Is There a Core Class for Almost Free Groups of Size $\aleph_1$?

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Abstract

We discuss the prospects for finding a "core class," i.e., a well-behaved class of non-free abelian groups of cardinality $\aleph_1$ such that every non-free abelian group of cardinality $\aleph_1$ has a subgroup in the core class.

1 Introduction

Let $\mathcal{F}$ be a family of abelian groups. Throughout this paper, all groups under consideration will be abelian and all families of groups will be closed under isomorphism. A *core class* $\mathcal{C}$ for $\mathcal{F}$ is intended to provide sensible reasons for the non-freeness of the non-free groups in $\mathcal{F}$. This means:

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1. $\mathcal{C}$ consists of some non-free groups in $\mathcal{F}$.

2. Every non-free group in $\mathcal{F}$ has a subgroup in $\mathcal{C}$.

3. $\mathcal{C}$ is sensible.

Of course (3) needs to be made precise, but we can define a *weak core class* to be a $\mathcal{C}$ satisfying (1) and (2). For example, the class of all non-free groups in $\mathcal{F}$ is a trivial example of a weak core class for $\mathcal{F}$. For most $\mathcal{F}$'s we would expect (3) to prevent this example from being a core class.

One way to make (3) precise is the following minimality requirement.

Definition 1 A *minimal* core class for $\mathcal{F}$ is a weak core class $\mathcal{C}$ for $\mathcal{F}$ such that no proper subclass of $\mathcal{C}$ is a weak core class for $\mathcal{F}$.

Another approach, not really precise but less vague than (3), is the following.

Definition 2 A *comprehensible* core class for $\mathcal{F}$ is a weak core class $\mathcal{C}$ for $\mathcal{F}$ such that there is a reasonable criterion (e.g., invariants) for isomorphism of groups from $\mathcal{C}$.

Example 3 If $\mathcal{F}$ is the class of finitely generated (abelian) groups, then the class of finite cyclic groups is a weak core class for $\mathcal{F}$, and so is the subclass of cyclic groups of prime order. The latter is a minimal core class, and both of these weak core classes are comprehensible core classes, since the cardinality of a group is surely a reasonable invariant.

Example 4 If $\mathcal{F}$ is the class of countable, torsion-free groups, then by Pontryagin's criterion [8, Theorem IV.2.3] the members of $\mathcal{F}$ that have finite rank but are not free (or, equivalently, are not finitely generated) form a weak core class. There seems to be no good way to make it minimal or comprehensible (see [12] for information about how hard it is to decide whether two such groups are isomorphic), but it may nevertheless deserve to be called a core class.

Recall that the rank-one torsion-free groups admit a reasonable isomorphism criterion [1], being classified by their characteristics. (The type of an element, a function from the set of primes to $\mathbb{N} \cup \{\infty\}$, tells how often the element is divisible by each prime; two types count as the same characteristic if they differ only by finite changes in finitely many components.) We can obtain a smaller weak core class by discarding all groups of rank > 1 except those in which all the rank 1 subgroups are free (so the characteristic is $(0, 0, \dots)$). More generally, we can discard all those rank k groups that have a non-free subgroup of lower rank.

Ideally, we would like a core class for the family of all abelian groups. Such a class would consist of

1. a core class for the finitely generated groups, presumably just the cyclic groups of prime order as in the first of the preceding examples,

2. a core class for the countable, torsion-free groups, presumably some subclass of the class of finite-rank non-free groups as in the second of the preceding examples, and

3. a core class for the $\aleph_1$-free groups.

In this paper, we shall be concerned with only the lowest level of (3), groups of size $\aleph_1$. In what follows, let

$$\mathcal{F} = \{\aleph_1\text{-free groups of cardinality } \aleph_1\}$$

and let $\mathcal{C}$ be a weak core class for this $\mathcal{F}$. We shall discuss the prospects for getting $\mathcal{C}$ to be a core class in the sense of either minimality or comprehensibility.

