

ON THE COFINALITY OF ULTRAPOWERS

ANDREAS BLASS
AND
HEIKE MILDENBERGER

ABSTRACT. We prove some restrictions on the possible cofinalities of ultrapowers of the natural numbers with respect to ultrafilters on the natural numbers. The restrictions involve three cardinal characteristics of the continuum, the splitting number \mathfrak{s} , the unsplitting number \mathfrak{r} , and the groupwise density number \mathfrak{g} . We also prove some related results for reduced powers with respect to filters other than ultrafilters.

1. INTRODUCTION

All ultrafilters considered in this paper are non-principal ultrafilters on the set ω of natural numbers. We shall be concerned with the possible cofinalities $\text{cf}(\mathcal{U}\text{-prod } \omega)$ of ultrapowers of ω with respect to such ultrafilters. We shall show that no cardinal below the groupwise density number \mathfrak{g} (see definition below) can occur as such a cofinality and that at most one cardinal below the splitting number \mathfrak{s} can so occur. The proof for \mathfrak{s} , when combined with a result of Nyikos, gives the additional information that all $P_{\mathfrak{b}+}$ -point ultrafilters are nearly coherent.

In Section 2, we review the necessary terminology and some previously known results. In Section 3, we prove the result concerning \mathfrak{g} . In Section 4, we prove the result concerning \mathfrak{s} , we show that in the statement of that result “at most one cardinal” cannot be improved to “no cardinal,” and we deduce the result about $P_{\mathfrak{b}+}$ -points. Section 5 is devoted to a dual result concerning the unsplitting number, and Section 6 contains some generalizations concerning filters that need not be ultrafilters.

We thank Simon Thomas for posing the question whether $\text{cf}(\mathcal{U}\text{-prod } \omega)$ can ever be smaller than \mathfrak{g} .

2. PRELIMINARIES

We write \exists^∞ and \forall^∞ for the quantifiers “there exist infinitely many” and “for all but finitely many,” respectively. Any ultrafilter (by which we always mean a non-principal ultrafilter on ω) \mathcal{U} will also be used as a quantifier meaning “for almost

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all with respect to \mathcal{U} ," i.e.,

$$(\mathcal{U}n) \varphi(n) \iff \{n \mid \varphi(n)\} \in \mathcal{U}.$$

Thus, the quantifier \mathcal{U} is intermediate between \forall^∞ and \exists^∞ in the sense that $(\forall^\infty n) \varphi(n) \implies (\mathcal{U}n) \varphi(n) \implies (\exists^\infty n) \varphi(n)$ for any predicate φ on natural numbers.

The ultrapower $\mathcal{U}\text{-prod } \omega$ is formed from the set ${}^\omega\omega$ of all functions $f : \omega \rightarrow \omega$ by identifying f with g whenever $(\mathcal{U}n) f(n) = g(n)$. It is linearly ordered by the relation

$$f \leq_{\mathcal{U}} g \iff (\mathcal{U}n) f(n) \leq g(n).$$

By $\text{cf}(\mathcal{U}\text{-prod } \omega)$ we mean the cofinality of this ordering, the smallest cardinality of a subset \mathcal{C} of ${}^\omega\omega$ such that every $f \in {}^\omega\omega$ is $\leq_{\mathcal{U}}$ some $g \in \mathcal{C}$.

This cofinality obviously satisfies $\mathfrak{b} \leq \text{cf}(\mathcal{U}\text{-prod } \omega) \leq \mathfrak{d}$, where the bounding number \mathfrak{b} and the dominating number \mathfrak{d} are defined as follows. (For more information on these and other cardinal characteristics of the continuum, see the survey papers [7,11].) \mathfrak{d} is the minimum size of a family $\mathcal{D} \subseteq {}^\omega\omega$ such that, for each $f \in {}^\omega\omega$ there is some $g \in \mathcal{D}$ satisfying $(\forall^\infty n) f(n) \leq g(n)$. The definition of \mathfrak{b} is the same except that \forall^∞ is replaced with \exists^∞ .

In addition to \mathfrak{b} and \mathfrak{d} , four other cardinal characteristics of the continuum, \mathfrak{s} , \mathfrak{r} , \mathfrak{g} , and $\text{cov}(\mathbb{B})$, will play a role in this paper.

The splitting number \mathfrak{s} is defined as the minimum size of a family \mathcal{S} of subsets of ω such that every infinite $X \subseteq \omega$ is split by some $Y \in \mathcal{S}$ in the sense that both $X \cap Y$ and $X - Y$ are infinite.

