ON THE GROUP OF EVENTUALLY DIVISIBLE INTEGER SEQUENCES

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Abstract. Let D be the group, under componentwise addition, of infinite sequences x of integers such that each positive integer divides all but finitely many terms of x. We study properties of D and related groups. We show, among other things, that all infinite-rank summands of D are isomorphic to D and that D has essentially indecomposable subgroups of all coranks from 1 to $2^{\aleph_0}$. Several of our theorems and proofs involve the notion of a basic subgroup, i.e., a pure, free subgroup with divisible quotient.

Introduction

This paper is about certain subgroups of the additive group $\Pi = \mathbb{Z}^{\aleph_0}$ of infinite sequences of integers. Our primary interest will be in the subgroup D consisting of those sequences $x \in \Pi$ that are *eventually divisible*, meaning that, for every positive integer d, all but finitely many terms of x are divisible by d. We write e_n for the element of Π whose n^{th} component is 1 while all other components are 0, and we write Σ for the subgroup of Π generated by these e_n's, i.e., the subgroup of sequences having only finitely many non-zero components. Then Σ is the free abelian group on the e_n's, and D is the $\mathbb{Z}$-adic closure of Σ in Π. In other words, D/Σ is the divisible part of Π/Σ.

All groups in this paper, except automorphism groups, are tacitly assumed to be abelian. By a *basic* subgroup [7, 8] of a group G, we mean a pure, free subgroup $H \subseteq G$ such that the quotient G/H is divisible. Thus, Σ is a basic subgroup of D. The group B of bounded sequences of integers is a basic subgroup of Π; the only difficult part of this is Nöbeling's theorem [11] that B is free. We remark for future reference that Nöbeling also showed that Σ is a direct summand of B. In Section 2 of this paper, we shall prove several results about basic subgroups of D and related groups, as well as some results which don't mention basic subgroups but whose proofs use them. In particular, we show that the automorphism group of D acts transitively on the set of basic subgroups of D, that every separable group

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with a countable basic subgroup can be embedded in D as a pure subgroup, and that Π is not the sum of fewer than $2^{\aleph_0}$ subgroups each isomorphic to D.

In Section 3, we discuss direct sums and summands. In particular, we show that every infinite-rank direct summand of D is isomorphic to D. We also construct essentially indecomposable subgroups of D of every co-rank from 1 to $2^{\aleph_0}$.

Finally, in Section 4, we present some examples to clarify the connection between copies of Σ, D, and Π inside Π.

1. Preliminaries

In this section, we list some known results that we shall need later. We write G^* for the dual $\mathrm{Hom}(G, \mathbb{Z})$ of a group G, and we write $\eta : G \to G^{**}$ for the canonical embedding of G in its double-dual.

We use the symbol $\mathfrak{c}$ for the cardinality of the continuum, $\mathfrak{c} = 2^{\aleph_0}$.

Proposition 1.1 (Specker [13]). *The dual of* Π *is isomorphic to* Σ*, via the pairing* $\Sigma \times \Pi \to \mathbb{Z} : \langle x, y\rangle \mapsto \sum_n x(n)y(n)$.

Proposition 1.2 (Yen, cf. [9], vol. II, p. 163, and Blass, Irwin, and Schlitt [3]). *The dual of* D *is isomorphic to* Σ*, via the (restriction of the) same pairing as in Proposition 1.1. The same applies to any pure subgroup* G *that has corank* $< \mathfrak{c}$ *in* D *and includes* Σ.

Since the dual of Σ is clearly isomorphic to Π via the same pairing, it follows that Π is reflexive and that the double-dual D^{**} (or G^{**} for any G as in the proposition) is identified with Π in such a way that the canonical embedding η becomes the inclusion map $D \to \Pi$ (or $G \to \Pi$).

Corollary 1.3. *Every endomorphism of* D *extends uniquely to an endomorphism of* Π.

Proof. The existence of the extension is immediate because double-dualization is a functor. The uniqueness comes from the facts that a homomorphism $\Pi \to \mathbb{Z}$ is determined by its action on Σ (clear from Prop. 1.1) and that a homomorphism $\Pi \to \Pi$ is determined by its components $\Pi \to \mathbb{Z}$. □

In the next proposition, we need the notion of a dense subset of Π with respect to the product topology, i.e., the Tychonoff topology on $\Pi = \mathbb{Z}^{\aleph_0}$ induced by the discrete topology on $\mathbb{Z}$. To avoid confusion with the $\mathbb{Z}$-adic topology, which will also play a role in our work, we write "product-dense," and similarly for other topological notions. Note that a subset X of Π is product-dense if and only if every finite sequence of integers occurs as an initial segment of some sequence $x \in X$.