Variant 5 It may be of some interest to consider a variant of the notions of core class and weak core class, requiring that every non-free group in $\mathcal{F}$ has a *pure* subgroup in $\mathcal{C}$. We shall refer to this as the *pure version* of the theory, and we shall describe it in brief remarks labeled (like this one) "Variant." Most of the time, the pure version of the theory is quite similar to the main (impure) version, but we shall encounter one aspect in which they diverge strongly.

Remark 6 Part of the motivation for this work comes from the papers [4, 10], which produced a somewhat comprehensible core class in the context of p-groups. In that context, the appropriate analog of "free" is "direct sum of cyclic groups." It is shown in [4] that every p-group without elements of infinite height either is a direct sum of cyclic groups or has a $p^{\omega+1}$-projective subgroup that is not a direct sum of cyclic groups. Furthermore, [10] gives an isomorphism criterion for $p^{\omega+1}$-projective groups, namely that two such groups are isomorphic if and only if their socles are isomorphic as valuated vector spaces. We used the phrase "somewhat comprehensible" for this, because it is not clear how to test valuated vector spaces for isomorphism.

2 Minimality

Define a pre-order $\preceq$ on the class of all abelian groups by

$$A \preceq B \iff A \text{ is isomorphic to a subgroup of } B,$$

and let $\approx$ be the associated equivalence relation

$$A \approx B \iff A \preceq B \preceq A.$$

Then the equivalence class of a group A is minimal (with respect to the induced partial order) among equivalence classes of non-free groups if and only if A is not free and every non-free subgroup of A contains an isomorphic copy of A. We call such a group *minimal.*

Notice that the equivalence relation $\approx$ defined here is strictly coarser than the relation of isomorphism, which we denote as usual by $\cong$.

Proposition 7 *If $\mathcal{C}$ is a minimal core class (for the class $\mathcal{F}$ of $\aleph_1$-free groups of size $\aleph_1$ as above), then $\mathcal{C}$ consists of one isomorphism class from each $\approx$-class of minimal groups in $\mathcal{F}$.*

Proof If A is minimal in $\mathcal{F}$ then, being a weak core class, $\mathcal{C}$ must contain some non-free $B \preceq A$. By minimality, $B \approx A$, so $\mathcal{C}$ contains a member of the $\approx$-class of A and hence a whole isomorphism class of such members.

Conversely, if $A \in \mathcal{C}$ then A is minimal. For suppose $B \preceq A$ and B is not free. $\mathcal{C}$ must contain some $C \preceq B$. As $\mathcal{C}$ is a minimal core class, the subclass $\mathcal{C}'$ obtained by removing the isomorphism class of A must fail to be a weak core class. So there is a non-free $G \in \mathcal{F}$ whose subgroups in $\mathcal{C}$ (which must exist as $\mathcal{C}$ is a weak core class) are all isomorphic to A. So $C \preceq B \preceq A \preceq G$. But then from $C \in \mathcal{C}$ and $C \subseteq G$ we obtain $C \cong A$. This proves that every $B \preceq A$ is also $\succeq A$, so A is minimal.

Thus, $\mathcal{C}$ must consist of some isomorphism classes of minimal groups $\in \mathcal{F}$, at least one from each minimal $\approx$-class. It cannot contain non-isomorphic $A \approx B$, for then removing the isomorphism class of B would leave a smaller weak core class, contrary to the minimality of $\mathcal{C}$. □

Corollary 8 *There is a minimal core class for $\mathcal{F}$ if and only if every non-free member of $\mathcal{F}$ is $\succeq$ a minimal member.*

We shall show next that it is consistent, relative to large cardinals, that there is no minimal core class for $\mathcal{F}$. We do not know whether it is also consistent that there is a minimal core class or even that there is a minimal group in $\mathcal{F}$. In other words, the following question remains open.

