I. Let $p : \mathbb{Z}^I \to \mathbb{Z}^J$ be the canonical projection map. By our choice of J, the elements $p(a_n)$ are, like the a_n, linearly independent. For any homomorphism $h : p(G) \to \mathbb{Z}$, we know that $hp : G \to \mathbb{Z}$ annihilates almost all a_n, so h annihilates almost all $p(a_n)$. Thus, $p(G)$ exhibits the Specker phenomenon witnessed by $(p(a_n))$. Since J is countable, $p(G)$ is isomorphic to a subgroup of Π, so we have an instance of the Specker phenomenon as originally defined, involving a group $p(G)$ no larger than the given G. Thus, $\mathfrak{se}$ defined with the relaxed condition on G in (a) is no smaller than $\mathfrak{se}$ as originally defined. That it is no larger is trivial, so (a) is proved.

(b) Consider any $G \subseteq \Pi$ exhibiting the Specker phenomenon witnessed by (a_n). The first part of the proof of Specker's [23, Satz I] can be used to show that, given any countably many linearly independent elements of Π, like the a_n, we have an endomorphism of Π mapping all these elements to independent values in the subgroup Σ of elements having only finitely many non-zero components. (See also Chase [7, Corollary 3.3].) Thus, we may assume without loss of generality, replacing G by its image under a suitable endomorphism of Π, that our witnesses a_n all lie in Σ. Let us write S_n for the (finite) set of coordinates where a_n has non-zero components. Each coordinate i is in only finitely many S_n's, because the projection of $G \subseteq \Pi$ to the ith coordinate is a homomorphism to $\mathbb{Z}$, to which the definition of the Specker phenomenon is applicable. This observation allows us to find an infinite subsequence of (a_n) for which the supports S_n are pairwise disjoint. Indeed, we can put a_0 into our subsequence, put out those finitely many other a_n whose S_n meets S_0, put in the first a_k different from all these, put out the finitely many other a_n whose S_n meets S_k, etc. Of course, the resulting subsequence, like any infinite subsequence of (a_n), still witnesses the Specker phenomenon for G, so we may assume, without loss of generality, that (a_n) is such that the supports S_n are pairwise disjoint. By applying suitable automorphisms of Π, composed from automorphisms of all the finite rank free groups $\mathbb{Z}^{S_n}$ separately, we can arrange that each a_n is an integer multiple of a standard unit vector e_n. Replacing G by its smallest supergroup pure in Π (which does not increase the cardinality), we can arrange that the Specker phenomenon is witnessed by a sequence of (some of the) standard unit vectors, say the e_n for n in a certain infinite subset T of ω. We can arrange further that G is a subgroup of $\mathbb{Z}^T$, just by replacing it with its image under the projection $\Pi = \mathbb{Z}^\omega \to \mathbb{Z}^T$. Finally, using a bijection between T and ω, we obtain a subgroup of Π, of no greater cardinality than the original G, for which the Specker property is witnessed by the sequence of all the standard unit vectors e_n. This proves (b). $\square$

The preceding theorem shows that the definition of $\mathfrak{se}$ is robust under fairly wide variations in the sort of group and the sort of witnessing sequence considered. Now we turn to our first estimate of $\mathfrak{se}$; the proof uses ideas from Eda [10].

Theorem 2. Π *has a subgroup of cardinality* $\mathfrak{b}$ *that exhibits the Specker phenom-*

enon (witnessed by the sequence (e_n) of standard unit vectors). Thus $\mathfrak{se} \le \mathfrak{b}$.

Proof. Let $\mathcal{B}$ be a family of $\mathfrak{b}$ functions $\omega \to \omega$ as in the definition (9) of $\mathfrak{b}$, i.e., for any $g : \omega \to \omega$ there is $f \in \mathcal{B}$ such that $g(n) < f(n)$ for infinitely many n. We can arrange that the functions in $\mathcal{B}$ are monotone non-decreasing and nowhere zero; just replace each $f \in \mathcal{B}$ with the larger function $f'(n) = 1 + \max\{f(k) \mid k \le n\}$. Henceforth, we assume that this replacement has been made, so $0 < f(m) \le f(n)$ whenever $m \le n$ and $f \in \mathcal{B}$.

Lemma. *For every infinite $A \subseteq \omega$ and every function $g : A \to \omega$, there is $f \in \mathcal{B}$ such that $g(n) < f(n)$ for infinitely many $n \in A$.*

Proof of lemma. Extend the given g to a function on all of ω by defining its value at any $k \notin A$ to be the (already defined) value of g at the next larger element of A. By our choice of $\mathcal{B}$, it contains an f that is greater than (the extended) g at infinitely many k. If infinitely many of these k's are in A, we are done. Otherwise, for each of these k's that is not in A, let n be the next larger element of A, and observe that, by definition of $g(k)$ and monotonicity of f, we have $g(n) = g(k) < f(k) \le f(n)$. So f is greater than g at infinitely many $n \in A$. $\square$

For each $f \in \mathcal{B}$, define a function $x : \omega \to \omega$ by the recursion $x(0) = 1$ and

$$x(n+1) = 2 \cdot f(n+1) \cdot x(n) \cdot \sum_{i=0}^{n} x(i). \tag{11}$$

(The motivation for this strange definition will become clear in the course of the proof.) We note for future reference that $x(n)$ is never zero, that $x(n) \to \infty$ as $n \to \infty$, and that, because of the factor $x(n)$ in the definition, $x(n)$ divides $x(m)$ whenever $n \le m$.

As $\mathcal{B}$ has cardinality $\mathfrak{b}$, we have obtained $\mathfrak{b}$ functions x in this manner. Let G be the smallest pure subgroup of Π containing all these x's and containing the (countably many) elements of Σ (the functions in Π with finite support). Then G has cardinality $\mathfrak{b}$, and we complete the proof by showing that G exhibits the Specker phenomenon witnessed by (e_n).

We proceed by contradiction, so suppose $h : G \to \mathbb{Z}$ is a homomorphism and the set $A = \{n \mid h(e_n) \neq 0\}$ is infinite. Define a function g by $g(n) = \max_{k \le n} |h(e_k)|$, and apply the lemma to obtain an $f \in \mathcal{B}$ such that $M = \{n \in A \mid f(n) > g(n)\}$ is infinite. Let x be the function defined, using this f, by (11). Thus $x \in G$. Since $x(n) \to \infty$ as $n \to \infty$ and since M is infinite, we can fix a non-zero $n \in M$ with $x(n) > 2|h(x)|$. We shall prove that $h(e_n) = 0$; this will be the desired contradiction, because $n \in M \subseteq A$.