Dually, the unsplitting number \mathfrak{r} (sometimes called the refining number or the reaping number) is defined as the minimum size of an unsplittable family, i.e., a family of infinite subsets of ω such that no single set splits them all.

To define \mathfrak{g} , we first need the notion of groupwise density. A family \mathcal{G} of infinite subsets of ω is said to be groupwise dense if it is closed under infinite subsets and finite modifications and if, whenever ω is partitioned into finite intervals, the union of some infinitely many of these intervals is in \mathcal{G} . Then \mathfrak{g} is defined as the minimum number of groupwise dense families with empty intersection. (See [3] for more information about groupwise density and \mathfrak{g} .)

Finally, $\text{cov}(\mathbb{B})$ is defined to be the minimum number of meager sets (i.e., sets of the first Baire category) needed to cover the real line.

We shall be concerned with restrictions, in terms of cardinal characteristics of the continuum, on the possible values of $\text{cf}(\mathcal{U}\text{-prod } \omega)$. The following theorem of Canjar [4,5] and Roitman [9] suggests that the trivial restriction $\mathfrak{b} \leq \text{cf}(\mathcal{U}\text{-prod } \omega) \leq \mathfrak{d}$ is all one can hope for.

Theorem 1 [4,5,9]. *It is consistent (relative to ZFC) that $\mathfrak{b} \ll \mathfrak{d}$ and every regular cardinal κ in the range $\mathfrak{b} \leq \kappa \leq \mathfrak{d}$ occurs as $\text{cf}(\mathcal{U}\text{-prod } \omega)$ for some \mathcal{U} .*

The model used to prove this theorem is the Cohen model, obtained by adding a large number of Cohen-generic reals to any model of ZFC. We shall see that the

trivial lower bound \mathfrak{b} for all $\text{cf}(\mathcal{U}\text{-prod } \omega)$ can be improved in some models (but not in all, by Theorem 1).

Canjar also showed that the trivial upper bound \mathfrak{d} cannot be improved in any model where \mathfrak{d} is regular.

Theorem 2 [6]. *There exists an ultrafilter \mathcal{U} with $\text{cf}(\mathcal{U}\text{-prod } \omega) = \text{cf}(\mathfrak{d})$. In particular, if \mathfrak{d} is regular then it occurs as $\text{cf}(\mathcal{U}\text{-prod } \omega)$ for some \mathcal{U} .*

For any ultrafilter \mathcal{U} and any function $f : \omega \rightarrow \omega$, the image $f(\mathcal{U})$ is defined as the ultrafilter $\{X \subseteq \omega \mid f^{-1}(X) \in \mathcal{U}\}$. (Contrary to our convention, this may be a principal ultrafilter, but only if f is constant on some set in \mathcal{U} ; we shall use $f(\mathcal{U})$ only for finite-to-one functions f , so no real difficulty arises.) Two ultrafilters \mathcal{U} and \mathcal{U}' are said to be nearly coherent if $f(\mathcal{U}) = f'(\mathcal{U}')$ for some finite-to-one functions f and f' . It is shown in [1] that the same relation of near coherence would be obtained if we required in the definition that $f = f'$ and that f be monotone. It is also shown there that near coherence is an equivalence relation and that, whenever \mathcal{U} and \mathcal{U}' are nearly coherent, then $\text{cf}(\mathcal{U}\text{-prod } \omega) = \text{cf}(\mathcal{U}'\text{-prod } \omega)$ (because both of these ultrapowers have cofinal submodels isomorphic to $f(\mathcal{U})\text{-prod } \omega$). The principle of near coherence of filters (NCF), introduced in [1] and proved consistent in [2], asserts that every two non-principal ultrafilters on ω are nearly coherent.

3. GROUPWISE DENSITY GIVES A LOWER BOUND

In this section, we prove the following answer to a question raised by Simon Thomas (private communication).

Theorem 3. *For every non-principal ultrafilter \mathcal{U} on ω , $\text{cf}(\mathcal{U}\text{-prod } \omega) \geq \mathfrak{g}$.*

Proof. Suppose $\mathcal{C} \subseteq {}^\omega\omega$ is cofinal with respect to $\leq_{\mathcal{U}}$. We shall associate to each $f \in \mathcal{C}$ a groupwise dense family \mathcal{G}_f in such a way that the intersection of these families is empty. Thus, we shall have $\mathfrak{g} \leq |\mathcal{C}|$, which establishes the theorem.