Proposition 1.4 (Chase [5]). *Every countable, pure, product-dense subgroup of* Π *can be mapped onto* Σ *by an automorphism of* Π*. Therefore, every countable subgroup of* Π *can be mapped into* Σ *by an automorphism of* Π.

Corollary 1.5 (Baer [1], Thm. 4.7 and Cor. 12.8). *Every countable subgroup of* Π *is free.*

Recall that a group G is *torsionless* if each non-zero element of it has a non-zero image under some homomorphism $G \to \mathbb{Z}$. Equivalently, G is isomorphic to a subgroup of $\mathbb{Z}^\kappa$ for some κ. If the subgroup here can be taken to be pure in $\mathbb{Z}^\kappa$ then G is *separable*.

Proposition 1.6 (Nunke [12]). *Every torsionless quotient of* Π *is isomorphic either to* Π *or to a finite power of* $\mathbb{Z}$.

Proposition 1.7 (Dugas and Irwin [7]). *A separable group with a basic subgroup of cardinality κ can be embedded as a pure subgroup of $\mathbb{Z}^\kappa$.*

Proposition 1.8 (Dugas and Irwin [8]). *Any two basic subgroups of the same group are isomorphic.*

Proposition 1.9 (Dugas and Irwin [8]). *If a separable group G has a countable basic subgroup, then so does every pure subgroup of G.*

Proposition 1.10 (Nöbeling [11]), see also [9], vol. 2, Thm. 97.3. *The group $B \subseteq \Pi$ of bounded sequences is free, and Σ is a direct summand of it.*

2. Basic Subgroups

We begin by showing that basic subgroups of D all "look alike," i.e., the automorphism group of D acts transitively on them.

Theorem 2.1. *If S_1 and S_2 are basic subgroups of D, then there is an automorphism h of Π that maps D onto itself and maps S_1 onto S_2.*

Remark. Once one has an automorphism of D sending S_1 onto S_2, the additional information in the theorem, namely that it extends to an automorphism of Π, could be obtained by applying Corollary 1.3 to the automorphism and its inverse. Our proof of the theorem will, however, give the automorphism of Π directly.

Proof of Theorem 2.1. It suffices to consider the case $S_2 = \Sigma$, for once this case is established we can obtain the general case by composing automorphisms. So let $S_1 = S$ be any basic subgroup of D. By definition, S is pure in D and therefore also in Π. Also, S is countable, by Prop. 1.8 since D has a countable basic subgroup Σ. We claim that S is product-dense in Π.

To prove this claim, suppose it were false, and fix a finite sequence u of integers that does not occur as an initial segment of any member of S. If n is the length of u, then the projection $p : \Pi \to \mathbb{Z}^n$ to the first n components sends S to a proper subgroup $p(S)$ of $\mathbb{Z}^n$, not containing u. So there is a non-trivial equation or congruence in n variables, satisfied by the first n components of any element of S but not satisfied by u. We may assume it's a congruence, since an equation could be replaced with a congruence having any sufficiently large modulus. Let m be the modulus of this congruence. Then, as S is basic in D, every element of D can be written as $s + mx$ for some $s \in S$ and some $x \in D$, and therefore every element of D satisfies our congruence. That is absurd, as u is an initial segment of some elements of D. This contradiction establishes the claim.

Being a countable, pure, product-dense subgroup of Π, S can be mapped to Σ by an automorphism h of Π, because of Prop. 1.4. It remains to prove that h maps D onto itself.

D is $\mathbb{Z}$-adically closed in Π, and each of Σ and S are $\mathbb{Z}$-adically dense in it since the quotients are divisible. So D is the $\mathbb{Z}$-adic closure of Σ and of S in Π. The automorphism h, being a $\mathbb{Z}$-adic homeomorphism and sending S to Σ, therefore sends D to D, as required. □

Remark. We cannot strengthen the theorem to say that every isomorphism between two basic subgroups of D extends to an automorphism of D. In fact, there are automorphisms of Σ that don't extend to automorphisms of D. See Example 1 in Section 4.

Corollary 2.2. *Every basic subgroup of D is a direct summand of a basic subgroup of* Π.