We split $x \in G$ as the sum

$$x = \sum_{i<n} x(i)e_i + y = \sum_{i<n} x(i)e_i + x(n)z,$$

where y is the element of Π that agrees with x from coordinate n on but is zero at the earlier coordinates. Since all components $x(m)$ for $m \ge n$ are divisible by

$x(n)$, we can write y as $x(n)z$ for some $z \in \Pi$. Furthermore, as G contains x and $\sum_{i<n} x(i)e_i$ (the latter because it is in Σ), it also contains their difference y and, being pure in Π, also contains z. Thus, we have

$$h(x) = \sum_{i<n} x(i)h(e_i) + x(n)h(z) \equiv \sum_{i<n} x(i)h(e_i) \pmod{x(n)}. \tag{12}$$

Recall that, by our choice of n, the left side $h(x)$ of this congruence has absolute value smaller than half the modulus. So does the right side, because

$$\left|\sum_{i<n} x(i)h(e_i)\right| \le \sum_{i<n} x(i)g(n) < f(n)\sum_{i<n} x(i) \le \frac{1}{2}x(n), \tag{13}$$

where we have used the definition of g, the fact that $n \in M$, and the definition (11) of x. But two numbers congruent modulo $x(n)$ and each smaller in absolute value than half of that modulus must be equal. So we have

$$h(x) = \sum_{i<n} x(i)h(e_i). \tag{14}$$

The preceding calculation can be re-done with $n+1$ in place of n. Some care is needed because $n+1$, unlike n, need not be in M. This affects only the calculation (13), which becomes

$$\left|\sum_{i<n+1} x(i)h(e_i)\right| \le \sum_{i\le n} x(i)g(n) < f(n)\sum_{i\le n} x(i) \le f(n+1)\sum_{i\le n} x(i) \le \frac{1}{2}x(n+1).$$

Then we obtain, by the same argument as for (14),

$$h(x) = \sum_{i<n+1} x(i)h(e_i).$$

Subtracting (14) from this, we obtain $x(n)h(e_n) = 0$. Since x is nowhere zero, we conclude that $h(e_n) = 0$, the desired contradiction to $n \in A$. □

3. Binding Subgroups

Definition. A group G *binds* a subgroup H if every homomorphism from G to a free group maps H into a group of finite rank.

This definition has some trivial cases, e.g., if H has finite rank or if G admits no non-zero homomorphisms to free groups. As in the discussion preceding Theorem 1, elements of G whose image under every homomorphism to $\mathbb{Z}$ is 0 are irrelevant distractions in connection with binding, so it is reasonable to divide G by the subgroup of these elements, i.e., to replace G by its largest torsionless quotient. Thus, when we discuss binding, we shall always assume that G is torsionless and therefore embeddable in $\mathbb{Z}^I$ for some set I. We shall also assume, to avoid triviality, that H has infinite rank.

Notice that, if G binds H, then every (torsionless) supergroup of G binds every (infinite rank) subgroup of H as well.

Definition. If a torsionless group G binds some subgroup of infinite rank, then we say G is *binding*. If G has infinite rank and binds itself, then we call it *self-binding*.

Clearly, every self-binding group is binding.

Proposition 3. *A torsionless group G binds a subgroup H if and only if every homomorphism $G \to \Sigma$ maps H to a group of finite rank. G binds itself if and only if it does not admit a homomorphism onto Σ, if and only if it does not have Σ as a direct summand.*

Proof. For the non-trivial half of the first statement, suppose $f : G \to \mathbb{Z}^{(I)}$ maps H to a group of infinite rank. We must achieve the same situation with a countable set in place of I. As $f(H)$ has infinite rank, we can choose a countable infinity of linearly independent elements in it and then choose, for each non-zero linear combination x of these elements, one element $i \in I$ with $x(i) \neq 0$. Let J consist of the countably many i so chosen, and compose f with the canonical projection $\mathbb{Z}^{(I)} \to \mathbb{Z}^{(J)}$. The resulting homomorphism $G \to \mathbb{Z}^{(J)} \cong \Sigma$ is as desired.

For the non-trivial half of the second statement, suppose G does not bind itself, and choose, by what we just proved, $f : G \to \Sigma$ with $f(G)$ of infinite rank. But any subgroup of Σ of infinite rank is isomorphic to Σ, so we can compose f with an isomorphism $f(G) \to \Sigma$ to obtain a surjection as claimed. The last part of the proposition, about direct summands, follows because any homomorphism onto a free group, such as Σ, splits. $\square$

We shall need to consider groups G with $\Sigma \subseteq G \subseteq \Pi$. We call such a group *$\Sigma$-binding* if it binds Σ, and we call it *weakly Σ-binding* if every homomorphism $G \to \Sigma$ that extends to an endomorphism of Π maps Σ to a group of finite rank. It is clear, by the first part of Proposition 3, that Σ-binding implies weakly Σ-binding. Although Σ-binding is a special case of the general notion of binding, the following theorem shows that it is not very special.

Theorem 4. *If a torsionless group G binds a subgroup of infinite rank, then G has a homomorphic image $G' \subseteq \Pi$ such that $G' + \Sigma$ binds Σ.*

Proof. Let $G \subseteq \mathbb{Z}^I$ bind a subgroup H of infinite rank. As G binds all subgroups of H, we may assume that H is countable. Choosing for each non-zero element $x \in H$ one $i \in I$ such that $x(i) \neq 0$, letting $J \subseteq I$ be the set of these countably many chosen i's, and replacing G and H by their projections in $\mathbb{Z}^J$, we arrange that, up to isomorphism, $G \subseteq \Pi$, G still binds H, and H still has infinite rank.

The next step in the proof resembles an argument in Specker's proof of [23, Satz I] and even more closely resembles Theorem 3.2 of [7].

Lemma. *If H is any subgroup of Π of countably infinite rank, then there exists an endomorphism f of Π and there exist positive integers $(d_n)_{n\in\omega}$ such that $f(H)$ includes the subgroup $\bigoplus_n d_n\mathbb{Z}$ of Σ.*

Proof of lemma. We define the endomorphism f as the composition of an infinite sequence of endomorphisms $f = \cdots \circ f_2 \circ f_1 \circ f_0$; the composition will be well-defined because f_k will not change the nth component of x for any $x \in \Pi$ and any $n < k$. Thus the nth component of $f(x)$ is the nth component of $f_n \circ \cdots \circ f_2 \circ f_1 \circ f_0(x)$.

To define f_0, choose a non-zero element $a \in H$ such that the greatest common divisor d of all its components $a(n)$ is as small as possible. Notice that d is the largest integer by which a is divisible in Π. This provides a description of d that is invariant under automorphisms of Π. Also, d is an integral linear combination of finitely many $a(n)$'s, say $\sum_{n<r} c_n a(n)$. Let g be the automorphism of Π that leaves the first r components of its argument unchanged but transforms the rest according to

$$g(x)(m) = x(m) - \frac{a(m)}{d} \sum_{n<r} c_n x(n) \quad \text{for } m \geq r.$$

Recall that d divides all $a(m)$, so this makes sense, and notice that $g(a)(m) = 0$ for all $m \geq r$. Now, by applying a suitable automorphism of $\mathbb{Z}^r$ to the first r coordinates (where $g(a)$ can be non-zero) and leaving all the other coordinates unchanged, we can send $g(a)$ to $d \cdot e_0$. (The components of $g(a)$ have greatest common divisor d, thanks to the automorphism-invariance of d noted above.)

Let f_0 be the composite of g and the automorphism just described. So $f_0(a) = d \cdot e_0$. Also, let $d_0 = d$.

Consider any element $b \in f_0(H)$. As f_0 is an automorphism of Π, the greatest common divisor of the components of b is the same as for $f_0^{-1}(b) \in H$, hence is at least d. In particular, $|b(0)|$ is at least d, unless it is zero. It follows that $b(0)$ is divisible by d, for otherwise we could write $b(0) = qd + r$ with $0 < r < d$, and then $b' = b - qde_0 \in f_0(H)$ would have 0th component r, a contradiction because the conclusion of the previous sentence applies to b' as well as to b.