By increasing them if necessary, we may assume without loss of generality that all the functions $f \in \mathcal{C}$ satisfy $f(n) \geq n$ for all n . To define \mathcal{G}_f , we first define, for each infinite $X \subseteq \omega$, the function $\nu_X : \omega \rightarrow \omega$ sending each natural number n to the next larger element of X . Then let

$$\mathcal{G}_f = \{X \subseteq \omega \mid X \text{ is infinite and } f <_{\mathcal{U}} \nu_X\}$$

for each $f \in \mathcal{C}$. Since these f 's are cofinal in $\mathcal{U}\text{-prod } \omega$, the intersection of the corresponding \mathcal{G}_f 's must be empty. It is also clear that each \mathcal{G}_f is closed under infinite subsets and under finite modifications. So to verify that each \mathcal{G}_f is groupwise dense, thus completing the proof, it remains only to check that, if f is fixed and if ω is partitioned into finite intervals then the union of some infinitely many of these intervals is in \mathcal{G}_f .

Inductively select intervals I_k from the given partition so that the first element of I_{k+1} is greater than $f(x)$ for all $x \in I_k$ and all smaller x . Let X be the union of the even-numbered intervals, I_{2j} , and Y the union of the odd-numbered ones.

For any natural number p in the interval $(\max I_{n-1}, \max I_n]$, one of $\nu_X(p)$ and $\nu_Y(p)$ (depending on the parity of n) will be $\min I_{n+1}$, which is greater than $f(p)$. Thus, every natural number p , except for the finitely many below $\max I_0$, is in one of the two sets $\{n \in \omega \mid f(n) < \nu_X(n)\}$ and $\{n \in \omega \mid f(n) < \nu_Y(n)\}$. Therefore, one of these sets is in \mathcal{U} , which means that one of X and Y is in \mathcal{G}_f . Since both X and Y are unions of infinitely many intervals from the given partition, this completes the proof that \mathcal{G}_f is groupwise dense and thus completes the proof of the theorem. \square

It is well-known (see [3]) that $\mathfrak{g} \leq \mathfrak{d}$. The following corollary gives an improvement when \mathfrak{d} is singular.

Corollary 4. $\mathfrak{g} \leq cf(\mathfrak{d})$.

Proof. Combine Theorems 2 and 3. \square

Encouraged by Theorem 3, one might look for additional cardinal characteristics that give lower bounds on the possible cofinalities of \mathcal{U} -prod ω . Such characteristics must be $\leq \mathfrak{d}$ and, to avoid trivialities, $\not\leq \mathfrak{b}$. Inspection of the diagrams of cardinal characteristics in [11] provides just two such characteristics, the splitting number \mathfrak{s} and the covering number for category $\text{cov}(\mathbb{B})$. (If one counts the somewhat artificial $\min\{\mathfrak{r}, \mathfrak{d}\}$ as a characteristic, then it also lies in the desired region. The following remark about $\text{cov}(\mathbb{B})$ applies to it as well.) If we add a large number κ of Cohen reals to a model of set theory, then the resulting model has $\text{cov}(\mathbb{B})$ large but has, by the proof of Theorem 1, ultrafilters with $cf(\mathcal{U}\text{-prod } \omega) = \aleph_1$. So $\text{cov}(\mathbb{B})$ cannot serve as a lower bound for $cf(\mathcal{U}\text{-prod } \omega)$. That leaves \mathfrak{s} as a possibility, which we analyze in the next section.

4. THE SPLITTING NUMBER

Unlike \mathfrak{g} , the splitting number \mathfrak{s} is not in general a lower bound for $cf(\mathcal{U}\text{-prod } \omega)$. The proof involves the notion of (pseudo-) P_κ point. An ultrafilter \mathcal{U} is called a P_κ point if, for every family $\mathcal{F} \subseteq \mathcal{U}$ with $|\mathcal{F}| < \kappa$, there is some $A \in \mathcal{U}$ with $A - F$ finite for all $F \in \mathcal{F}$. Pseudo- P_κ points are defined similarly, except that A is not required to be in \mathcal{U} , only to be infinite. We shall need the following results of Nyikos, folklore, and Shelah, respectively. (Although Nyikos's paper [8] is not yet published, Proposition 5 and its proof were in a 1984 letter from Nyikos to the first author.)

Proposition 5 [8]. *If \mathcal{U} is a pseudo- P_κ point and $\kappa > \mathfrak{b}$, then $cf(\mathcal{U}\text{-prod } \omega) = \mathfrak{b}$.*

Proposition 6. *If \mathcal{U} is a pseudo- P_κ point then $\mathfrak{s} \geq \kappa$.*

Proposition 7 [2]. *It is consistent relative to ZFC that $\mathfrak{b} = \aleph_1$ and there is a P_{\aleph_2} -point.*

Since the first two of these propositions are fairly easy, we give their proofs. For Proposition 7, we refer to Theorem 6.1 of [2], which gives (more than) a model with a P_{\aleph_2} -point and another ultrafilter generated by \aleph_1 sets. The latter gives us $\mathfrak{b} = \aleph_1$ because, by a theorem of Solomon [10], no ultrafilter can be generated by fewer than \mathfrak{b} sets.