Question 9 Assume A is an $\aleph_1$-free abelian group of cardinality $\aleph_1$, and assume that every non-free subgroup of A contains an isomorphic copy of A. Does it follow that A is free?

Let PFA be the proper forcing axiom [8, page 170]. It is known that PFA is consistent with the usual (ZFC) axioms of set theory provided supercompact cardinals are consistent.

Theorem 10 *PFA implies the existence of a non-free group in $\mathcal{F}$ that has no subgroup minimal in $\mathcal{F}$.*

Proof The proof consists mainly of quoting results from [8]. By Theorem VII.1.3, let A be a strongly $\aleph_1$-free (hence $\aleph_1$-free), non-free abelian group of cardinality $\aleph_1$. Then $A \in \mathcal{F}$, and we claim that A has no subgroup minimal in $\mathcal{F}$.

Let B be any subgroup of A; we shall show it isn't minimal in $\mathcal{F}$. By Exercise VIII.1, B is strongly $\aleph_1$-free. If it were free, then it wouldn't be minimal and we'd be done; so assume B is not free. By Theorem VIII.3.3 (which uses PFA), B is $\aleph_1$-separable. Since it isn't free, its Γ-invariant is the equivalence class $\tilde{E}$ of a stationary set $E \subseteq \aleph_1$, by Proposition IV.1.7. E can be split into two (or even $\aleph_0$) stationary pieces, and each of these is the Γ-invariant of some subgroup (in fact, of some direct summand) of B by Corollary VIII.3.4 (using PFA again). So fix a subgroup $C \subsetneq B$ with $\Gamma(C) \lneqq \Gamma(B)$. Every subgroup of C has Γ-invariant $\leq \Gamma(C) \lneqq \Gamma(B)$ and thus cannot be isomorphic to B. Therefore B is not minimal in $\mathcal{F}$. □

Corollary 11 *PFA implies that there is no minimal core class for $\mathcal{F}$.*

Variant 12 For the pure variant of the theory, redefine $A \preceq B$ to mean that A is isomorphic to a pure subgroup of B. The equivalence relation $\approx$

is defined as before but in terms of the new $\preceq$. Minimality of A now means that every non-free, pure subgroup of A has a pure subgroup isomorphic to A. With these modifications, all the proofs in this section still work. The analog of Question 9 is also open.

Notice that the proof of Theorem 10 actually establishes a stronger form that mixes the pure and impure versions: PFA implies the existence of a non-free $\aleph_1$-free group A such that every non-free subgroup B has a pure non-free subgroup (in fact a direct summand) C no subgroup of which is isomorphic to B.

3 Comprehensibility

The definition of comprehensibility, involving a "reasonable" isomorphism criterion, is too vague to be the subject of proofs. We can make it more precise by importing from Shelah's classification theory [13] the idea that a class of structures admits no reasonable classification if it contains 2^κ non-isomorphic structures of cardinality κ for all infinite cardinals κ. Since we are dealing only with groups of size $\aleph_1$, we might take non-classifiability for a class $\mathcal{C}$ to mean the existence of $2^{\aleph_1}$ non-isomorphic groups in $\mathcal{C}$. We shall show that a weak core class $\mathcal{C}$ for $\mathcal{F}$ must contain $2^{\aleph_1}$ non-isomorphic groups; in this sense, $\mathcal{F}$ has no comprehensible core class.

In fact, we shall prove somewhat more, to compensate for the distortion of Shelah's ideas in using only one cardinal $\aleph_1$ instead of all cardinals. We shall describe a procedure Φ for converting any $\aleph_1$-free group A of size $\aleph_1$ into a subset $\Phi(A)$ of $\aleph_1$, such that isomorphic groups produce the same set, and we shall show that, as A ranges over any weak core class $\mathcal{C}$ for $\mathcal{F}$, $\Phi(A)$ ranges over *all* subsets of $\aleph_1$. Thus $\mathcal{C}$ must contain groups that encode, via Φ, arbitrary subsets of $\aleph_1$.