Proof. By Prop. 1.10, Σ is a direct summand of B, the group of bounded sequences, which is a basic subgroup of Π. So the corollary is true for Σ, and the theorem allows us to transfer it from Σ to any other basic subgroup of D. $\square$

Theorem 2.3. *Every separable group with a countable basic subgroup can be embedded as a pure subgroup of D.*

Proof. Let G be a separable group with a countable basic subgroup H. By Prop. 1.7, G can be embedded as a pure subgroup in Π. By Prop. 1.4, we can arrange (by composing with an automorphism of Π) that the embedding sends H to a subgroup of Σ. Identifying (without loss of generality) G and H with their images in Π, we have that $(G+\Sigma)/\Sigma \cong G/(G\cap\Sigma)$ is a quotient of G/H, hence is divisible. So $G+\Sigma \subseteq D$ and, in particular, $G \subseteq D$. Finally, since G is pure in Π, it is pure in D as well. $\square$

Remark. The proof establishes a stronger form of the theorem where a particular basic subgroup of G is specified and the embedding is required to send that basic subgroup into Σ.

Π is the union of its subgroups that are isomorphic to D. Indeed, every element of Π can be sent into Σ by an automorphism of Π. Pulling back D along that automorphism, we get a subgroup of Π isomorphic to D and containing the given element. The next theorem shows that we cannot represent Π as the union of a small number of copies of D. In fact, it shows more, replacing "union" with "sum."

Theorem 2.4. Π *is not the sum of fewer than $\mathfrak{c}$ subgroups isomorphic to D.*

Proof. Suppose Π were the sum of a family of fewer than $\mathfrak{c}$ copies of D. In each of these copies of D, fix a countable basic subgroup, and let H be their sum. Then $|H| < \mathfrak{c}$ and Π/H is divisible. This is impossible by the proof of Theorem 2 in [8]; that result would apply directly if H were free and pure in Π, hence basic, but the proof applies in any case, as follows.

Since Π/H is divisible, we have $\Pi = H + 2\Pi$. Therefore, $\Pi/2\Pi = (H+2\Pi)/2\Pi \cong H/(H\cap 2\Pi)$. But the group $\Pi/2\Pi$ on the left here has the cardinality of the continuum, while the group on the right, a quotient of H, is smaller, a contradiction. $\square$

The representation of Π as a union of $\mathfrak{c}$ copies of D can be tidied up a bit, as the following theorem shows; a corollary will give an even tidier representation when the continuum hypothesis holds.

Theorem 2.5. Π *is the union of a countably-directed family of pure subgroups isomorphic to D.*

Recall that a family is *countably directed* if every countable subfamily has an upper bound.

Proof. For each countable, pure, product-dense $S \subseteq \Pi$, let $D(S)$ be its $\mathbb{Z}$-adic closure in Π. Then each $D(S)$ is isomorphic to D because of Prop. 1.4. Each element of Π is in some such S and therefore in the corresponding $D(S)$. Finally, each countably many S's, say S_n, are in a single such S, say S^+, namely the purification of their sum. Then all the $D(S_n)$ are subgroups of $D(S^+)$. $\square$

Corollary 2.6. *Assume the continuum hypothesis. Then* Π *is the union of a chain, of length* $\aleph_1$*, of pure subgroups isomorphic to* D.

Proof. The countably directed system in Theorem 2.5 clearly has the cardinality of the continuum, so under the continuum hypothesis it has cardinality $\aleph_1$. But any countably directed partial ordering of cardinality $\aleph_1$ with no maximal element has a cofinal well-ordered subset of length $\aleph_1$. □

3. Direct Sums

This section is about direct sum decompositions of D and certain subgroups. We show first that D has no "strange" summands. Notice that Nunke's Prop. 1.6 above implies the analogous result for Π.

Theorem 3.1. *Every direct summand of* D *of infinite rank is isomorphic to* D.

Proof. Suppose $D = D_1 \oplus D_2$. Since double-dualization is an additive functor, we have $\Pi = D^{**} = D_1^{**} \oplus D_2^{**}$. By Prop. 1.6, each D_i^{**} is isomorphic to Π or to a finite power of $\mathbb{Z}$. If D_i has infinite rank, then the latter possibility is excluded and we have $D_i^{**} \cong \Pi$.

The topology on D_i^{**} corresponding to the product topology on Π via such an isomorphism will be called the product topology on D_i^{**}. Notice that it doesn't depend on the choice of the isomorphism, since any two such isomorphisms differ by an automorphism of Π and any such automorphism is, thanks to Prop. 1.1, a homeomorphism of product spaces.

Since D has a countable basic subgroup Σ, Prop. 1.9 says that each D_i has a countable basic subgroup, say S_i. Then $S_1 \oplus S_2$ is clearly a basic subgroup of D. Applying an automorphism obtained from Theorem 2.1, we may assume without loss of generality that $S_1 \oplus S_2 = \Sigma$.