Thus, every element of $f_0(H)$ is expressible as $qd_0e_0 + x$, where $x \in f_0(H)$ has $x(0) = 0$. Therefore, $f_0(H) = \langle d_0e_0 \rangle \oplus H_1$, where H_1 is a subgroup of $\Pi_1 = \mathbb{Z}^{\omega \setminus \{0\}}$.

Now proceed with H_1 just as we did with H in the preceding paragraphs, obtaining an automorphism f_1 of Π_1 (which we extend to Π by letting it act trivially on the 0 coordinate) sending H_1 to $\langle d_1e_1 \rangle \oplus H_2$, where H_2 is a subgroup of $\Pi_2 = \mathbb{Z}^{\omega \setminus \{0,1\}}$ and where d_1 is the smallest g.c.d. of all the components of any element of H_1.

Continuing this process inductively, we obtain f_n's whose composite sends H to a subgroup containing all the d_ne_n. (Notice that, although each f_n is an automorphism of Π, we can only claim that the composite f is an endomorphism.) This completes the proof of the lemma. □

Returning to the proof of the theorem, we apply the lemma to the H that was bound by our (modified) $G \subseteq \Pi$, and we set $G' = f(G)$, where f is given by the lemma. Then G' is a homomorphic image of our modified G, hence also of our original G. It binds $f(H)$, so its supergroup $G' + \Sigma$ binds the subgroup $S = \bigoplus_n d_n\mathbb{Z}$ of $f(H)$. But S is a subgroup of Σ such that the quotient Σ/S is a torsion group, because S contains a multiple of each of the generators e_n of Σ. It follows immediately that any homomorphism that is defined on Σ and maps S into a group of finite rank must also map Σ into a group of (the same) finite rank. Thus, binding S implies binding Σ, and the proof is therefore complete. □

Corollary 5. *Each of the following cardinals is less than or equal to the next:*

(1) $\aleph_1$
(2) *the smallest cardinality of a weakly* Σ*-binding group*
(3) *the smallest cardinality of a binding group*
(4) *the smallest cardinality of a self-binding group*
(5) *the cardinality* $\mathfrak{c}$ *of the continuum.*

Proof. (1)≤(2): We must show that, if $\Sigma \subseteq G \subseteq \Pi$ and G is countable, then there is an endomorphism of Π mapping G into Σ and mapping Σ onto a group of infinite rank. For this, it more than suffices to get an automorphism of Π mapping G into Σ. Furthermore, it suffices to do this under the additional assumption that G is a pure subgroup of Π, because G can be enlarged to a pure subgroup without increasing its cardinality. But now what we need is given by [7], Corollary 3.3.

(2)≤(3): By Theorem 4, the cardinal in (3) is also the smallest cardinality of a Σ-binding group. Since Σ-binding implies weakly Σ-binding, the desired inequality follows.

(3)≤(4): This is immediate, as every self-binding group is binding.

(4)≤(5): Π has cardinality $\mathfrak{c}$ and binds itself. (If it had Σ as a direct summand, then its dual, which is isomorphic to Σ by [23, Satz III], [14, Cor. 94.6], would have a summand isomorphic to the dual of Σ, i.e., to Π. This is absurd as Σ is countable and Π is not. The same conclusion also follows instantly from the more detailed information in [14, pp.159–160] about homomorphisms from products to sums.) $\square$

The next two results relate binding to the Specker phenomenon discussed in Section 2.

Theorem 6. *A group that exhibits the Specker phenomenon is binding.*

Proof. Suppose G exhibits the Specker phenomenon witnessed by (a_n), and let H be the subgroup of G generated by these witnesses a_n. Suppose also, toward a contradiction, that G does not bind H. So let $f : G \to \Sigma$ map H to a group of infinite rank. We inductively construct an element $x \in \Pi$ and an increasing sequence of natural numbers k_m as follows.

At stage m of the construction, we have already defined k_i for $i < m$, and we have already defined some finite initial segment of x. As the group $f(H)$ generated by the $f(a_n)$'s has infinite rank, we can choose k_m so that $f(a_{k_m})$ has a non-zero component in some position, say the ith, such that $x(i)$ is not yet defined. Then we can easily extend x (or, rather, the finite part of x already defined) so that (a) $x(j)$ becomes defined for all j such that $f(a_{k_m})(j) \neq 0$ and so that (b) the inner product $\langle f(a_{k_m}), x\rangle$ is not zero. (The inner product will not depend on future steps in the definition of x, because of (a). Making it non-zero is easy, by appropriately choosing the value of $x(i)$.)

Notice that all the k_m are distinct, because the mth stage is the first one where x is defined at all coordinates where $f(a_{k_m})$ has a non-zero component, so we can recover m from k_m.

Now the homomorphism $z \mapsto \langle z, x\rangle$ maps Σ and hence $f(G)$ into $\mathbb{Z}$ and maps none of the $f(a_{k_m})$'s to zero. So its composite with f violates the choice of the a_n as witnessing the Specker phenomenon. $\square$

This theorem immediately implies that $\mathfrak{se}$ is at least as big as the cardinal labeled (3) in the Corollary 5. In fact, we can do a bit better, replacing (3) with (4).

Theorem 7. *The smallest cardinality of a self-binding group is $\leq \mathfrak{se}$.*

Thus, $\mathfrak{se}$ could be inserted into Corollary 5 as item (4.5). Theorems 6 and 7 cannot be combined to assert that a group exhibiting the Specker phenomenon binds itself; a counterexample is given by $\Sigma \oplus \Pi$ regarded as a subgroup of Π in the obvious way (sequences in which only finitely many odd-numbered components are non-zero).

Proof of Theorem 7. By Theorem 1, let G_1 be a group of cardinality $\mathfrak{se}$ such that $\Sigma \subseteq G_1 \subseteq \Pi$ and G_1 exhibits the Specker phenomenon witnessed by the sequence of standard unit vectors e_n. By Theorem 6, G_1 is binding, and by Corollary 5 there is a weakly Σ-binding group G_2 of cardinality at most $\mathfrak{se}$. Let $G_3 = G_1 + G_2$. Then $\Sigma \subseteq G_3 \subseteq \Pi$, G_3 exhibits the Specker phenomenon witnessed by the e_n's, G_3 is weakly Σ-binding, and $|G_3| = \mathfrak{se}$.

The next step of the proof is based on an idea from [10]. Fix two disjoint increasing sequences of prime numbers, say (p_n) and (q_n). For each n, obtain from the Chinese remainder theorem an integer d_n that is divisible by all p_k for $k \leq n$ but is congruent to -1 modulo all q_k for $k \leq n$. Extend G_3 to a pure subgroup G of Π that is closed under (component-by-component) multiplication by the sequence $d = (d_n)$. This can be done with $|G| = \mathfrak{se}$, because we are simply closing G_3 under countably many partial functions (multiplication by d and division by each positive integer). The following lemma is due to Eda [10].