To describe Φ, we begin with two pieces of coding apparatus.

First, let $\{0,1\}^*$ be the set of all finite sequences of zeros and ones. We often visualize this set as a tree, ordered by the relation "initial segment of." It is countable, so fix a one-to-one map of it into the set of prime numbers. Let p_s be the prime associated to the sequence s.

Second, fix for each countable ordinal number α a one-to-one map $j_\alpha : \alpha \to \mathbb{N}$. Here, as usual, an ordinal α is identified with the set of all smaller ordinals.

Notice that every set $S \subseteq \mathbb{N}$ gives rise to a path $\hat{S}$ through the tree

$\{0,1\}^*$, namely the set of all initial segments of the characteristic function of S. It is trivial that we can recover S from $\hat{S}$, since the union of all the finite sequences in $\hat{S}$ is just the characteristic function of S. In exactly the same way, we can recover S if we are merely given an infinite subset of $\hat{S}$. Better yet, we can recover S if we are given a set $X \subseteq \{0,1\}^*$ that differs from an infinite subset of $\hat{S}$ by only finitely many elements. Indeed, in this situation we can first recover $\hat{S}$ as

$$\hat{S} = \{s \in \{0,1\}^* : s \text{ is an initial segment of infinitely many members of } X\}$$

and then recover S from $\hat{S}$. We describe this situation by saying that S is *tree-coded* by X.

We are now ready to describe the action of Φ on an $\aleph_1$-free group A of size $\aleph_1$. Given A, proceed as follows.

Step 1: Fix a continuous filtration $(A_\nu)_{\nu<\aleph_1}$ of A by countable subgroups. (The final result will be independent of the choice of filtration.)

Step 2: For each $\nu < \aleph_1$, if the quotient A/A_ν has a non-free rank-1 subgroup B such that $(A/A_\nu)/B$ is $\aleph_1$-free, then notice that B is uniquely determined because it consists of exactly those elements $b \in A/A_\nu$ that have non-zero characteristic (i.e. those elements such that the pure closure $\langle b\rangle_*$ is not isomorphic to $\mathbb{Z}$). Choose some $b \in B - \{0\}$ and let

$$X_\nu = \{s \in \{0,1\}^* : b \text{ is divisible by } p_s \text{ in } B\}.$$

A different choice of b would differ from b by a rational factor (since B has rank 1), so X_ν would change by only a finite amount. (Finite changes in X_ν won't change $\Phi(A)$.) If no such B exists, then set $X_\nu = \varnothing$.

Step 3: If X_ν differs only finitely from an infinite subset of a path through the tree $\{0,1\}^*$, then let S_ν be the subset of $\mathbb{N}$ tree-coded by X_ν. If, on the other hand, X_ν doesn't tree-code any set, then let $S_\nu = \varnothing$. Since tree-coding is unaffected by finite changes in the coding set, we have kept the promise in Step 2 that finite changes in X_ν won't matter in the final result $\Phi(A)$. (This is why tree-coding was defined to be impervious to finite changes.)

Step 4: If there is a unique nonempty $Y \subseteq \aleph_1$ such that

$$\{\nu < \aleph_1 : Y \cap \nu = j_\nu^{-1}(S_\nu)\}$$

is stationary, then let $\Phi(A) = Y$. Otherwise, let $\Phi(A) = \varnothing$.

Had we started with a different filtration of A, there would be a closed unbounded set C of ordinals ν such that the two filtrations have the same

A_ν (see [8, II.4.12]) and therefore have the same S_ν. Since a subset of $\aleph_1$ is stationary if and only if its intersection with C is stationary, it is clear that Step 4 will yield the same $\Phi(A)$ no matter which filtration we began with. This observation completes the definition of Φ.

