Sums, Products, and Choice for Finite Sets

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Abstract

We work in set theory without the axiom of choice, so infinite sums and products of cardinal numbers may not be well defined. We consider the special case of sums or products of countably many copies of the same finite cardinal number n. We characterize the sets of integers that can, consistently with ZF, occur as the set of n such that the product (or the sum) of countably many copies of n is well defined.

1 Introduction and Notation

The natural way to define the sum of an infinite sequence $\langle n_i \rangle_{i \in \mathbb{N}}$ of cardinals is to take sets A_i of cardinalities n_i, form their disjoint union

$$\bigsqcup_i A_i = \{\langle i, a \rangle : a \in A_i\},$$

and take the cardinality of the result. The product of the sequence of cardinals $\langle n_i \rangle_{i \in \mathbb{N}}$ is defined similarly, using the cartesian product

$$\prod_i A_i = \{f : f \text{ is a function on } \mathbb{N} \text{ with each } f(i) \in A_i\}$$

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of the sets A_i in place of their disjoint union.

In the absence of the axiom of choice, these definitions may be ambiguous in the sense that the cardinal one obtains may depend on the particular system of representative sets A_i. Another system $\langle B_i \rangle$ would have bijections $A_i \cong B_i$ for all i. But in order to combine these bijections to produce a bijection

$$\bigsqcup_i A_i \cong \bigsqcup_i B_i \quad \text{or} \quad \prod_i A_i \cong \prod_i B_i,$$

we need to select one bijection $A_i \cong B_i$ for each i, and this may be impossible if the axiom of choice is unavailable.

We say that an infinite sum $\sum_i n_i$ is well defined if, for all systems A_i and B_i as above (so $|A_i| = |B_i| = n_i$ for all i), there is a bijection $\bigsqcup_i A_i \cong \bigsqcup_i B_i$. Similarly, we say that the product $\prod_i n_i$ is well defined if, for all such systems, there is a bijection $\prod_i A_i \cong \prod_i B_i$.

We shall be interested in the special case where all the n_i are finite and equal.

Definition 1 Let n be a natural number.

- $DS(n)$ denotes the statement that $\sum_{i \in \mathbb{N}} n$ is well-defined, i.e., that whenever $\langle A_i \rangle_{i \in \mathbb{N}}$ and $\langle B_i \rangle_{i \in \mathbb{N}}$ satisfy $|A_i| = |B_i| = n$ for all i, then $\bigsqcup_i A_i \cong \bigsqcup_i B_i$.
- $DP(n)$ denotes the statement that $\prod_{i \in \mathbb{N}} n$ is well-defined, i.e., that whenever $\langle A_i \rangle_{i \in \mathbb{N}}$ and $\langle B_i \rangle_{i \in \mathbb{N}}$ satisfy $|A_i| = |B_i| = n$ for all i, then $\prod_i A_i \cong \prod_i B_i$.

The notations DS and DP abbreviate "defined sum" and "defined product."

Adib Ben-Jebara raised the question whether it is consistent to have $DP(n)$ holding for all odd n but for no even $n > 0$. Paul Howard answered this question negatively by showing that $DP(n+1)$ implies $DP(n)$. We reproduce his proof below, and we show that his result gives almost, but not quite, all the implications between the statements $DP(n)$ for different n. For the sake of completeness, we also present the corresponding, easier results for $DS(n)$.

In addition to DS and DP, we shall also need to use the following two statements.

Definition 2
- $CC(n)$ is the statement that, if $\langle A_i \rangle_{i\in\mathbb{N}}$ is a sequence of n-element sets, then there is a choice function, i.e., $\prod_i A_i \neq \varnothing$.
- $LO(n)$ is the statement that, if $\langle A_i \rangle_{i\in\mathbb{N}}$ is a sequence of n-element sets, then there is a function assigning to each $i \in \mathbb{N}$ a linear ordering $\leq_i$ of A_i.

The notations CC and LO abbreviate "countable choice"and "linear order."

2 Elementary Facts

In this section, we collect some easy observations about the statements DS, DP, CC, and LO. The first (which we won't actually need) is that all of these statements follow, for arbitrary n, from the axiom of choice. The observations that we will need are as follows.