Proof of Proposition 5. Let \mathcal{U} be a pseudo- P_κ point with $\kappa > \mathfrak{b}$, and let $\mathcal{C} \subseteq {}^\omega\omega$ be a family of cardinality \mathfrak{b} such that for every $f \in {}^\omega\omega$ there is $g \in \mathcal{C}$ with $(\exists^\infty n) f(n) \leq g(n)$. By increasing each $g \in \mathcal{C}$ if necessary, we can assume that g is a monotone non-decreasing function. To complete the proof, we show that \mathcal{C} is cofinal with respect to the linear ordering $\leq_{\mathcal{U}}$ of $\mathcal{U}\text{-prod } \omega$.

Suppose to the contrary that $h \in {}^\omega\omega$ is such that $g \leq_{\mathcal{U}} h$ for all $g \in \mathcal{C}$. This means that the sets $M_g = \{n \in \omega \mid g(n) \leq h(n)\}$ are in \mathcal{U} for all $g \in \mathcal{C}$. Since $|\mathcal{C}| = \mathfrak{b} < \kappa$ and since \mathcal{U} is a pseudo- P_κ point, there is an infinite set $X \subset \omega$ such that each $X - M_g$ is finite. As in the proof of Theorem 3, let $\nu_X(n)$ denote the next member of X after n . By our original choice of \mathcal{C} , there is $g \in \mathcal{C}$ such that $h(\nu_X(n)) < g(n)$ for infinitely many n . For each such n we have, since g is non-decreasing, $h(\nu_X(n)) < g(\nu_X(n))$ and therefore $\nu_X(n) \in X - M_g$. But this applies to infinitely many n , giving infinitely many $\nu_X(n)$, contrary to the fact that $X - M_g$ is finite. \square

Proof of Proposition 6. Let \mathcal{U} be a pseudo- P_κ point and let \mathcal{S} be a family of fewer than κ subsets of ω . We must find an infinite set $X \subseteq \omega$ that is not split by any member of \mathcal{S} .

For each $Y \in \mathcal{S}$, let Y' be Y or $\omega - Y$, whichever is in \mathcal{U} . As \mathcal{U} is a pseudo- P_κ point, there is an infinite X such that $X - Y'$ is finite for all $Y \in \mathcal{S}$. This X is clearly not split by any such Y . \square

Corollary 8. *It is consistent, relative to ZFC, that there is a non-principal ultrafilter \mathcal{U} on ω with $\text{cf}(\mathcal{U}\text{-prod } \omega) < \mathfrak{s}$.*

Proof. In the model given by Proposition 7, let \mathcal{U} be a P_{\aleph_2} point. Its existence gives $\mathfrak{s} \geq \aleph_2$ by Proposition 6, and we also have, by Propositions 5 and 7, $\text{cf}(\mathcal{U}\text{-prod } \omega) = \mathfrak{b} = \aleph_1$. \square

Although Corollary 8 shows that it is consistent for the set of cofinalities of ultrapowers of ω to contain a cardinal below \mathfrak{s} , we shall see that this set cannot contain two cardinals below \mathfrak{s} . That will be a consequence of the following theorem.

Theorem 9. *Suppose \mathcal{U} and \mathcal{U}' are non-principal ultrafilters on ω such that both $\text{cf}(\mathcal{U}\text{-prod } \omega)$ and $\text{cf}(\mathcal{U}'\text{-prod } \omega)$ are smaller than \mathfrak{s} . Then \mathcal{U} and \mathcal{U}' are nearly coherent.*

Proof. Let \mathcal{U} and \mathcal{U}' satisfy the hypotheses of the theorem, and suppose these ultrafilters are not nearly coherent. Let \mathcal{C} and \mathcal{C}' be subfamilies of ${}^\omega\omega$, each of size $< \mathfrak{s}$, and cofinal with respect to $\leq_{\mathcal{U}}$ and $\leq_{\mathcal{U}'}$ respectively. Let \mathcal{D} be the set of functions of the form $\max\{g, g'\}$, where $g \in \mathcal{C}$, $g' \in \mathcal{C}'$, and \max means the pointwise maximum of the functions. Then, for each $f \in {}^\omega\omega$, there is an $h \in \mathcal{D}$ such that both inequalities $f \leq_{\mathcal{U}} h$ and $f \leq_{\mathcal{U}'} h$ hold.