The divisible part of Π/Σ is, on the one hand,

$$\operatorname{Div}\left[\frac{\Pi}{\Sigma}\right] = \frac{D}{\Sigma} = \frac{D_1 \oplus D_2}{S_1 \oplus S_2} = \frac{D_1}{S_1} \oplus \frac{D_2}{S_2}$$

and, on the other hand

$$\begin{aligned}\operatorname{Div}\left[\frac{\Pi}{\Sigma}\right] &= \operatorname{Div}\left[\frac{D_1^{**} \oplus D_2^{**}}{S_1 \oplus S_2}\right] \\ &= \operatorname{Div}\left[\frac{D_1^{**}}{S_1} \oplus \frac{D_2^{**}}{S_2}\right] \\ &= \operatorname{Div}\left[\frac{D_1^{**}}{S_1}\right] \oplus \operatorname{Div}\left[\frac{D_2^{**}}{S_2}\right]\end{aligned}$$

After checking that all the identifications here are coherent, we conclude that

$$\operatorname{Div}\left[\frac{D_i^{**}}{S_i}\right] = \frac{D_i}{S_i}.$$

We claim that S_i is countable, pure, and product-dense in D_i^{**}. (Recall that the product topology on D_i^{**} was defined above by transferring the product topology of Π via an isomorphism. We assumed there that D_i has infinite rank; in the finite rank case, the product topology would be discrete.) Countability was already pointed out when S_i was introduced, and purity follows from the fact that D_i^{**}/S_i is a subgroup of the torsion-free group Π/Σ. As for product-density, notice that the projection of Π onto D_i^{**} (coming from $\Pi = D_1^{**} \oplus D_2^{**}$) sends $\Sigma = S_1 \oplus S_2$ onto S_i, that Σ is product-dense in Π, and that the projection is product-continuous (since its components are, by Prop. 1.1). But a continuous surjection sends dense sets to dense sets, so the claim is established.

Now assume that D_i has infinite rank, so there is an isomorphism $D_i^{**} \to \Pi$. Such an isomorphism, the claim just established, and Prop. 1.4 yield a new isomorphism $D_i^{**} \to \Pi$, sending S_i to Σ. But then this isomorphism also sends the pre-image in D_i^{**} of the divisible part of D_i^{**}/S_i onto the pre-image in Π of the divisible part of Π/Σ. That is, it sends D_i onto D. In particular, D_i is isomorphic to D. $\square$

It seems reasonable to conjecture for D the full analog of Prop. 1.6, namely that any torsionless quotient of D is isomorphic to either D or a finite power of $\mathbb{Z}$. We do not know how to prove this conjecture, but the following proposition reduces it, at least for separable groups, to the special case where the quotient in question is of a rather special form.

Proposition 3.2. *Every separable, infinite-rank, homomorphic image of D is isomorphic to $h(D)$ for some surjective endomorphism h of Π that maps Σ onto itself.*

Proof. Let G be a separable group of infinite rank with a surjective homomorphism $h : D \to G$. We shall modify h in a number of steps, composing it with isomorphisms either on the left or on the right, until we obtain the conclusion of the proposition.

In the first step, we arrange that $h(\Sigma)$ is pure in G. If it isn't, then we can take a countable Σ' with $\Sigma \subseteq \Sigma' \subseteq D$, with Σ' pure in D, and with $h(\Sigma')$ pure in G. Finding such a Σ' is easy, as we just need to close Σ under countably many functions. As both Σ and Σ' are basic in D, Theorem 2.1 gives us an automorphism of D sending Σ onto Σ'. Composing such an automorphism of D with h we get a new h, still mapping D onto G, and now mapping Σ onto a pure subgroup of G.

We claim that $h(\Sigma)$ is a countable, basic subgroup of G. It is clearly countable and therefore free, since G is separable. We've just arranged for it to be pure in G. And the quotient is divisible because, with K denoting the kernel of h, we have

$$\frac{G}{h(\Sigma)} \cong \frac{D}{K+\Sigma} \cong \frac{D/\Sigma}{(K+\Sigma)/\Sigma},$$

a quotient of the divisible group D/Σ.

Thus, we can apply Theorem 2.3 to assume, without loss of generality, that G is a pure subgroup of D and therefore of Π. Thus, we regard h as a homomorphism $D \to \Pi$. As such, it extends naturally to the double-duals, and we use the same symbol h for this extension, $h : \Pi \to \Pi$.

The range of this extended h is a torsionless (because it's $\subseteq \Pi$) homomorphic image of Π, and it has infinite rank because it includes G. By Prop. 1.6, it is isomorphic to Π. Composing h with such an isomorphism, we may assume that h maps Π onto Π. So we have a surjective endomorphism h of Π that maps D onto G.