Lemma 3 *$DS(n)$ and $DP(n)$ are each equivalent to the special case where $B_i = \{1, 2, \ldots, n\}$ for all i.*

Proof Assume the special case for, say, $DP(n)$. To prove the general case, let $\langle A_i \rangle$ and $\langle B_i \rangle$ be as there. By the special case, $\prod_i A_i$ and $\prod_i B_i$ are each in one to one correspondence with $\prod_i \{1, 2, \ldots, n\}$. So they are in one to one correspondence with each other. The proof for $DS(n)$ is similar. □

Lemma 4 *$LO(n)$ implies $DS(n)$ and $DP(n)$.*

Proof Given $\langle A_i \rangle$ with all $|A_i| = n$, apply $LO(n)$ to get linear orders $\leq_i$ of the sets A_i. These determine specific bijections $A_i \cong \{1, 2, \ldots, n\}$, namely the unique order-preserving bijections. In turn, these bijections induce bijections

$$\bigsqcup_i A_i \cong \bigsqcup_i \{1, 2, \ldots, n\} \quad \text{and} \quad \prod_i A_i \cong \prod_i \{1, 2, \ldots, n\}.$$

□

Lemma 5 *$DP(n)$ implies $CC(n)$.*

Proof If $\langle A_i \rangle$ is a sequence of n-element sets, then $DP(n)$ gives a bijection between $\prod_i A_i$ and $\prod_i \{1, 2, \ldots, n\}$. Since the latter set is obviously nonempty, containing the constant functions, so is the former. $\square$

Corollary 6 *$LO(n)$ implies $CC(n)$.*

Proof Immediate from the preceding two lemmas. $\square$

Lemma 7 *$LO(n+1)$ implies $LO(n)$*

Proof Given n-element sets A_i for all $i \in \mathbb{N}$, let x be some object that is in none of them, and let $A_i' = A_i \cup \{x\}$ for all i. Since each A_i' is an $(n+1)$-element set, $LO(n+1)$ provides a sequence of linear orderings $\leq_i$ for these sets. Restricting each $\leq_i$ to A_i gives what $LO(n)$ requires. $\square$

Lemma 8 *$LO(n+1)$ is equivalent to the conjunction of $LO(n)$ and $CC(n+1)$.*

Proof The implication from left to right is given by Corollary 6 and Lemma 7. For the converse, assume $LO(n)$ and $CC(n+1)$, and let a sequence $\langle A_i \rangle$ of $(n+1)$-element sets be given. By $CC(n+1)$, let $\langle a_i \rangle$ be a sequence of elements $a_i \in A_i$. Apply $LO(n)$ to the sequence of n-element sets $A_i - \{a_i\}$ to get a sequence $\langle \leq_i \rangle$ of linear orderings of them. Extend each $\leq_i$ to a linear ordering of A_i by putting a_i at the end of the ordering. $\square$

Lemma 9 *$LO(n)$ is equivalent to the conjunction $\bigwedge_{k \leq n} CC(k)$.*

Proof Apply Lemma 8 repeatedly and use that $LO(1)$ is trivially true. $\square$

3 Defined Sums or Products Give Linear Orders

The next result is due to Paul Howard and is the crucial ingredient in his answer to Ben-Jebara's question.

Proposition 10 *$DP(n+1)$ implies $LO(n)$.*

Proof Observe that the cartesian product $\prod_{i\in\mathbb{N}}\{1,2,\ldots,n,n+1\}$ has a canonical linear ordering, namely the lexicographic one defined as follows. For any two elements f and g of the product, we set $f \prec g$ if $f \neq g$ and, for the smallest $i \in \mathbb{N}$ such that $f(i) \neq g(i)$, we have $f(i) < g(i)$.

Now assume $DP(n+1)$ and let a sequence $\langle A_i\rangle$ of n-element sets be given. Let x be an object that is in none of the A_i, and let $A'_i = A_i \cup \{x\}$. By $DP(n+1)$ we have a bijection between the sets $\prod_i A'_i$ and $\prod_i\{1,2,\ldots,n,n+1\}$. Use this bijection to transport the lexicographic linear ordering $\preceq$ from the latter set to the former. We use the resulting linear order $\preceq'$ of $\prod_i A'_i$ to linearly order all the A_i as follows.

For any $j \in \mathbb{N}$ and any $a \in A_j$, let $f_{j,a} \in \prod_i A'_i$ be the function sending j to a and all other elements of $\mathbb{N}$ to x. Then define the ordering $\leq_j$ on A_j by $a \leq_j b \iff f_{j,a} \preceq' f_{j,b}$. □

Corollary 11 *$DP(n)$ is equivalent to $LO(n)$.*

Proof By Proposition 10 and Lemma 5, $DP(n+1)$ implies both $LO(n)$ and $CC(n+1)$. And these two together imply, by Lemma 8, $LO(n+1)$. Applying this to n in place of $n+1$ gives the left to right direction of the required equivalence. The converse is part of Lemma 4. □

The analog of the corollary with sums in place of products is also true and somewhat easier to prove.