Theorem 13 *If $\mathcal{C}$ is a weak core class for $\mathcal{F}$ and if $Y \subseteq \aleph_1$ then there is a group $A \in \mathcal{C}$ with $\Phi(A) = Y$.*

Proof In view of the definition of weak core class, it suffices to construct, for any given $Y \subseteq \aleph_1$, a non-free group $G \in \mathcal{F}$ such that every non-free subgroup A of G has $\Phi(A) = Y$.

So let $Y \subseteq \aleph_1$ be given. For each countable limit ordinal δ, let

$$Q_\delta = j_\delta(Y \cap \delta),$$

a subset of $\mathbb{N}$, let $\hat{Q}_\delta$ be (as before) the corresponding path through $\{0,1\}^*$, and let

$$P_\delta = \{p_s : s \in \hat{Q}_\delta\}$$

be the corresponding set of primes. By the method of [8, VIII.1.1], build an $\aleph_1$-free (in fact $\aleph_1$-separable), non-free (in fact having Γ-invariant 1) group G of cardinality $\aleph_1$, equipped with a continuous filtration by countable (and therefore free) subgroups $(G_\nu)_{\nu<\aleph_1}$ such that

- $G/G_{\nu+1}$ is $\aleph_1$-free for all ν, and
- for limit δ, $G_{\delta+1}/G_\delta$ is isomorphic to the direct sum of a free abelian group and the rank-1 group

$$\mathbb{Q}_{P_\delta} = \{\frac{m}{n} \in \mathbb{Q} : n \text{ is square-free and all its prime factors } \in P_\delta\}.$$

This construction is exactly like the one in [8] except that the denominators in $\mathbb{Q}_{P_\delta}$ are products of distinct primes from P_δ rather than powers of a fixed prime p_δ.

Now let A be any non-free subgroup of G; we must show that $\Phi(A) = Y$, and we shall do this by following, step for step, the definition of Φ.

At Step 1, we could choose any filtration of A; we use $A_\nu = G_\nu \cap A$.

At Step 2, we must analyze the structure of A/A_δ for all limit ordinals $\delta < \aleph_1$. (Officially, Step 2 looks at all ordinals, not just limits, but the limit ordinals form a closed unbounded set, so only limit ordinals will be relevant when we consider stationary sets at Step 4.) We have

$$\frac{A}{A_\delta} = \frac{A}{G_\delta \cap A} \cong \frac{G_\delta + A}{G_\delta} \subseteq \frac{G}{G_\delta}.$$

We use this to identify A/A_δ with a subgroup of G/G_δ.

By the construction of G, the quotient G/G_δ includes the subgroup $G_{\delta+1}/G_\delta$ which has the form $\tilde{B} \oplus \text{free}$, where $\tilde{B} \cong \mathbb{Q}_{P_\delta}$.

We claim that $(G/G_\delta)/\tilde{B}$ is $\aleph_1$-free. Indeed, to verify that all its countable subgroups are free, it suffices to consider those of the form $(G_\mu/G_\delta)/\tilde{B}$ for $\delta < \mu < \aleph_1$. In the exact sequence

$$0 \to \frac{G_{\delta+1}}{G_\delta} \to \frac{G_\mu}{G_\delta} \to \frac{G_\mu}{G_{\delta+1}} \to 0,$$

the term $G_\mu/G_{\delta+1}$ is free, as the construction of G makes $G/G_{\delta+1}$ $\aleph_1$-free. So the sequence splits and

$$\frac{G_\mu}{G_\delta} \cong \frac{G_{\delta+1}}{G_\delta} \oplus \text{free} \cong (\tilde{B} \oplus \text{free}) \oplus \text{free}.$$

So $(G_\mu/G_\delta)/\tilde{B}$ is free, and the claim is thereby proved.