Lemma. *A homomorphism $f : G \to \mathbb{Z}$ is completely determined if the values $f(e_n)$ are known.*

Proof. It suffices to show that, if $f(e_n) = 0$ for all n then $f(x) = 0$ for all $x \in G$. Fix any such f and x. Also, temporarily fix a positive integer n. Decompose the componentwise product dx as the sum of $\sum_{i<n} d_i x(i) e_i$ and the rest, which, by choice of d_n, is divisible by p_n in Π, hence also in G by purity. The first part of this decomposition is annihilated by f, as all $f(e_i) = 0$. So $f(dx)$ is divisible by p_n. Now un-fix n. $f(dx)$ is divisible by arbitrarily large primes p_n, hence is zero. We can repeat the same argument with the sequence $d + 1 = (d_n + 1)$ in place of d and with q_n in place of p_n (since q_k divides $d_n + 1$ when $k \leq n$). We obtain $f((d + 1)x) = 0$. But then, as $x = (d + 1)x - dx$, it follows that $f(x) = 0$, as desired. $\square$

Combining the lemma with the Specker phenomenon witnessed by the e_n's, we find that every homomorphism $f : G \to \mathbb{Z}$ has the form $f(x) = \sum_{i<n} f(e_i)x(i)$ for some n. Therefore f extends to a homomorphism $\Pi \to \mathbb{Z}$. It follows, by applying this observation to each component, that every homomorphism $G \to \Sigma$ extends to

a homomorphism $\Pi \to \Pi$. We use this to complete the proof of the theorem by showing that G binds itself.

Let $f : G \to \Sigma$. As f extends to a homomorphism $\Pi \to \Pi$ and as G is weakly Σ-binding, $f(\Sigma)$ must have finite rank. Therefore, for all but finitely many $n \in \omega$, the nth component $f_n : G \to \mathbb{Z}$ of f is zero on Σ. But, by the lemma, this implies that these (all but finitely many) f_n are zero on G. So $f(G)$ has finite rank, as required. $\square$

The following corollary extends Corollary 5 to incorporate Theorems 2 and 7.

Corollary 8. *Each of the following cardinals is less than or equal to the next:*

(1) $\aleph_1$
(2) *the smallest cardinality of a weakly Σ-binding group*
(3) *the smallest cardinality of a binding group*
(4) *the smallest cardinality of a self-binding group*
(5) $\mathfrak{se}$
(6) $\mathfrak{b}$
(7) $\mathfrak{c}$. $\square$

We are now in a position to fulfill our promise, from Section 2, to give another proof, not using Bell's theorem, of the inequality $\mathfrak{p} \leq \mathfrak{se}$. In fact, we obtain a stronger result, showing that $\mathfrak{p} \leq$ all the cardinals except $\aleph_1$ mentioned in Corollary 8.

Theorem 9. *A group of cardinality $< \mathfrak{p}$ cannot be weakly Σ-binding.*

Proof. Suppose $\Sigma \subseteq G \subseteq \Pi$ and $|G| < \mathfrak{p}$. Recall from Section 1 that if C is a countable set, if $\mathcal{F}$ is a family of fewer than $\mathfrak{p}$ subsets of C, and if every finite subfamily of $\mathcal{F}$ has infinite intersection, then there is an infinite subset A of C such that $A \subseteq^* F$ for all $F \in \mathcal{F}$. We apply this with $C = \Sigma \setminus \{0\}$ and with the following $|G|$ sets Z_x as the family $\mathcal{F}$. For each $x \in G$, let

$$Z_x = \{c \in C \mid \langle c, x \rangle = 0\}.$$

To check that the family $\mathcal{F} = \{Z_x \mid x \in G\}$ has the strong finite intersection property, let any finitely many of its elements, say $Z_{x_1}, Z_{x_2}, \dots, Z_{x_n}$, be given; we must show that their intersection is infinite. For each $k \in \omega$, let y_k be the n-component integer vector $(x_1(k), x_2(k), \dots, x_n(k))$. These infinitely many vectors y_k, lying in the finite-dimensional vector space $\mathbb{Q}^n$, must be linearly dependent; fix a non-zero r-tuple $(c(0), c(1), \dots, c(r))$ of rational numbers such that $\sum_{k=0}^{r} c(k)x_i(k) = 0$ for all $i = 1, 2, \dots, n$. Clearing denominators, we can arrange that the $c(k)$ are integers. Extending the definition of $c(k)$ to $k > r$ by making these $c(k) = 0$, we obtain an element $c \in \Sigma \setminus \{0\}$ with $\langle c, x_i \rangle = 0$ for all $i = 1, 2, \dots, n$. This shows that the intersection of the Z_{x_i} is non-empty; in fact, it is infinite because we can multiply c by any non-zero integer.

Having checked the strong finite intersection property, we use the fact that $|\mathcal{F}| \leq |G| < \mathfrak{p}$ to obtain an infinite set $A = \{a_0, a_1, \dots, a_n, \dots\} \subseteq \Sigma \setminus \{0\}$ such that $A \subseteq^* Z_x$ for all $x \in G$. Now we define a homomorphism $f : \Pi \to \Pi$ by $f(x)(n) = \langle a_n, x \rangle$.

For each $x \in G$, we have, for all but finitely many $n \in \omega$, that $a_n \in Z_x$, which means that $f(x)(n) = 0$. Thus, $f(x) \in \Sigma$. We have shown that (the restriction of) f maps G into Σ and extends to an endomorphism (namely the unrestricted f) of Π. We shall show that $f(\Sigma)$ has infinite rank, thereby completing the proof that G is not weakly Σ-binding.

Suppose, toward a contradiction, that $f(\Sigma)$ had finite rank. Then there would be an $m \in \omega$ such that, for all $n \geq m$ and all $k \in \omega$, $f(e_k)(n) = 0$. This equation means $0 = \langle a_n, e_k \rangle = a_n(k)$. This would make all the a_n for $n > m$ equal to zero, contradicting the fact that the a_n are all in C which doesn't contain zero. □

This theorem allows us to insert $\mathfrak{p}$ as item (1.5) in Corollaries 5 and 8.

We close this section by connecting the notion of binding to a question studied by Eklof and Shelah in [11]. One of their results (not the main one) is that, under the assumption of Martin's axiom, if G is a group of cardinality $< \mathfrak{c}$ and if $G \cong G \oplus \mathbb{Z}^m$ for some positive integer m, then $G \cong G \oplus \Sigma$ (and therefore $G \cong G \oplus \mathbb{Z}^m$ for every positive integer m). In fact, their application of Martin's axiom involved a σ-centered partially ordered set, so, by Bell's theorem, the hypotheses that Martin's axiom holds and $|G| < \mathfrak{c}$ can be weakened to the hypothesis that $|G| < \mathfrak{p}$. (Here Bell's theorem is used in the form "Martin's axiom holds for σ-centered partial orders and fewer than $\mathfrak{p}$ dense sets," as in [13, Theorem 14C]. As the referee pointed out, Bell [5] stated only the special case $\mathfrak{p} = \mathfrak{c}$, although his proof gives the general form that we need.) Here is an alternate approach, not using Bell's theorem, to (a stronger form of) this result.

Assume $G \cong G \oplus \mathbb{Z}^m$ for some positive integer m. Instead of assuming $|G| < \mathfrak{p}$, we assume only that $|G|$ is smaller than the smallest cardinality of any self-binding group. (This hypothesis would follow if $|G| < \mathfrak{p}$, by virtue of Corollary 8 and Theorem 9.) Let N be, as before, the subgroup of G consisting of elements mapped to 0 by all homomorphisms $G \to \mathbb{Z}$. The analogous subgroup of $G \oplus \mathbb{Z}^m$ is clearly $N \oplus 0$, as $\mathbb{Z}^m$ is torsionless. So the assumed isomorphism from G onto $G \oplus \mathbb{Z}^m$ must send N to $N \oplus 0$ and must therefore induce an isomorphism from the largest torsionless quotient G/N onto $(G/N) \oplus \mathbb{Z}^m$. By comparing the ranks of these groups, we conclude that G/N has infinite rank. (This is the only use we need to make of the assumption that $G \cong G \oplus \mathbb{Z}^m$.) As $|G/N| \leq |G| <$ the smallest cardinal of any self-binding group, Proposition 3 tells us that G/N admits a surjection to Σ, and therefore so does G, and therefore $G \cong A \oplus \Sigma$ for some A. But since $\Sigma \cong \Sigma \oplus \Sigma$, it follows that $G \cong G \oplus \Sigma$, as desired.