Lemma 12 *$DS(n)$ is equivalent to $LO(n)$.*

Proof Observe that $\bigsqcup_{i\in\mathbb{N}}\{1,2,\ldots,n\}$ has a canonical linear ordering, namely the lexicographic one, in which $(i,p) \prec (j,q)$ if and only if either $i < j$ or $i = j$ and $p < q$ (where $<$ is the usual ordering of natural numbers).

Now assume $DS(n)$ and let $\langle A_i\rangle$ be a sequence of n-element sets. By $DS(n)$ we have a bijection between $\bigsqcup_i A_i$ and $\bigsqcup_i\{1,2,\ldots,n\}$. Transporting the lexicographic order of the latter set, via a bijection, we get a linear order, say $\preceq'$, of $\bigsqcup_i A_i$. Then we can linearly order all the sets A_i by setting, for $a, b \in A_i$,

$$a \leq_i b \iff (i,a) \preceq' (i,b).$$

This proves the left to right half of the required equivalence, and the converse is part of Lemma 4. □

The following theorem summarizes the preceding results.

Theorem 13 *For any positive integer n, the following are equivalent.*

1. $DS(n)$

2. $DP(n)$

3. $LO(n)$

4. *For all $k \leq n$, $CC(n)$.*

Proof See Lemmas 9 and 12 and Corollary 11. □

It follows immediately from the theorem that $\{n \in \mathbb{N} : DP(n)\}$ is an initial segment of $\mathbb{N}$, for clause (4) of the theorem is obviously preserved when n is decreased. Of course this initial segment contains 1. Are there any other (provable) constraints on what this initial segment might be? The next section gives a complete answer to this question.

4 Possibilities for $DP(n)$

It is consistent that the initial segment $\{n \in \mathbb{N} : DP(n)\}$ be all of $\mathbb{N}$, for example if the axiom of choice holds. It can also be all of $\mathbb{N}$ even when the axiom of choice fails, in fact even when the axiom of choice for countably many finite sets fails. See [2] for a model of set theory where $CC(n)$ holds for all $n \in \mathbb{N}$, yet there is a countable sequence of finite sets (of different cardinalities) with no choice function.

In the remainder of this section, we consider the contrary situation, that the initial segment in question is finite. Determining this segment amounts to determining the first number n not in it, i.e., the first number for which any and therefore (by Theorem 13) all of $DP(n)$, $DS(n)$, $LO(n)$, and $CC(n)$ fail. We have already remarked that this n is at least 2. It turns out that there is just one additional constraint.

Theorem 14 *The first n for which $CC(n)$ fails is not 4. That is, $CC(2)$ and $CC(3)$ together imply $CC(4)$.*

This result is proved in [1, page 266], but for completeness we repeat the proof here.

Proof Assume $CC(2)$ and $CC(3)$, and let $\langle A_i \rangle$ be a sequence of 4-element sets. Define a new sequence $\langle B_i \rangle$ by letting B_i be the set of all partitions of A_i into two 2-element pieces. Thus, if $A_i = \{w, x, y, z\}$ then

$$B_i = \Big\{ \{\{w,x\},\{y,z\}\}, \{\{w,y\},\{x,z\}\}, \{\{w,z\},\{x,y\}\} \Big\}.$$

As each B_i is a 3-element set, $CC(3)$ provides a choice function, i.e., a sequence $\langle C_i \rangle$ where each C_i is a partition of A_i into two 2-element pieces. Then $CC(2)$ lets us choose one member of each C_i, so we get a sequence $\langle D_i \rangle$ where each D_i is a 2-element subset of A_i. Another application of $CC(2)$ lets us choose an element from each D_i. But then these chosen elements constitute a choice function for the original $\langle A_i \rangle$. □

Remark 15 A famous theorem of Tarski (see [2, page 107]) asserts that the axiom of choice for families of 2-element sets implies the axiom of choice for families of 4-element sets. But to establish the axiom of choice for *countable* families of 4-element sets, Tarski's proof needs the axiom of choice for some not necessarily countable families of 2-element sets. Thus, Tarski's theorem does not allow us to drop the hypothesis $CC(3)$ in Theorem 14. In fact, dropping this hypothesis would make the theorem false.

Finally, we show that Theorem 14 is the only constraint on the first n for which $CC(n)$ fails. We give the proof somewhat sketchily, assuming familiarity with permutation models as described in [2].