Temporarily fix some $h \in \mathcal{D}$. Partition ω into finite intervals $I_n = [a_n, a_{n+1})$ such that $h(x) < a_{n+1}$ for all $x < a_n$. (It is trivial to produce such $a_0 = 0 < a_1 < a_2 < \dots$ inductively.) Let $p : \omega \rightarrow \omega$ be the function that sends all points in I_n to n , for all n . Since p is finite-to-one and since \mathcal{U} and \mathcal{U}' are not nearly coherent, the ultrafilters $p(\mathcal{U})$ and $p(\mathcal{U}')$ are distinct, so one contains a set whose complement

is in the other. Pulling these sets back along p , we get two sets, say $A \in \mathcal{U}$ and $A' \in \mathcal{U}'$, each a union of some I_n 's, but with no I_n in common.

Define $q(x) = p(x) + 1$. Applying again the fact that \mathcal{U} and \mathcal{U}' are not nearly coherent, we have $q(\mathcal{U}) \neq p(\mathcal{U}')$, so we can get a set in $q(\mathcal{U})$ whose complement is in $p(\mathcal{U}')$. Pulling these sets back along q and p respectively, we get $B \in \mathcal{U}$ and $B' \in \mathcal{U}'$, each a union of some I_n 's, and such that we never have an $I_n \subseteq B$ and $I_{n+1} \subseteq B'$.

Arguing analogously with $p(\mathcal{U}) \neq q(\mathcal{U}')$, we get $C \in \mathcal{U}$ and $C' \in \mathcal{U}'$, each a union of some I_n 's, such that we never have an $I_n \subseteq C'$ and $I_{n+1} \subseteq C$.

Let $D = A \cap B \cap C$ and $D' = A' \cap B' \cap C'$. Then $D \in \mathcal{U}$, $D' \in \mathcal{U}'$, and both are unions of some I_n 's. Furthermore, if a particular I_n is included in D then neither it nor its neighbors $I_{n\pm 1}$ can be included in D' .

Let E be the union of all the I_n 's and I_{n+1} 's such that $I_n \subseteq D$, i.e., the union of the intervals that constitute D and their right neighbor intervals. Define E' similarly from D' , and note that E and E' are disjoint.

We claim that, if X is an infinite subset of ω and if $\nu_X \leq_{\mathcal{U}} h$, then $X \cap E$ is infinite. To see this, notice first that the set $\{k \in \omega \mid \nu_X(k) \leq h(k)\}$, being in \mathcal{U} , must contain infinitely many points $k \in D$ because $D \in \mathcal{U}$. For each of these infinitely many k , there is an element of X , namely $\nu_X(k)$, in the interval $[k, h(k)]$. By our choice of the intervals I_n , this element of X is either in the same interval as k or in its right neighbor. In either case, it is in E because $k \in D$. Thus, we have infinitely many (since k can be arbitrarily large) elements of $X \cap E$, as claimed.

Similarly, if $\nu_X \leq_{\mathcal{U}'} h$, then $X \cap E'$ is infinite and therefore so is $X - E$ since E and E' are disjoint.

Now un-fix h . For each $h \in \mathcal{D}$, the preceding discussion produces an E , which we now call E_h to indicate its dependence on the (previously fixed) h . For any infinite subset X of ω , the function ν_X is majorized, with respect to both $\leq_{\mathcal{U}}$ and $\leq_{\mathcal{U}'}$, by some $h \in \mathcal{D}$. Then the preceding discussion shows that X is split by the corresponding E_h . Therefore, $\{E_h \mid h \in \mathcal{D}\}$ is a splitting family. But this is absurd, as $|\mathcal{D}| < \mathfrak{s}$. \square

Corollary 10. *At most one cardinal smaller than \mathfrak{s} can occur as $\text{cf}(\mathcal{U}\text{-prod}\omega)$.*

Proof. Combine Theorem 9 and the fact that nearly coherent ultrafilters produce ultrapowers of the same cofinality. \square

Corollary 11. *Any two pseudo- $P_{\mathfrak{b}+}$ points are nearly coherent.*

Proof. If two ultrafilters are pseudo- $P_{\mathfrak{b}+}$ points, then the corresponding ultrapowers have cofinality \mathfrak{b} by Proposition 5, and this is smaller than \mathfrak{s} by Proposition 6. So Theorem 9 applies and gives the required near coherence. \square