4. Predicting and Evading

This section is devoted to combinatorial concepts, prediction and evasion, motivated by some of the group-theoretic concepts of the preceding section, particularly the weakest of these, weak Σ-binding.

Definition. For any set S, an *S-valued predictor* is a pair $\pi = (D_\pi, (\pi_n)_{n \in D_\pi})$ where D_π is an infinite subset of ω and π_n is, for each $n \in D_\pi$, a function $S^n \to S$. We say that the predictor π *predicts* the function $x : \omega \to S$ if, for all but finitely many $n \in D_\pi$, we have $x(n) = \pi_n(x \upharpoonright n)$. Otherwise, we say that x *evades* π.

Here $x \upharpoonright n$ means the n-tuple $(x(0), x(1), \ldots, x(n-1))$. Intuitively, we regard predictors π as follows. The values of an unknown function $x : \omega \to S$ are being revealed, one at a time, in order, and we are to guess $x(n)$ just before it is to be revealed, i.e., just after we have seen $x \upharpoonright n$. The predictor π provides a strategy for making these guesses when $n \in D_\pi$, namely, if we have seen the n-tuple t so far, then we are to guess $\pi_n(t)$. The functions predicted by π are just those for which all but finitely many of the guesses provided by this strategy are correct.

Definition. A $\mathbb{Q}$-valued predictor π is called *linear* if, for each $n \in D_\pi$, the function $\pi_n : \mathbb{Q}^n \to \mathbb{Q}$ is a linear function with rational coefficients. A predictor is *forgetful* if, whenever $m < n$ are consecutive elements of D_π, the function π_n depends only on the last $n - m - 1$ components of its argument n-tuple.

In terms of the intuitive picture of predictors, forgetfulness means that, when guessing $x(n)$, the strategy considers only the values of x revealed since its last guess, namely $x(m+1)$ through $x(n-1)$.

Definition. $\mathfrak{e}$, the *evasion number*, is the smallest possible cardinality of a family $\mathcal{E}$ of functions $\omega \to \omega$ such that every ω-valued predictor is evaded by some $x \in \mathcal{E}$. $\mathfrak{e}_l$ (resp. $\mathfrak{e}_f$, resp. $\mathfrak{e}_{fl}$) is the smallest possible cardinality of a family $\mathcal{E}$ of functions $\omega \to \mathbb{Z}$ such that every linear (resp. forgetful, resp. forgetful linear) $\mathbb{Q}$-valued predictor is evaded by some $x \in \mathcal{E}$.

It may seem strange to use linear predictors, which are rational-valued, to predict only integer-valued functions. In fact, we are interested only in integer values; the rationals are involved only in order to get enough predictions of integers without violating linearity. Consider, for example, a guessing strategy which guesses that $x(1) = x(0)/2$ if $x(0)$ is even but has nothing intelligent to say about $x(1)$ if $x(0)$ is odd and therefore makes some arbitrary guess in this case. The essential part of this is linear, sending a one-term sequence (n) with even n to $n/2$, but it clearly cannot be regarded as a linear function over the integers. (I thank the referee for pointing out this difficulty with my original definition of linear predictors.) By using $\mathbb{Q}$ instead, we can make such predictions honestly linear. The non-integer values of a predictor will, of course, never be correct because we are trying to predict integer-valued functions, so these values can be regarded as the result of the predictor's giving up because it has nothing intelligent to say. Any linear predictor π can be converted into a non-linear $\mathbb{Z}$-valued predictor π' by redefining the non-integer outputs of each π_n (on integer inputs) to be zero; then any integer-valued function predicted by π will also be predicted by π'.

Note that $\mathfrak{e}$ would be unchanged if its definition were phrased in terms of functions $\omega \to D$ and D-valued predictors, where D is any countably infinite set. In particular, D could be $\mathbb{Z}$, and so it is clear that $\mathfrak{e}$ is greater than or equal to both $\mathfrak{e}_l$ and $\mathfrak{e}_f$, which are in turn greater than or equal to $\mathfrak{e}_{fl}$.

The general notion of predictor and the associated cardinal $\mathfrak{e}$ seem quite natural, but it is the more specialized notions involving linearity and the associated cardinals that connect directly with the group-theoretic concepts of the preceding section, as the following theorem shows.

Theorem 10. *The following three cardinals are equal.*

1. *The smallest cardinality of a weakly* Σ*-binding group*
2. $\mathfrak{e}_l$
3. $\mathfrak{e}_{fl}$

Proof. (1)$\geq$(2): Let G be a weakly Σ-binding group. In particular, $G \subseteq \Pi$, so G is a family of functions $\omega \to \mathbb{Z}$. We shall show that every linear predictor π is evaded by some element of G, so $\mathfrak{e}_l \leq |G|$, as required.

Suppose, toward a contradiction, that π is linear and predicts every element of G. Let $D_\pi = \{d_0, d_1, \dots\}$. Thus, for each $x \in G$, all but finitely many $k \in \omega$ have $x(d_k) = \pi_{d_k}(x \restriction d_k)$. As π_{d_k} is linear with rational coefficients, we can clear denominators to rewrite this equation as a linear relation with integer coefficients

$$\sum_{i=0}^{d_k} c_{ki} x(i) = 0. \tag{15_k}$$

Note that the coefficients c_{ki} here depend only on π_{d_k}, not on x. Note also that $c_{kd_k} \neq 0$, because the rational linear relation from which we got (15_k) really involved $x(d_k)$.

We define an endomorphism C of Π by setting

$$C(x)(k) = \sum_{i=0}^{d_k} c_{ki} x(i)$$

for all $x \in \Pi$. Regarding the elements of Π as infinite column vectors, this endomorphism is given by left multiplication by the infinite matrix $C = (c_{ni})$. If $x \in G$, then (15_k) holds, and therefore $C(x)(k) = 0$, for all but finitely many k, i.e., $C(x) \in \Sigma$. So C is an endomorphism of Π mapping G into the free group Σ. As G is weakly Σ-binding, C must map Σ into a group of finite rank. But $C(\Sigma)$ includes the elements $C(e_n)$, the columns of the matrix C; among these are the columns indexed by the elements d_k of D_π. The kth of these columns has a non-zero entry c_{kd_k} in the kth row and zero entries in all earlier rows. Thus, these columns by themselves form a lower triangular matrix with non-zero diagonal entries. They are therefore linearly independent, contrary to the fact that they lie in a group of finite rank.

(2)$\geq$(3): trivial.

(3)$\geq$(1): Suppose $\mathcal{E}$ is a family of functions $\omega \to \mathbb{Z}$ such that every forgetful linear predictor is evaded by some element of $\mathcal{E}$. So $\mathcal{E}$ is a subset of Π; let G be the subgroup that it and Σ generate. G has the same cardinality as $\mathcal{E}$, so to complete the proof it suffices to show that G is weakly Σ-binding.