To complete the proof, we want to arrange that h maps Σ onto itself. We do this by composing h with an automorphism of Π that sends $h(\Sigma)$ back to Σ. The existence of the required automorphism will follow from Prop. 1.4 once we check that $h(\Sigma)$ is a countable, pure, product-dense subgroup of Π. It is obviously countable, and it is pure in Π because it is pure in G which in turn is pure in Π. Finally, it is product-dense because Σ is product-dense in Π, h is product-continuous, and continuous surjections preserve density. $\square$

Remark. In view of the conjecture preceding the proposition, one might hope to prove that, if h is as in the proposition, then $h(D)$ is actually equal to D. We shall see an example in Section 4 where this is not so.

We turn now from direct summands (or separable quotients) of D to direct summands of certain subgroups of D. Dugas, Irwin, and Khabbaz [6] constructed subgroups of Π that are essentially indecomposable, i.e., when they are split into a direct sum of two subgroups then one of the summands must be free of finite rank. In fact, the groups they constructed are subgroups of D, and they have the property, stronger than essential indecomposability, that their only endomorphisms are (1) multiplication by integer scalars, (2) endomorphisms of finite rank, and (3) sums of (1) and (2). We shall construct such subgroups with any prescribed co-rank (except 0) in D.

To avoid interrupting the argument later, we first give a lemma combining some results from [2] and [4] that we shall need.

Lemma 3.3. *Suppose that h is an endomorphism of D, that α and β are integers with $\beta \neq 0$, and that the range of the endomorphism $\beta h - \alpha$ has cardinality $< \mathfrak{c}$. Then h is the sum of an (integer) scalar multiplication and an endomorphism of D of finite rank.*

Proof. It is shown in [4] that any torsionless quotient of D of cardinality $< \mathfrak{c}$ is in fact countable. In the present situation, this means that the range of $\beta h - \alpha$ is countable, hence free, and hence isomorphic to a direct summand of D. But D has no free summands of infinite rank (because its dual Σ is too small to have as a summand the dual $\mathbb{Z}^\kappa$ of a free group of infinite rank κ). So the range of $\beta h - \alpha$ has finite rank.

Passing to the divisible hull of D, where multiplication by rational scalars makes sense, we have that h and the rational scalar multiplication α/β differ by an endomorphism of finite rank. It is shown in Lemma 2 of [2] that, in this situation, the scalar α/β must be an integer. $\square$

Theorem 3.4. *For every cardinal number in the range $1 \leq \kappa \leq \mathfrak{c}$, there exists a pure subgroup of co-rank κ in D, all of whose endomorphisms are sums of scalar multiplications and endomorphisms of finite rank.*

Proof. The case $\kappa = \mathfrak{c}$ is easy. First find inside D a pure copy of D of the desired co-rank, for example $\{x \in D \mid x(n) = 0 \text{ for all even } n\}$. Then apply the construction from [6] inside this copy of D.

We assume from now on that $1 \leq \kappa < \mathfrak{c}$. The idea of our proof is very similar to the idea in [6]; we shall list all possible endomorphisms h not of the desired form, and we shall get rid of each one by putting into the group G under construction some element x while keeping $h(x)$ out of G. The new twist is that, to keep the co-rank of G small, we mustn't keep too many elements out of G. To get around the difficulty, we put a few elements permanently out of G and then, when we want to keep some y out of G, we instead put *into* G the difference between y and one of the permanently out elements. Here are the details.

There are $\mathfrak{c}$ endomorphisms h of D, because each one is determined by where it sends the countably many elements of Σ. List these endomorphisms in a well-ordered sequence $(h_\xi)_{\xi<\mathfrak{c}}$ in which each element has $< \mathfrak{c}$ predecessors. We shall build, by a transfinite induction of this same length $\mathfrak{c}$, an increasing sequence of pure subgroups G_ξ of D. Each G_ξ will have cardinality smaller than $\mathfrak{c}$; in fact, $|G_\xi| \leq (1 + |\xi|)\aleph_0$.

The beginning of the construction and the limit stages are easy. $G_0 = \Sigma$ and, for limit ordinals λ, $G_\lambda = \bigcup_{\xi<\lambda} G_\xi$. Before describing the successor stages, we need some preliminary work.

Fix a pure subgroup $J \subseteq D$ of rank κ such that $J \cap \Sigma = 0$. One way to get such an J is to choose κ infinite subsets S of $\mathbb{N}$ that are almost disjoint, i.e., the intersection of any two is finite. Such sets exist; cf. Lemma 23.9 in [10]. Then associate to each chosen S the element $j(S) \in D$ defined by $j(S)_n = n!$ for $n \in S$ and $j(S)_n = 0$ for $n \notin S$. The purification of the subgroup generated by these $j(S)$'s will serve as J.