Remark. For an ultrafilter \mathcal{U} to have a small system of generators and for its ultrapower $\mathcal{U}\text{-prod}\omega$ to have small cofinality are in some sense antithetical properties. Specifically, the proof of Theorem 16 in [1] shows that the number of generators of \mathcal{U} and $\text{cf}(\mathcal{U}\text{-prod}\omega)$ cannot both be smaller than \mathfrak{d} . Yet each property, when it holds of two ultrafilters (with an appropriate sense of “small”) implies near coherence. For $\text{cf}(\mathcal{U}\text{-prod}\omega)$, the appropriate sense of “small” is $< \mathfrak{s}$ and the relevant result

is Theorem 9 above. For the number of generators of \mathcal{U} , the appropriate sense of “small” is $< \mathfrak{d}$, for Corollary 13 of [1] says that any two ultrafilters generated by fewer than \mathfrak{d} sets are nearly coherent.

5. THE UNSPLITTING NUMBER

The following result is in some sense a dual to Theorem 9. It involves the charactersitic \mathfrak{r} dual to \mathfrak{s} , the inequalities are reversed, and the proof uses the same ideas as that of Theorem 9. What is perhaps surprising is that the notions of ultrapower cofinality and of near coherence do not change under this dualization.

Theorem 12. *Suppose \mathcal{U} and \mathcal{U}' are non-principal ultrafilters on ω such that both $cf(\mathcal{U}\text{-prod}\omega)$ and $cf(\mathcal{U}'\text{-prod}\omega)$ are greater than \mathfrak{r} . Then \mathcal{U} and \mathcal{U}' are nearly coherent.*

Proof. Suppose \mathcal{U} and \mathcal{U}' satisfy the hypothesis but are not nearly coherent. Let \mathcal{X} be an unsplittable family of cardinality \mathfrak{r} , and consider the family of functions ν_X for $X \in \mathcal{X}$. (As before, $\nu_X(n)$ is the first element of X that is larger than n .) This family cannot be cofinal in either $\mathcal{U}\text{-prod}\omega$ or $\mathcal{U}'\text{-prod}\omega$, since it has only \mathfrak{r} members. So there is a function $h : \omega \rightarrow \omega$ with $\nu_X <_{\mathcal{U}} h$ and $\nu_X <_{\mathcal{U}'} h$ for all $X \in \mathcal{X}$.

Using this h , proceed exactly as in the proof of Theorem 9 to produce a set $E \subseteq \omega$ such that, for all infinite $X \subseteq \omega$, if $\nu_X <_{\mathcal{U}} h$ then $X \cap E$ is infinite and if $\nu_X <_{\mathcal{U}'} h$ then $X - E$ is infinite.

Thus, E splits all the sets X in the unsplittable family \mathcal{X} . This contradiction completes the proof of the theorem. \square

Corollary 13. *If $\mathfrak{r} < \mathfrak{s}$ then there are at most two near-coherence classes of ultrafilters.*

Proof. The ultrafilters \mathcal{U} with $cf(\mathcal{U}\text{-prod}\omega) < \mathfrak{s}$ (if any) form a single near-coherence class by Theorem 9; those with $cf(\mathcal{U}\text{-prod}\omega) > \mathfrak{r}$ (if any) form another single near-coherence class. The hypothesis of the corollary implies that every ultrafilter is of one or the other of these sorts. \square

Corollary 14. *It is consistent that there are exactly two near-coherence classes of ultrafilters.*

Proof. Consider the model from Theorem 6.1 of [2], which we used to establish Proposition 7 and Corollary 8 above. As pointed out in the proof of Corollary 8, it has $\mathfrak{s} = \aleph_2$. Since it also has an ultrafilter generated by \aleph_1 sets and since a base for an ultrafilter is obviously unsplittable, it has $\mathfrak{r} = \aleph_1$. So Corollary 13 applies to this model; there are at most two near-coherence classes.

To see that there are exactly two, suppose instead that there were only one, i.e., that NCF holds. Then, by Theorem 16 of [1], all ultrapowers would have cofinality $\mathfrak{d} = \aleph_2$. But we saw in the proof of Corollary 8 that there are ultrapowers of cofinality $\mathfrak{b} = \aleph_1$. \square

Corollary 15. *No ultrafilter \mathcal{U} can satisfy $\mathfrak{r} < \text{cf}(\mathcal{U}\text{-prod } \omega) < \mathfrak{s}$.*