Suppose it were not. Fix an endomorphism f of Π mapping G into Σ and mapping Σ onto a group of infinite rank. Each component of f, mapping Π to $\mathbb{Z}$, has, by Specker's theorem [23 Satz III], the form $f(x)(n) = \sum_i c_{ni} x(i)$, where for each fixed n only finitely many c_{ni} are non-zero. Thus, in the matrix $C = (c_{ni})$, each row has only finitely many non-zero entries. So does each column, for the ith column is $f(e_i) \in f(G) \subseteq \Sigma$. (Recall that G was defined so as to contain all the e_i.)

We inductively choose infinitely many rows, the rows indexed by i_0, i_1, etc., as follows. Choose i_0 so that the i_0th row of C isn't zero. (This is possible as f is not identically zero.) For the induction step, suppose we have already chosen i_k, an index of a non-zero row in C. Let d_k be the number of the column in which the last non-zero entry of row i_k occurs. Then choose i_{k+1} so that row i_{k+1} of C is non-zero, but its entries in columns 0 through d_k are all zero. The first half of this constraint is satisfied by infinitely many rows, for otherwise $f(\Pi)$ would have finite rank, whereas we are assuming that even $f(\Sigma)$ has infinite rank. The second half of the constraint is satisfied by all but finitely many rows, because each column of C has only finitely many non-zero entries. So the choice of i_{k+1} is possible.

Define a forgetful linear predictor π by $D_\pi = \{d_k \mid k \in \omega\}$ (where d_k is, as in the preceding paragraph, the largest d for which $c_{i_k d} \neq 0$) and

$$\pi_{d_k}(t) = -\frac{1}{c_{i_k d_k}} \sum_{r<d_k} c_{i_k r} t(r).$$

This is clearly a linear predictor; it is forgetful because $c_{i_{k+1} r} = 0$ for $r \leq d_k$ by the second half of the constraint on the choice of i_{k+1}.

As G includes $\mathcal{E}$ which has elements to evade any given forgetful linear predictor, fix an $x \in G$ evading this π. This means that, for infinitely many k,

$$x(d_k) \neq -\frac{1}{c_{i_k d_k}} \sum_{r<d_k} c_{i_k r} x(r).$$

Clearing denominators and transposing the negative terms, we find that the inner product of x and the i_kth row of C is non-zero. This inner product is the i_kth component of $f(x)$, so its being non-zero for infinitely many k means that $f(x) \notin \Sigma$. But this is a contradiction because $x \in G$ and $f(G) \subseteq \Sigma$. $\square$

Corollary 11. $\mathfrak{p} \leq \mathfrak{e}_l$

Proof. Theorems 9 and 10. $\square$

By Theorem 10, any lower bound for $\mathfrak{e}_l = \mathfrak{e}_{fl}$ is also a lower bound for all the cardinals except $\aleph_1$ listed in Corollary 8, in particular for $\mathfrak{se}$. Apart from $\mathfrak{p}$, we have one more such bound, given by the following theorem.

Theorem 12. $\mathbf{add}(L) \leq \mathfrak{e}_l$.

Proof. We use the combinatorial description of $\mathbf{add}(L)$, at the end of Section 1, as the smallest number of of functions that do not all go through a single slalom. We observe that, although that description is phrased in terms of functions whose values are elements of ω and slaloms whose values are subsets of ω, it would make no difference if ω were replaced in both places with some other countably infinite set A. (The domains of the slaloms and of the functions under consideration are still ω; only the ranges are modified. Thus, the requirement in the definition of slalom that $|s(n)| = (n+1)^2$ still makes sense.)

We begin by describing a convenient A for this proof. Partition ω into finite blocks of consecutive numbers, say $I_0 = [0, a_1)$, $I_1 = [a_1, a_2)$, etc., in such a way that I_n has more than $(n+1)^2$ elements. (For example, one could take $a_n = 2n^3$.) Let A be the set of all functions with domain equal to one of the I_n's and with values in $\mathbb{Z}$. As A is countably infinite, the remarks in the preceding paragraph apply to it. The proof will be complete if we find $\mathfrak{e}_l$ functions $\omega \to A$ that do not all go through any single slalom s (whose values are subsets of A).

Let $\mathcal{E}$ be a family of $\mathfrak{e}_l$ functions $\omega \to \mathbb{Z}$ as in the definition of $\mathfrak{e}_l$, i.e., every linear predictor is evaded by some $f \in \mathcal{E}$. For each $f \in \mathcal{E}$, define $f' : \omega \to A$ by $f'(n) = f \restriction I_n$. Suppose s were a slalom that all these f''s go through; we shall complete the proof by deducing a contradiction.

Temporarily fix some $n \in \omega$. Let T be the set of functions $I_n \to \mathbb{Z}$ that are elements of $s(n)$. ($s(n)$ may also have elements with domain I_m for $m \neq n$, but these are irrelevant to our purpose as well as to the assumption that all f' go through s.) T consists of at most $|s(n)| = (n+1)^2$ elements of the rational vector space of rational-valued functions on I_n. This vector space has dimension $|I_n| > (n+1)^2$, so there is a non-zero linear functional ϕ_n on this vector space annihilating all the elements of T. It has the form $\phi_n(t) = \sum_{i \in I_n} c_{ni} t(i)$, for some rational coefficients c_{ni}, not all zero. Let d_n be the largest i for which $c_{ni} \neq 0$. We record for future use the fact that, if $f'(n) \in s(n)$ then $\phi_n(f \restriction I_n) = 0$. This is because $f \restriction I_n = f'(n)$ belongs to $s(n)$ and maps I_n into $\mathbb{Z}$ and therefore belongs to T.

Now unfix n, but keep the notations ϕ_n, c_{ni}, and d_n from the preceding paragraph. (The subscripts n, superfluous before, now prevent ambiguity.) Define a predictor π by setting $D_\pi = \{d_n \mid n \in \omega\}$ and, for each $d_n \in D_\pi$, setting

$$\pi_{d_n}(t) = -\frac{1}{c_{nd_n}} \sum_{\substack{i < d_n \\ i \in I_n}} c_{ni} t(i).$$

This is a forgetful linear predictor. If some $f : \omega \to \mathbb{Z}$ and some $n \in \omega$ satisfy $\phi_n(f \restriction I_n) = 0$, then trivial algebraic manipulation of this equation gives $f(d_n) = \pi_{d_n}(f \restriction d_n)$.

For each $f \in \mathcal{E}$, the assumption that f' goes through s means that for all but finitely many n we have $f'(n) \in s(n)$, and so $\phi_n(f \restriction I_n) = 0$, and so $f(d_n) = \pi_{d_n}(f \restriction d_n)$. But that means that π predicts every $f \in \mathcal{E}$, contrary to the choice of $\mathcal{E}$. $\square$

Upper bounds for the evasion cardinals $\mathfrak{e}$ and $\mathfrak{e}_l$ are less interesting than lower bounds, since they do not imply corresponding bounds for the more natural group-theoretic cardinals in Corollary 8. Nevertheless, we record for the sake of completeness the upper bounds which we have been able to obtain.

Theorem 13. *All three of* $\mathbf{unif}(L)$, $\mathbf{unif}(B)$, *and* $\mathfrak{d}$ *are upper bounds for* $\mathfrak{e}$. *Furthermore,* $\min\{\mathfrak{e}, \mathfrak{b}\} \leq \mathbf{add}(B)$.