We shall choose the G_ξ so as to have zero intersection with J. Note that this is the case for $\xi = 0$ and will be preserved at limit stages, according to the description already given for these G_ξ.

We intend to carry out the successor steps of the inductive construction in such a way that, unless h_ξ is of the form scalar plus finite rank, there will be $x, y \in G_{\xi+1}$ such that $h_\xi(x) - y \in J - \{0\}$. Before showing how to do this, we show that it will complete the proof.

Suppose then that G_ξ has been defined for all $\xi < \mathfrak{c}$ in conformity with the preceding specification. Let $G_\mathfrak{c}$ be the union of all the G_ξ. Its intersection with J is 0. By Zorn's Lemma, enlarge $G_\mathfrak{c}$ to a maximal $G \subseteq D$ with zero intersection with J. Then this G is pure in D because of its maximality. It has co-rank κ in D. Indeed, the projection from D to D/G is one-to-one on J (as $G \cap J = 0$), and every element of D/G has a multiple in the image of J by maximality of G, so the rank of D/G equals the rank of J, namely κ. It remains to show that every endomorphism of G has the form scalar plus finite rank. Suppose h were a counterexample. It extends to an endomorphism, still called h, of the second-dual of G, which is Π by Prop. 1.2. And that extension maps D into itself, since it is $\mathbb{Z}$-adically continuous and D is the $\mathbb{Z}$-adic closure of G in Π (because $\Sigma \subseteq G \subseteq D$). So h occurs as some h_ξ in our list. If it has the form scalar plus finite rank (on D, therefore also on G), there is nothing to prove. Otherwise, let $x, y \in G_{\xi+1} \subseteq G$ be such that $h(x) - y \in J - \{0\}$. As h is an endomorphism of G, we'd have $h(x) - y \in G$, contradicting $G \cap J = 0$. That contradiction shows that G is as required.

All that remains, then, is to carry out the successor steps of the induction so as to satisfy the specifications above. Consider, therefore, the step from G_ξ to $G_{\xi+1}$. We shall let $G_{\xi+1}$ be the purification of the subgroup of D generated by G_ξ and some $x, y \in D$ chosen so that $h_\xi(x) - y \in J - \{0\}$. Of course, we must be careful lest this $G_{\xi+1}$ have a non-trivial intersection with J.

Let H be the purification of $G_\xi + J$ in D, and note that its cardinality is $< \mathfrak{c}$. We shall choose an appropriate $x \in D - H$ and $y \in D$. Let us temporarily consider an arbitrary $x \in D - H$ and consider under what circumstances we *cannot* find an appropriate y. That would mean that, for any element $j \in J - \{0\}$ (the intended $h_\xi(x) - y$), adjoining x and $y = h_\xi(x) - j$ to G_ξ and purifying the result leads to a non-trivial intersection with J. In other words, for any such j, there are integers α and β and there is an element $g \in G_\xi$ such that $g + \alpha x + \beta h_\xi(x) - \beta j \in J - \{0\}$. Then $\alpha x + \beta h_\xi(x) \in H$. Since we are considering only x's not in H, we have $\beta \neq 0$ here.

Let p be the projection map from D to D/H. We have seen that, if $x \in D - H$ and if no appropriate y can be found, to go with this x and produce $G_{\xi+1}$, then $\alpha p(x) + \beta p(h_\xi(x)) = 0$ for suitable integers α and $\beta \neq 0$. We now assume that *no* $x \in D - H$ admits an appropriate y, and we show that then h_ξ is of the form scalar plus finite rank. This will show that the construction of $G_{\xi+1}$ can be carried out whenever necessary, i.e., whenever h_ξ is not of that form.

By our assumption, we have, for each $x \in D - H$, integers α and $\beta \neq 0$ with $\alpha p(x) + \beta p(h_\xi(x)) = 0$. Let us work in the quotient D/H, which is a torsion-free, divisible group, because H is pure and includes $G_0 = \Sigma$. Writing ρ for $-\alpha/\beta$, we have, for each non-zero element $p(x) \in D/H$, a rational scalar ρ such that $p(h_\xi(x)) = \rho p(x)$. There could be several such ρ for one $p(x)$ as $p(x) = p(x')$ for many $x' \neq x$; in addition, it seems likely that ρ would be different for different $p(x)$'s. We shall show that a single ρ works for all non-zero $p(x) \in D/H$ simultaneously.