Theorem 16 *For every positive integer n other than 1 and 4, there is a Fraenkel-Mostowski permutation model of set theory in which $CC(n)$ is false but $CC(k)$ is true for all $k < n$.*

Proof For $n = 2$, the second Fraenkel model (see [2, Chapter 4]) is as required. For $n = 3$ or $n \geq 5$, we use the following variant of that model. The set of atoms is (in the ambient universe where choice holds) the disjoint union of countably many n-element sets A_i ($i \in \mathbb{N}$). For each i, let G_i be the group of even permutations of A_i (the alternating group), and let G be the direct product of the G_i's, acting in the obvious way on the set $A = \bigsqcup_i A_i$ of all atoms; the i^{th} component of a member of G acts on A_i. Let M be the permutation model determined by this G and finite supports. Then $CC(n)$ fails because the model contains the sequence $\langle A_i \rangle$ but no choice function for it (because the action of G_i on A_i has no fixed point — here we use that $n > 2$ and this is why we treated the case $n = 2$ separately at the beginning of the proof). It remains to prove that $CC(m)$ holds for every $m < n$.

Fix a positive integer $m < n$ and let $\langle B_i \rangle_{i \in \mathbb{N}}$ be a sequence in M of m-element sets. Let E be a finite subset of $\mathbb{N}$ such that $\bigsqcup_{i \in E} A_i$ supports the sequence $\langle B_i \rangle$ and therefore supports each B_i (as natural numbers i are fixed by G).

Temporarily fix some $j \in \mathbb{N} - E$ and regard G_j as a subgroup of G (namely the subgroup of elements whose components are all the identity map except for the j^{th} component). As $j \notin E$, G_j acts as a permutation group on each B_i. That is, we have, for each i, a homomorphism h_i from G_j to the group of permutations of B_i. As B_i has cardinality m, its group of permutations has order $m!$, which is smaller than the order of G_j which is $n!/2$ (remember that $m < n$ and $n > 2$). So h_i must have a nontrivial kernel. But the alternating group G_j is simple — here we use that $n \neq 4$ — so the kernel of h_i is all of G_j. That is, each element of G_j fixes all elements of $B = \bigsqcup_i B_i$.

It follows that, if an element of G has only finitely many non-identity components, all at coordinates $j \notin E$, then it fixes all elements of B. Indeed, such an element is the product of finitely many factors, each of which has only one non-identity component and is therefore covered by the argument of the preceding paragraph.

It follows further that every element $b \in B$ is fixed by every element $g \in G$ whose components at coordinates in E are the identity. To see this, let any such b and g be given. Let F be a finite subset of $\mathbb{N}$ such that $\bigsqcup_{i \in F} A_i$ supports b. Let $g' \in G$ be the element that agrees with g at coordinates in F and has the identity in all other coordinates. So $g = g'h$ where h has the identity in all components in $E \cup F$. In particular, h fixes all points in the support $\bigsqcup_{i \in F} A_i$ of b and therefore fixes b. Also, g' has only finitely many non-identity components, all at coordinates not in E, so it also fixes b. Therefore, $g = g'h$ fixes b, as claimed.

We have thus shown that one finite set, namely $\bigsqcup_{i \in E} A_i$, supports all the elements of B. Therefore, it also supports any function from natural numbers into B. Thus, any choice function for $\langle B_i \rangle$ in the ambient universe is in M. □

Corollary 17 *For every positive integer n other than 1 and 4, it is consistent relative to Zermelo-Fraenkel set theory that n is the smallest integer for which $CC(n)$ fails.*

Proof The statement that $CC(m)$ holds for all $m < n$ and fails for $m = n$ is injectively boundable in the sense of [3]. (The reason is essentially that no uncountable ordinal admits a one to one map into a countable union of finite sets.) According to the main result of [3], the Fraenkel-Mostowski model from the theorem ensures the Zermelo-Fraenkel consistency claimed in the corollary. □

Remark 18 We pointed out in the proof of Theorem 16 that the hypothesis $n \neq 4$ is used to infer from the fact that h_i has a nontrivial kernel that h_i must be trivial. For $n = 4$, this inference fails, as the alternating group on 4 objects has a normal subgroup (the Klein 4-group) of index 3 and thus has a nontrivial homomorphism into the symmetric group on 3 objects. Thus, the argument for $CC(3)$ in the permutation model fails.

But the alternating group on 4 objects has no subgroup of index 2 and therefore no nontrivial homomorphism into the symmetric group on 2 objects. Thus, the argument for $CC(2)$ still works when $n = 4$.

Summarizing, we have a permutation model that satisfies $CC(2)$ but not $CC(4)$ and therefore (thanks to Theorem 14) not $CC(3)$. This justifies our assertion, in an earlier remark, that Theorem 14 would become false if we omitted the $CC(3)$ hypothesis.

References

[1] Andreas Blass, "Cohomology detect failures of the axiom of choice," *Trans. Amer. Math. Soc.* 279 (1983) 257–269.

[2] Thomas Jech, *The Axiom of Choice*, North-Holland, 1973.

[3] David Pincus, "Zermelo-Fraenkel consistency results by Fraenkel-Mostowski methods," *J. Symbolic Logic* 37 (1972) 721–743.