Proof. $\mathfrak{e} \leq \mathbf{unif}$: We handle the measure and category cases together. It is easy to verify that, for any predictor π, the set of functions $x \in 2^\omega$ (i.e., functions on ω with only 0 and 1 as values) that are predicted by π is a set of first category and measure zero in 2^ω. The desired inequalities follow immediately.

$\mathfrak{e} \leq \mathfrak{d}$: Let $\mathcal{D}$ be a family of $\mathfrak{d}$ functions from $\omega \times \omega$ (the set of pairs of natural numbers) to ω such that every function $\omega \times \omega \to \omega$ is majorized (everywhere) by a function from $\mathcal{D}$. (See the remarks in Section 1 after the definition of $\mathfrak{d}$.) To each function $g \in \mathcal{D}$, we associate a function $x : \omega \to \omega$ by the recursion

$$x(n) = g(n, 1 + \max\{x(p) \mid p < n\}). \tag{16}$$

We shall show that every ω-valued predictor π is evaded by one of these $\mathfrak{d}$ functions x.

Let $\pi = (D_\pi, (\pi_n))$ be given, and define a function $f : \omega \times \omega \to \omega$ by

$$f(n,k) = \begin{cases} \max\{\pi_n(t) \mid t \in \omega^n \text{ and all values of } t \text{ are } < k\}, & \text{if } n \in D_\pi \\ \text{arbitrary}, & \text{otherwise.} \end{cases}$$

Choose $g \in \mathcal{D}$ such that $g(n,k) > f(n,k)$ for all n and k, and let x be the function obtained from g by (16). We shall show that this x evades π.

Consider any $n \in D_\pi$. Let $k = 1 + \max\{x(p) \mid p < n\}$ and note that $x \restriction n$, being in ω^n and having all its values $< k$, is one of the t's involved in the definition of $f(n,k)$. So $f(n,k) \geq \pi_n(x \restriction n)$. On the other hand, by the definition (16) of x and the choice of g, we also have $x(n) \geq g(n,k) > f(n,k)$. Comparing these inequalities, we find that $x(n) \neq \pi_n(x \restriction n)$. As n was an arbitrary element of D_π, this shows that x evades π. (It actually shows more: not only are infinitely many of π's predictions wrong, as required for evasion, all of them are wrong.)

$\min\{\mathfrak{e}, \mathfrak{b}\} \leq \mathbf{add}(B)$: Since $\mathbf{add}(B) = \min\{\mathbf{cov}(B), \mathfrak{b}\}$, it suffices to prove that $\min\{\mathfrak{e}, \mathfrak{b}\} \leq \mathbf{cov}(B)$. For this we use the combinatorial description of $\mathbf{cov}(B)$ at the end of Section 1. We must show that, if $\mathcal{F}$ is a family of functions $\omega \to \omega$ of cardinality smaller than $\min\{\mathfrak{e}, \mathfrak{b}\}$, then there exists $g : \omega \to \omega$ such that every $x \in \mathcal{F}$ is infinitely often equal to g.

Let such an $\mathcal{F}$ be given. As its cardinality is smaller than $\mathfrak{b}$, fix $d : \omega \to \omega$ eventually majorizing (i.e., $\geq^*$) every $x \in \mathcal{F}$. For any natural number k we will write $\max(d,k)$ for the function whose value at any n is $\max\{d(n), k\}$. For any $x \in \mathcal{F}$, we can find k so large that x is majorized everywhere by $\max(d,k)$. (d majorizes all but finitely many values of x, so we just choose k to majorize those finitely many values.)

Partition ω into finite blocks of consecutive numbers, say $I_0 = [0, a_1)$, $I_1 = [a_1, a_2)$, etc., in such a way that the cardinality of I_k is

$$|I_k| = \prod_{j < a_k} \max\{d(j), k\}, \tag{17}$$

i.e., the number of functions $I_0 \cup \cdots \cup I_{k-1} \to \omega$ that are majorized on their domain by $\max(d,k)$. Note that none of these blocks are empty.

Let A be the set of functions into ω whose domains are blocks in this partition. For each $x \in \mathcal{F}$, define an associated function $x' : \omega \to A$ by $x'(k) = x \restriction I_k$. As $|\mathcal{F}| < \mathfrak{e}$, all these functions x' are predicted by a single A-valued predictor $\pi = (D_\pi, (\pi_n))$.

We use π to define the desired g (infinitely often equal to each $x \in \mathcal{F}$) as follows. If $k \notin D_\pi$, define $g \restriction I_k$ arbitrarily. If $k \in D_\pi$, then proceed as follows. List the functions $I_0 \cup \cdots \cup I_{k-1} \to \omega$ that are majorized (on their domain) by $\max(d, k)$ as $t_0, t_1, \ldots\ t_{r-1}$; the number of such functions, here called r, is the cardinality of I_k, by (17). For each $j < r$, obtain t'_j from t_j analogously to the way x' was defined from x, i.e., $t'_j : \{0, 1, \ldots, k-1\} \to A$ and $t'_j(i) = t_j \restriction I_i$. Consider the set X of those $\pi_k(t'_j)$ (for our fixed k and arbitrary $j < k$) that are functions $I_k \to \omega$. This is a set of at most r functions on a set I_k of size r. So there is a function $I_k \to \omega$ that agrees at least once with each of the functions in X. Let $g \restriction I_k$ be such a function.

To show that g has the desired property, consider any $x \in \mathcal{F}$; we must show that g and x are infinitely often equal. Temporarily fix some $k \in D_\pi$ so large that $\max(d, k)$ majorizes x everywhere and so large that π correctly predicts $x'(k)$, i.e., $x'(k) = \pi_k(x' \restriction k)$. Then, in the definition of $g \restriction I_k$, one of the t_j's under consideration was $x \restriction I_0 \cup \cdots \cup I_{k-1}$ and the corresponding t'_j was $x' \restriction k$. So, as π correctly predicted $x'(k)$, we have that

$$x \restriction I_k = x'(k) = \pi_k(x' \restriction k) = \pi_k(t'_j) \in X$$

is one of the functions with which we made $g \restriction I_k$ agree at least once. This proves that g and x agree at least once in I_k. Now un-fix k. The preceding argument applies to infinitely many k's, namely all sufficiently large elements of D_π. So g and x are infinitely often equal. □

Corollary 14. $\mathfrak{e}_l \leq \mathbf{add}(B)$

Proof. Combine Theorem 13 with the facts that $\mathfrak{e}_l \leq \mathfrak{e}$ (as we remarked right after defining these cardinals) and $\mathfrak{e}_l \leq \mathfrak{b}$, which follows from Theorem 10 and Corollary 8. □

Remark. Referring to the intuitive interpretation of predictors, which was described just after their definition, we point out that there is a natural extension of the concept of predictor that matches even better the intuition of a guessing strategy. Instead of having a fixed set D_π of numbers n for which the strategy tries to predict the value $x(n)$ of an unknown function, given the list $x \restriction n$ of prior values, we could let the decision whether to attempt a prediction of $x(n)$ for a particular n depend also on $x \restriction n$. Of course, we should either require that, for each x, the strategy attempts infinitely many predictions or else declare that any x for which this fails is deemed to have evaded π. The cardinal $\mathfrak{e}+$ associated to this concept, namely the smallest number of functions $\omega \to \omega$ needed to evade all ω-valued predictors of this generalized sort, is clearly $\geq \mathfrak{e}$. All the upper bounds we have given for $\mathfrak{e}$ apply also to $\mathfrak{e}+$, with only minor modifications of the proofs.