To this end, suppose ρ and ρ' work for (non-zero) $p(x)$ and $p(x')$, respectively. Multiplying x' by 2 if necessary, we can assume that $x + x' \notin H$, so some τ works for $x + x'$. (Note that the multiplication by 2 does not interfere with the assumption that ρ' works for x'.) Thus we have both

$$ph_\xi(x + x') = ph_\xi(x) + ph_\xi(x') = \rho p(x) + \rho' p(x')$$

and

$$ph_\xi(x + x') = \tau p(x + x') = \tau p(x) + \tau p(x').$$

If $\rho \neq \rho'$ then these two equations say that $p(x)$ and $p(x')$ are linearly dependent in D/H. In other words, if ρ works for $p(x)$, then no other ρ' can work for any $p(x')$ linearly independent of $p(x)$. So ρ must work for these $p(x')$ and therefore also, by the same argument, for all $p(x'')$ independent of any such $p(x')$. But that's all elements of D/H. (We've used that there are enough linearly independent elements in D/H. The argument would fail if the rank of D/H were 1. But the rank is $\mathfrak{c}$, since H has smaller cardinality.) So a single ρ works simultaneously for all non-zero $p(x) \in D/H$.

Returning to D, we see that we have a single pair of integers α and $\beta \neq 0$ such that, for all $x \in D - H$, $\alpha x + \beta h_\xi(x) \in H$. This actually holds for all $x \in D$, since every such x is the sum of two elements of $D - H$.

Since $|H| < \mathfrak{c}$, Lemma 3.3 now tells us that h is the sum of a scalar multiplication and a finite-rank endomorphism, as required. □

4. Examples

Many of the results in the previous sections are based on the rather tight connections that exist between Π, D, and Σ. The fact that Π is the double-dual of D allows us to extend endomorphisms from D to Π (Cor. 1.3). That D is the $\mathbb{Z}$-adic closure of Σ in Π ensures that automorphisms of Π that preserve Σ also preserve D. That Σ is $\mathbb{Z}$-adically dense in D and product-dense in Π means that homomorphisms are often determined by their restrictions to Σ.

In this section, we explore the limitations of the connections between Π, D, and Σ by giving examples where the three groups exhibit different behavior. These provide counterexamples to some natural extensions of our earlier results.

Example 1. Let δ be the endomorphism of Π defined by

$$(\delta(x))_n = x_n - x_{n+1}.$$

This is a surjective endomorphism of Π. It is almost but not quite one-to-one, for its kernel is the subgroup, isomorphic to $\mathbb{Z}$, generated by the sequence all of whose components are 1.

If we restrict δ to D or to Σ, it becomes one-to-one, since these groups have zero intersection with the kernel. In fact, the restriction of δ to Σ is easily seen to be an automorphism of Σ. Its inverse is the function σ defined by

$$(\sigma(x))_n = \sum_{k=n}^{\infty} x_k.$$

Notice that σ is well-defined on Σ (since the apparently infinite sum is finite for $x \in \Sigma$) but not on D or on Π. In fact, σ cannot be extended to an automorphism or even an endomorphism of D or of Π. This is because its first component, $\Sigma \to \mathbb{Z} : x \mapsto \sum_{k=0}^{\infty} x_k$, cannot be extended to a homomorphism $D \to \mathbb{Z}$, since Prop. 1.2 would require such an extension to depend on only finitely many components of its input. Thus we have the example claimed in the remark following the proof of Theorem 2.1.

If we restrict δ to D, the result is a one-to-one map of D into itself. That $\delta(D) \subseteq D$ follows from $\delta(\Sigma) = \Sigma \subseteq D$ and the fact that D is the $\mathbb{Z}$-adic closure of Σ in Π. Notice, however, that the $\mathbb{Z}$-adic closure relationship between Σ and D does not allow us to conclude that δ maps D onto itself just because it maps each of Π and Σ onto itself. In fact, $\delta(D)$ is, in a sense, a rather small part of D; it has co-rank $\mathfrak{c}$ in D. In another sense though, it is large; it is dense in D with respect to both the product and $\mathbb{Z}$-adic topologies. Thus, we have a dense isomorphic copy of D, between Σ and Π, and having co-rank $\mathfrak{c}$ in D.

These facts follow easily from an explicit description of $\delta(D)$. To give this description, we first notice that there is a "summation" homomorphism θ, from D onto the $\mathbb{Z}$-adic completion $\hat{\mathbb{Z}}$ of $\mathbb{Z}$, sending every sequence $x \in D$ to $\sum_{k=0}^{\infty} x_k$; this sum converges ($\mathbb{Z}$-adically) in $\hat{\mathbb{Z}}$ precisely because $x \in D$. Then $\delta(D)$ is the inverse image of $\mathbb{Z}$ under θ.

Finally, we observe that, when h is as in Theorem 3.2, we cannot conclude that $h(D) = D$, not even if h restricts to an automorphism of Σ and is one-to-one on D; a counterexample is given by $h = \delta$.