Recall that we have identified A/A_δ with a subgroup of G/G_δ. Let $B \subseteq A/A_\delta$ be the intersection of $\tilde{B} \subseteq G/G_\delta$ with A/A_δ. As $\tilde{B}$ has rank 1, B must have rank 1 or 0. Also, $(A/A_\delta)/B$ is a subgroup of $(G/G_\delta)/\tilde{B}$ and is therefore $\aleph_1$-free by the claim above. Thus, if B isn't free then it is the group called B in Step 2 of the definition of $\Phi(A)$.

Define

$$E = \{\delta < \aleph_1 : \delta \text{ a limit and } B \text{ is not free}\}.$$

Since $(A/A_\delta)/B$ is $\aleph_1$-free, we see that A/A_δ is $\aleph_1$-free if and only if B is free, i.e., if and only if $\delta \notin E$. Therefore E, or rather its equivalence class modulo nonstationary differences, is $\Gamma(A)$. Since A isn't free, E is stationary.

For limit ordinals $\delta \notin E$, A/A_δ is $\aleph_1$-free, so Step 2 of the definition of $\Phi(A)$ will find no appropriate B and will set $X_\delta = \varnothing$.

For $\delta \in E$, Step 2 finds the B that we obtained above, a subgroup of $\tilde{B}$ and therefore isomorphic to a non-free (since $\delta \in E$) subgroup of $\mathbb{Q}_{P_\delta}$. Thus, $B \cong \mathbb{Q}_T$ for some set T of primes almost (i.e., modulo finite sets) included in P_δ. And T is infinite because B isn't free. So X_δ is, modulo finite differences, an infinite subset of $\hat{Q}_\delta$.

So for $\delta \in E$ Step 3 of the definition of $\Phi(A)$ produces $S_\delta = Q_\delta$. For limit ordinals $\delta \notin E$ it produces $S_\delta = \varnothing$.

Finally, in Step 4 we have

$$j_\delta^{-1}(S_\delta) = j_\delta^{-1}(Q_\delta) = Y \cap \delta$$

for $\delta \in E$ and $j_\delta^{-1}(S_\delta) = \varnothing$ for limit ordinals $\delta \notin E$. So, if Y is nonempty, then it is the unique set with all the properties required in Step 4 and therefore $\Phi(A) = Y$. If, on the other hand, Y is empty, then there is no set as required in Step 4, so $\Phi(A) = \varnothing = Y$. Thus, in every case, $\Phi(A) = Y$. $\square$

Variant 14 Since every weak core class in the pure sense is also a weak core class in the original sense, Theorem 13 immediately implies its pure variant.

4 Basic Subgroups

In this section, we consider whether particularly nice groups, namely those with basic subgroups, could serve as a core class. We begin by recalling the definition (see [5, 7] for more information).

Definition 15 A *basic subgroup* of a torsion-free abelian group G is a pure, free subgroup B of G such that G/B is divisible.

The next two theorems exhibit a difference between the main theory (dealing with general subgroups) and the pure version.

Theorem 16 *Every non-free torsion-free abelian group of infinite rank has a non-free subgroup that has a basic subgroup.*

Proof Let G be any non-free torsion-free group of infinite rank κ, let $\bar{G} = G \otimes \mathbb{Q}$ be its divisible hull (in which G is canonically embedded by $g \mapsto g \otimes 1$), and let S be a basis for $\bar{G}$ as a rational vector space. Multiplying elements of S by suitable integers, we arrange that $S \subseteq G$. Partition S into a countable infinity of subsets S_n, each of cardinality κ, let $\bar{G}_n$ be the subspace of $\bar{G}$ spanned by $\bigcup_{k \leq n} S_k$, and let $G_n = \bar{G}_n \cap G$. Then G is the union of the countable increasing sequence of pure subgroups G_n. If each G_n were free, then Hill's theorem [11] would imply that G is free, contrary to hypothesis. So not all G_n are free. Fix, for the remainder of this proof, an n such that

G_n is not free. Notice that, by our choice of the sets S_n, the cardinality of $\bar{G}_n$ is κ.