Proof. Suppose $\mathfrak{r} < \text{cf}(\mathcal{U}\text{-prod } \omega) < \mathfrak{s}$. For any other ultrafilter \mathcal{U}' we have either $\mathfrak{r} < \text{cf}(\mathcal{U}'\text{-prod } \omega)$ or $\text{cf}(\mathcal{U}'\text{-prod } \omega) < \mathfrak{s}$. In either case, \mathcal{U}' is nearly coherent with \mathcal{U} , by Theorem 9 or 12. Thus, NCF holds. By Theorem 16 of [1], it follows that $\text{cf}(\mathcal{U}\text{-prod } \omega) = \mathfrak{d}$. But it is well known that $\mathfrak{d} \geq \mathfrak{s}$ (see [7] or [11]), so we have a contradiction to the hypothesis $\text{cf}(\mathcal{U}\text{-prod } \omega) < \mathfrak{s}$. \square

6. SMALLER FILTERS

In this final section, we indicate how some of our previous results can be extended to deal with filters more general than ultrafilters. We shall need several definitions and a lemma. We shall give the proofs somewhat sketchily, because they are quite similar to the ultrafilter proofs already given.

All filters considered here will be proper filters on ω that contain all the cofinite sets. The quantifier associated to a filter and the image of a filter under a function are defined just as for ultrafilters in Section 2. A filter \mathcal{F} is called feeble if there is a finite-to-one $f : \omega \rightarrow \omega$ such that $f(\mathcal{F})$ consists of only the cofinite sets. Equivalently, feebleness means that ω can be partitioned into finite pieces in such a way that every set in \mathcal{F} meets all but finitely many of the pieces. A filter is called nearly ultra if there is a finite-to-one $f : \omega \rightarrow \omega$ such that $f(\mathcal{F})$ is an ultrafilter.

If a filter \mathcal{F} is not an ultrafilter, then the reduced power of ω with respect to \mathcal{F} is not linearly ordered. That is, there are functions from ω to ω that are incomparable with respect to the ordering

$$f \leq_{\mathcal{F}} g \iff (\mathcal{F}n)f(n) \leq g(n).$$

Thus, the cofinality of an ultraproduct corresponds to two cardinals associated to a reduced product. We write $\mathfrak{d}(\mathcal{F})$ for the minimum cardinality of a family $\mathcal{D} \subseteq {}^{\omega}\omega$ such that every function in ${}^{\omega}\omega$ is $\leq_{\mathcal{F}}$ one in \mathcal{D} . We write $\mathfrak{b}(\mathcal{F})$ for the minimum cardinality of a family $\mathcal{B} \subseteq {}^{\omega}\omega$ such that no single function is $\geq_{\mathcal{F}}$ all members of \mathcal{B} . Notice that when \mathcal{F} is the filter of cofinite sets then these cardinals are \mathfrak{d} and \mathfrak{b} as defined in Section 2. For ultrafilters \mathcal{U} , we have $\mathfrak{d}(\mathcal{U}) = \mathfrak{b}(\mathcal{U}) = \text{cf}(\mathcal{U}\text{-prod } \omega)$. If $\mathcal{F} \subseteq \mathcal{F}'$ then

$$\mathfrak{b}(\mathcal{F}) \leq \mathfrak{b}(\mathcal{F}') \leq \mathfrak{d}(\mathcal{F}') \leq \mathfrak{d}(\mathcal{F}).$$

If f is a finite-to-one function then it is easy to verify that $\mathfrak{d}(f(\mathcal{F})) = \mathfrak{d}(\mathcal{F})$ and $\mathfrak{b}(f(\mathcal{F})) = \mathfrak{b}(\mathcal{F})$.

The following theorem directly generalizes Theorem 3.

Theorem 16. *If the filter \mathcal{F} is not feeble, then $\mathfrak{b}(\mathcal{F}) \geq \mathfrak{g}$.*

Proof sketch. Proceeding as in the proof of Theorem 3, we find that it suffices to prove that

$$\mathcal{G}_f = \{X \subseteq \omega \mid X \text{ is infinite and } f \leq_{\mathcal{F}} \nu_X\}$$

is groupwise dense for every $f : \omega \rightarrow \omega$. Given f and given a partition of ω into consecutive finite intervals I_n , we find an infinite union of I_n 's in \mathcal{G}_f as follows. By

merging adjacent intervals, we may assume that, for every x , $f(x)$ lies at most one interval beyond x , i.e., $(x, f(x))$ never includes a whole I_n . Consider the partition of ω whose pieces are the double intervals $I_{2n} \cup I_{2n+1}$. Since \mathcal{F} isn't feeble, it contains a set A missing infinitely many of these double blocks. Let X be the union of the "second halves" I_{2n+1} of these double blocks missed by A . Then for every $a \in A$, there is a whole I_k included in $(a, \nu_X(a))$, namely the first half of the double block whose second half contains $\nu_X(a)$. Since there isn't a whole I_k included in $(a, f(a))$, we have $f(a) < \nu_X(a)$, and since $A \in \mathcal{F}$ the proof is complete. \square

In order to generalize Theorems 9 and 12, we shall need the following lemma.