Example 2. This example looks very similar to Example 1, but some of its properties are quite different. We define $\delta : \Pi \to \Pi$ (recycling the notation δ) by

$$(\delta(x))_0 = x_0 \qquad (\delta(x))_{n+1} = x_{n+1} - x_n.$$

(The reversal of signs relative to Example 1 is convenient but inessential; it makes the diagonal entries of the matrix representing δ all 1's rather than -1's. The essential change is the 0-component.) This δ is (unlike the one in Example 1) an automorphism of Π. Its inverse is given by

$$(\sigma(x))_n = \sum_{k=0}^{n} x_k.$$

The restrictions of δ to D and to Σ are one-to-one endomorphisms of these groups, but they are not surjective. $\delta(\Sigma)$ is the subgroup of Σ consisting of those elements x for which $\sum_{k=0}^{\infty} x_k = 0$. (Notice that the apparently infinite sum is finite for $x \in \Sigma$.) This $\delta(\Sigma)$ is a co-rank 1 direct summand of Σ, a complementary summand being the subgroup generated by $e_0 = (1, 0, 0, \ldots)$.

One might think that, since δ is an automorphism of Π and maps Σ onto something that is "almost" Σ, it ought to map D onto something that is "almost" D, but that is not the case. $\delta(D)$, the $\mathbb{Z}$-adic closure of $\delta(\Sigma)$ in Π, is a co-rank $\mathfrak{c}$ subgroup of D and not a direct summand. In fact, it is exactly the kernel of the summation map $\theta : D \to \hat{\mathbb{Z}}$ defined in Example 1.

This $\delta(D)$ is product-dense but not $\mathbb{Z}$-adically dense in D.

An easy way to get a pure subgroup of Π that is isomorphic to D but properly includes D is to take $\delta^{-1}(D)$. In fact, by iterating this, one gets a tower of such subgroups $\delta^{-n}(D)$. The process can be iterated transfinitely; at countable limit stages, take the $\mathbb{Z}$-adic closure of the union of the previous stages. (Each previous stage has a countable, $\mathbb{Z}$-adically dense subset; therefore so does the union; therefore the $\mathbb{Z}$-adic closure of the union is still isomorphic to D by Prop. 1.4.) $\delta^{-1}(D)$ admits an alternative description; it is the $\mathbb{Z}$-adic closure of the group generated by D and the all-ones sequence in Π.

We close our dicussion of this example by showing that the direct sum decomposition $\Sigma = \delta(\Sigma) \oplus \langle e_0 \rangle$ does not extend to a direct sum decomposition of D or of Π. Indeed, suppose $D = X \oplus Y$ were such an extension. (The proof with Π in place of D is the same.) So $X \cap \Sigma = \delta(\Sigma)$ and $Y \cap \Sigma = \langle e_0 \rangle$. Define a homomorphism $f : D \to \mathbb{Z}$ by composing the projection $p : D \to Y$ (with kernel X), the inclusion $Y \hookrightarrow \Pi$, and the first component map $\Pi \to \mathbb{Z} : x \mapsto x_0$. For each n, we have $e_n = (e_n - e_0) + e_0$ where $e_n - e_0 \in \delta(\Sigma) \subseteq X$ and $e_0 \in Y$. So $p(e_n) = e_0$ and $f(e_n) = 1$ for all n. That contradicts Prop. 1.2 (or 1.1 if D is replaced by Π).

If we take the $\mathbb{Z}$-adic closures (in Π) of the two summands $\delta(\Sigma)$ and $\langle e_0 \rangle$ of Σ, then the sum of the resulting groups is not the $\mathbb{Z}$-adic closure D of Σ but only the subgroup $\delta(\Sigma)$.

Example 3. This final example is less relevant to the study of D, but we include it to show how a minor change in the definition can influence the behavior of δ. We define

$$(\delta(x))_n = x_n - 2x_{n+1},$$

modifying Example 1 only by inserting a factor 2. As in Example 1, we have an automorphism of Σ, the inverse now being

$$(\sigma(x))_n = \sum_{k=n}^{\infty} x_k 2^{k-n}.$$

In contrast to Example 1, however, this δ, considered as an endomorphism of Π, is not surjective but rather injective. Its range can be described as the pre-image of $\mathbb{Z}$ under a "power-series" map ϕ from Π to the group of 2-adic integers, namely $\phi(x) = \sum_{k=0}^{\infty} x_k 2^k$. (The sum clearly converges 2-adically.) Since ϕ maps Π onto the group of 2-adic integers, whose cardinality is $\mathfrak{c}$, the co-rank of $\delta(\Pi)$ in Π is also $\mathfrak{c}$.

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