Let F be the subgroup of G generated by $T = \bigcup_{k>n} S_n$, the part of the basis S not in G_n. Being part of a basis, T is linearly independent, so it generates F freely. Notice also that T has cardinality κ and that $G_n \cap F = \{0\}$.

We can extend the identity map of G_n to a homomorphism h of $G_n \oplus F$ onto $\bar{G}_n$. Indeed, we can take an arbitrary function from T onto $\bar{G}_n$ (which exists because both have cardinality κ) and extend it linearly to map F onto $\bar{G}_n$. Then this extension and the identity map of G_n together determine the required homomorphism on $G_n \oplus F$.

Since h is one-to-one on G_n, its kernel K is disjoint from G_n. But if a subgroup of a direct sum is disjoint from one summand then it is isomorphic (by projection) to a subgroup of the other summand. Thus, K is isomorphic to a subgroup of F and is therefore free. Furthermore, since $(G_n \oplus F)/K$ is isomorphic to $\bar{G}_n$, a torsion-free divisible group, it follows that K is a basic subgroup of $G_n \oplus F$. □

Corollary 17 *The non-free, $\aleph_1$-free groups of size $\aleph_1$ that have basic subgroups form a weak core class for the family $\mathcal{F}$ of all $\aleph_1$-free abelian groups of size $\aleph_1$.*

Variant 18 In contrast to what happened in previous sections, this last result is completely reversed in the pure variant. Notice that the preceding proof does not apply to the pure variant, because F and therefore $G_n \oplus F$ need not be pure in G.

Theorem 19 *There is a non-free, $\aleph_1$-free group of size $\aleph_1$ (a pure subgroup of $\mathbb{Z}^{\aleph_0}$) that has no non-free pure subgroup with a basic subgroup.*

Proof The group we construct to have the properties in the theorem will be a subgroup of the group $\Pi = \mathbb{Z}^{\aleph_0}$ of sequences $\langle x_n \rangle_{n \in \mathbb{N}}$ of integers with componentwise addition. We begin by recalling some well-known facts about Π and its subgroup Σ consisting of those sequences in which only finitely many terms are non-zero. All the facts we cite here (except for obvious ones) can be found in [9, pages 94 and 177]; the original sources are [1, 2, 3, 14]. Σ is the free abelian group generated by the sequences in which a single term is 1 and all the rest are 0. All countable subgroups of Π are free, but Π itself is not free.

The quotient Π/Σ is the direct sum of a divisible part (a $2^{\aleph_0}$-dimensional vector space over the rationals) and the direct product, over all prime numbers p, of $(J_p)^{\aleph_0}$, where J_p is the additive group of p-adic integers. Since J_p is q-divisible for all primes $q \neq p$, it can be viewed as a module over the ring $\mathbb{Z}_{(p)}$ of rational numbers whose denominators are prime to p. So $(J_p)^{\aleph_0}$ is also a $\mathbb{Z}_{(p)}$-module, and this module contains pure, free $\mathbb{Z}_{(p)}$-submodules of rank equal to the cardinality of the continuum.

Let $\hat{G} \subseteq \Pi/\Sigma$ be a pure, free $\mathbb{Z}_{(p)}$-submodule of $(J_p)^{\aleph_0}$ of rank $\aleph_1$, and let G be its pre-image in Π. We shall show that this G has the properties required in the theorem.

G is $\aleph_1$-free because Π is. Its cardinality is $\aleph_1$ because $\hat{G} \cong G/\Sigma$, being a free module of rank $\aleph_1$ over a countable ring, has cardinality $\aleph_1$ and G must have the same cardinality because Σ is countable. We check next that G is not free. If it were free, then its countable subgroup Σ would be included in a countable direct summand G_0, and therefore G/Σ would be the direct sum of a countable group G_0/Σ and a free group. But $G/\Sigma \cong \hat{G}$ is q-divisible for primes $q \neq p$ and therefore has no free summand.