We also remark that, if we were to weaken the definition of "predicts" by requiring only that infinitely many (rather than all but finitely many) predictions are correct, then the new $\mathfrak{e}$ would be $\geq$ the old one, but it would still be $\leq \mathfrak{d}$ and its minimum with $\mathfrak{b}$ would still be $\leq \mathbf{add}(B)$, by the same proofs as for the original $\mathfrak{e}$.

5. The Slender Property

In this section, we relate the results proved earlier to results and questions about the slender property of complete Boolean algebras, introduced by Eda [10]. Unlike the preceding sections of this paper, the present section presupposes familiarity with forcing in the context of Boolean-valued models of set theory. We refer to Bell [4] for background information and notation.

Instead of using Eda's definition of the slender property, we use an equivalent formulation, essentially given by Proposition 6 of [10].

Definition. A complete Boolean algebra $\mathcal{B}$ has the *slender property* if it is true in $V^{\mathcal{B}}$ (with truth value 1) that $\check{\Pi}$ (the Π of the ground model) exhibits the Specker phenomenon witnessed by the sequence of standard unit vectors e_n.

The methods used to prove upper and lower bounds for $\mathfrak{se}$ in the preceding sections can also be used to obtain positive and negative results (respectively) about the slender property. The two theorems in this section are based in this manner on Theorems 2 and 12.

Theorem 15. *If forcing with $\mathcal{B}$ does not adjoin a dominating real, then $\mathcal{B}$ has the slender property.*

To say that forcing with $\mathcal{B}$ does not adjoin a dominating real means that in $V^{\mathcal{B}}$, with truth value 1, no single real is $\geq^*$ all the reals of the ground model.

Proof of Theorem. The proof of Theorem 2 actually shows the following (where we change the notation $\mathcal{B}$ used there to $\mathcal{Z}$ to avoid confusion with the Boolean algebra in the present context). If a family $\mathcal{Z}$ of non-decreasing, nowhere zero functions $\omega \to \omega$ is unbounded (i.e., no single function is $\geq^*$ all the members of $\mathcal{Z}$), and if G is a pure subgroup of Π containing, for each $f \in \mathcal{Z}$, the x defined by the recursion (11) of Section 2, then G exhibits the Specker phenomenon witnessed by (e_n). We apply this fact inside $V^{\mathcal{B}}$, taking as $\mathcal{Z}$ the family of all non-decreasing, nowhere zero functions $\omega \to \omega$ in the ground model (an unbounded family by hypothesis), and taking as G the Π of the ground model. Since the ground model is closed under primitive recursions such as (11) and under division by constants, this G satisfies all the assumptions above, and we conclude that it exhibits the Specker phenomenon witnessed by (e_n). That is, $\mathcal{B}$ has the slender property. $\square$

This theorem subsumes Eda's result [10, Theorem 3] (on whose proof that of Theorem 2 was modeled) that any Boolean algebra satisfying the (ω, ω) weak distributive law (i.e., adjoining no unbounded reals) has the slender property, Kamo's result [16, Theorem 2] that the Cohen algebra (Borel subsets of the real line modulo first category sets) has the slender property, and the following improvement of Theorem 3 of [16] (an improvement because we have $\mathfrak{d}$ instead of $\mathfrak{b}$).

Corollary 16. *If a complete Boolean algebra has a dense subset of cardinality $< \mathfrak{d}$, then it has the slender property.*

Proof. The hypothesis implies that forcing with the algebra adjoins no dominating reals; see [6, Lemma 8]. □

Theorem 17. *Suppose $\mathcal{F}$ is a family of measure zero sets such that every measure zero set is a subset of a member of $\mathcal{F}$. Suppose further that the cardinality of $\mathcal{F}$ is collapsed to $\aleph_0$ in $V^{\mathcal{B}}$. Then $\mathcal{B}$ does not have the slender property.*

Proof. Since every measure zero set has a Borel superset of measure zero, we may assume that $\mathcal{F}$ consists of Borel sets. The Borel sets in $V^{\mathcal{B}}$ with the same Borel codes as the members of $\mathcal{F}$ constitute, in $V^{\mathcal{B}}$, a countable family of measure zero sets, having among their subsets all the measure zero Borel sets coded in the ground model. (For the notions of coding and the absoluteness results used here, see [22].) Thus, the union U of all measure zero Borel sets coded in the ground model is, in $V^{\mathcal{B}}$, a set of measure zero.

Lemma 7 of [12] provides two operations, one sending each function $f : \omega \to \omega$ to a measure zero Borel set V_f, and the other sending each measure zero set E to a slalom s_E, such that if $V_f \subseteq E$ then f goes through s_E. Furthermore, inspection of the proof in [12] reveals that the code for V_f is obtained from f by an absolute construction. Thus, as f ranges over all functions in the ground model, V_f ranges over some ground model coded Borel sets of measure zero. As all the latter are subsets of U, all f's in the ground model must go through the slalom s_U.

Now that we have a slalom through which all ground model functions $f : \omega \to \omega$ must go, we can use the proof of Theorem 12 (particularly the last three paragraphs) to produce, in $V^{\mathcal{B}}$, a forgetful linear predictor π that predicts every such f. Then, using π, proceed as in the proof of Theorem 10 to define an endomorphism C of Π mapping $\check{\Pi}$ into Σ, and define $h : \check{\Pi} \to \mathbb{Z}$ by $h(x) = \sum_k C(x)(k)$. Since π is forgetful, one easily sees that $h(e_n) \neq 0$ whenever $n \in D_\pi$. Thus, h shows that $\check{\Pi}$ does not exhibit the Specker phenomenon witnessed by (e_n). □

It is consistent with the negation of the continuum hypothesis that there is an $\mathcal{F}$ of cardinality $\aleph_1$ satisfying the first hypothesis of Theorem 17. (Several models for this are indicated in [3], for example the model obtained by an $\aleph_2$ stage countable support iteration of Sacks forcing over a model of the continuum hypothesis.) In such a model, any complete Boolean algebra collapsing $\aleph_1$ to $\aleph_0$ satisfies the second hypothesis of Theorem 17 and therefore lacks the slender property. This answers a question of Kamo [16].

6. Questions

The results presented in this paper raise a multitude of questions, for we have introduced numerous cardinals and many possible connections between them remain undecided. Here are some rather strong conjectures, whose proof would greatly simplify our picture of these cardinals.

(1) $\mathfrak{e}_l = \mathfrak{s}\mathfrak{e}$. (Notice that this would imply equality of cardinals (2) through (5) in Corollary 8.)

(2) $\mathfrak{e} \leq \mathfrak{b}$.
(3) $\mathfrak{se} = \mathbf{add}(B)$.

A more specific problem, whose solution might throw considerable light on the general situation, is to compute the value of $\mathfrak{se}$ in the model obtained by an $\aleph_2$-stage, countable support iteration of Mathias forcing over a model of the generalized continuum hypothesis. This model has $\mathfrak{b}$ and both **unif**'s equal to $\aleph_2$ while both **cov**'s are $\aleph_1$. Therefore $\mathfrak{e} = \aleph_1$ by Theorem 13, but our results do not determine $\mathfrak{se}$.

Addendum. Since this paper was written, Jörg Brendle has obtained numerous results about prediction and evasion, including the consistency of $\mathfrak{e} < \mathbf{add}(B)$. Since $\mathfrak{e}_l \leq \mathfrak{e}$, it follows that conjectures (1) and (3) cannot both be provable.

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