Lemma 17. *Suppose that the filter \mathcal{F} is not nearly ultra, and let ω be partitioned into finite intervals I_n . Then there are sets $D, D' \subseteq \omega$ with the properties:*

- (1) *Every set in \mathcal{F} intersects both D and D' .*
- (2) *Each of D and D' is a union of intervals I_n .*
- (3) *If $I_n \subseteq D$ then I_n and its neighbors $I_{n\pm 1}$ are disjoint from D' .*

Proof sketch. If every set in \mathcal{F} meets all but finitely many I_n (so \mathcal{F} is feeble), then we can take D to be the union of all I_{4n} and D' to be the union of all I_{4n+2} . So assume that some $A \in \mathcal{F}$ misses infinitely many I_n . By merging adjacent intervals and enlarging A , we can assume that A consists of the I_n for all even n . Let $f : \omega \rightarrow \omega$ be constant on each I_n with value n . Since $f(\mathcal{F})$ is not an ultrafilter, there are two disjoint sets C and C' each meeting every set in $f(\mathcal{F})$. Then $D = f^{-1}(C) \cap A$ and $D' = f^{-1}(C') \cap A$ are as required in the lemma. \square

Theorem 18. *If \mathcal{F} is not nearly ultra then $\mathfrak{d}(\mathcal{F}) \geq \mathfrak{s}$ and $\mathfrak{b}(\mathcal{F}) \leq \mathfrak{r}$.*

Before proving this theorem, we point out how it subsumes Theorems 9 and 12. Suppose \mathcal{U} and \mathcal{U}' are ultrafilters whose ultrapowers both have cofinality $< \mathfrak{s}$. Let $\mathcal{F} = \mathcal{U} \cap \mathcal{U}'$. Then it is easy to check that $\mathfrak{d}(\mathcal{F}) = \max\{\text{cf}(\mathcal{U}\text{-prod } \omega), \text{cf}(\mathcal{U}'\text{-prod } \omega)\}$. So Theorem 18 says that \mathcal{F} is nearly ultra. But if f is a finite-to-one function such that $f(\mathcal{F})$ is an ultrafilter then $f(\mathcal{U})$ and $f(\mathcal{U}')$, which both include this ultrafilter, must be equal. So \mathcal{U} and \mathcal{U}' are nearly coherent. This proves Theorem 9, and the deduction of Theorem 12 from Theorem 18 is analogous, using $\mathfrak{b}(\mathcal{F}) = \min\{\text{cf}(\mathcal{U}\text{-prod } \omega), \text{cf}(\mathcal{U}'\text{-prod } \omega)\}$.

Proof sketch for Theorem 18. We shall indicate how to modify the proof of Theorem 9 to obtain $\mathfrak{d}(\mathcal{F}) \geq \mathfrak{s}$. The other inequality is obtained dually, i.e., by analogously modifying the proof of Theorem 12.

Suppose toward a contradiction that \mathcal{F} is not nearly ultra but \mathcal{D} is a family of fewer than \mathfrak{s} functions such that every function in ${}^\omega\omega$ is $\leq_{\mathcal{F}}$ one from \mathcal{D} . Temporarily fix some $h \in \mathcal{D}$, and partition ω into intervals I_n as in the proof of Theorem 9. Let D and D' satisfy the conclusion of Lemma 17 for this partition. As in the proof of Theorem 9, let E be the union of the intervals I_n in D and their right neighbor intervals, and similarly for E' . Then E and E' are disjoint. Also, if X is an infinite subset of ω and $\nu_X \leq_{\mathcal{F}} h$ then X has infinite intersection with both E and E' . The proof of this is just as in the proof of Theorem 9, except that where we formerly

used that $D \in \mathcal{U}$ and $D' \in \mathcal{U}'$, we now use (1) of Lemma 17. Thus, every such X is split by E .

Finally, un-fix h and observe that, as before, the sets E associated to the functions $f \in \mathcal{D}$ form a splitting family of cardinality at most $|\mathcal{D}|$. This contradicts $|\mathcal{D}| < \mathfrak{s}$. \square

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MATHEMATICS DEPT., UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109, U.S.A.

MATHEMATISCHES INSTITUT DER UNIVERSITÄT BONN, BERINGSTRASSE 1, 53115 BONN, GERMANY

E-mail address: ablass@umich.edu, heike@math.uni-bonn.de