To complete the proof, suppose toward a contradiction that G had a non-free, pure subgroup H which had, in turn, a basic subgroup B.

Temporarily fix a prime $q \neq p$, and let D_q be the subgroup of Π consisting of those sequences $\langle x_n \rangle$ in which the terms are eventually divisible by arbitrarily large powers of q. An equivalent description is that D_q is the pre-image in Π of the q-divisible part of Π/Σ. In view of the structure of Π/Σ as described above, that q-divisible part is the direct sum of the divisible part of Π/Σ and the product of all the $(J_r)^{\aleph_0}$ for primes $r \neq q$. In particular, it includes $(J_p)^{\aleph_0}$ and therefore $\hat{G}$ as pure subgroups. So D_q includes G and therefore H and B as pure subgroups. But it was shown in [6, Example 3.2] that all pure, free subgroups of D_q are countable. So we know that B must be countable.

Since both B and Σ are countable, so is $B + (\Sigma \cap H)$. On the other hand, H is uncountable, because it is a non-free subgroup of Π. So the quotient $H/(B + (\Sigma \cap H))$ is uncountable. This quotient is also a homomorphic image of the divisible (because B is basic) group H/B, so it is also divisible. Of course, since $B \subseteq H$,

$$B + (\Sigma \cap H) = (B + \Sigma) \cap H,$$

and so the quotient that we have shown to be uncountable and divisible can be rewritten as

$$\frac{H}{B+(\Sigma\cap H)} = \frac{H}{(B+\Sigma)\cap H} \cong \frac{H+B+\Sigma}{B+\Sigma} \subseteq \frac{G}{B+\Sigma} \cong \frac{\hat{G}}{\hat{B}},$$

where $\hat{B}$ denotes the image of B in $\hat{G} = G/\Sigma$. Thus, we find that $\hat{G}/\hat{B}$ has an uncountable divisible part.

Remember, however, that $\hat{G}$ is a free module of uncountable rank over $\mathbb{Z}_{(p)}$ and that B and therefore $\hat{B}$ are countable. So $\hat{B}$ lies in a countable direct summand (qua $\mathbb{Z}_{(p)}$-module) of $\hat{G}$. Therefore $\hat{G}/\hat{B}$ is the direct sum of a countable $\mathbb{Z}_{(p)}$-module and a free $\mathbb{Z}_{(p)}$-module. The divisible part of $\hat{G}/\hat{B}$ must lie entirely in the first of these two summands, for a free $\mathbb{Z}_{(p)}$-module has no p-divisible part. Therefore, the divisible part of $\hat{G}/\hat{B}$ is countable. This contradicts the result of the preceding paragraph and thus completes the proof. □

5 Conclusion

The prospects for a core class, either minimal or comprehensible, look rather bleak, though not totally hopeless. For minimality, Theorem 10 implies that, short of refuting the consistency of the large cardinals (supercompact) used in proving the consistency of PFA, the best we can hope for is a consistency result. Proving that an affirmative answer to Question 9 is consistent would be a significant step in that direction, but we have our doubts whether even this much can be achieved.

For comprehensibility, the result in Section 3 makes it unlikely that much can be done. A genuine isomorphism criterion would have to include, in some sense, a complete understanding of all the subsets of $\aleph_1$.

The one bright spot is Theorem 16, saying that a weak core class can consist of groups having at least one pleasant property, namely having a basic subgroup. Unfortunately, the *way* in which these groups ($G_n \oplus F$ in the proof of Theorem 16) have basic subgroups, namely by having a large free summand, does not look promising for an attempt to understand *non-freeness.*

The partial result from [4, 10] for p-groups, described in Remark 6 above, suggests that there might be a weak core class with a "relative comprehensibility" property. That is, the question whether two given groups in the class are isomorphic might be reducible to another question that looks simpler, though it may still turn out to be intractable.

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