SUBGROUPS OF THE BAER-SPECKER GROUP WITH FEW ENDOMORPHISMS BUT LARGE DUAL

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Abstract. Assuming the continuum hypothesis, we construct a pure subgroup G of the Baer-Specker group $\mathbb{Z}^{\aleph_0}$ with the following properties. Every endomorphism of G differs from a scalar multiplication by an endomorphism of finite rank. Yet G has uncountably many homomorphisms to $\mathbb{Z}$.

§1 Introduction

The Baer-Specker group, which is the additive group $P = \mathbb{Z}^{\aleph_0}$ of all integer-valued sequences, and its obvious generalization $\mathbb{Z}^{\kappa}$, the cartesian product of $\mathbb{Z}$ for any infinite cardinal κ, have attracted continuous research over many years. This is not too surprising because many questions on abelian groups, like the famous Whitehead problem, can be stated in terms of subgroups of such products. Recall that

(†) *P is $\aleph_1$-free, i.e. all countable subgroups of P are free*

by a result of Baer, cf. Fuchs [15, Vol. 1, p. 94, Theorem 19.2]. Besides the Whitehead problem, many other properties of these products turn out to depend on the set theoretic assumptions, which make these groups a favored playground for logicians as well as algebraists. Examples of results of this kind can be found in [1–3, 8, 13, 17–19]; see [14] for general references. Moreover, P carries many algebraic pathologies which also occur elsewhere in abelian group theory but which can be studied more explicitly in P because of the sequence representation of elements in P. Such properties quite often can be phrased in the language of the endomorphism ring of certain subgroups of products, cf. [5, 7, 9–12, 16, 20].

The most useful tool for constructing such groups is a fundamental result on P which characterizes slender groups, which is due to Nunke and based on Specker [22], see [15, Vol. 2, pp. 158–163]. Let $S = \mathbb{Z}^{(\aleph_0)} = \oplus_{i\in\omega} e_i\mathbb{Z}$ denote the free group on $\aleph_0$ generators, i.e., the canonical direct sum in the product $P = \prod_{i\in\omega} e_i\mathbb{Z}$. Recall that a group G is *slender* if any homomorphism from P into G maps almost all elements $e_i (i \in \omega)$ onto 0. Specker [22] showed that $\mathbb{Z}$ is slender. This has many consequences for dual groups, including the following, where we use the notation

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G^* for the dual $\mathrm{Hom}(G, \mathbb{Z})$ of G. Specker [22] showed that $P^* \cong S$, and obviously $S^* \cong P$. It follows that P has no direct summand isomorphic to S. More generally, any group G whose dual is isomorphic to S cannot have a summand isomorphic to S, for $G^* \cong S$ has no summand isomorphic to $S^* \cong P$. It is natural to ask how generally "summand $\cong S$" is the only obstruction to "dual $\cong S$." Specifically, John Irwin asked whether a pure subgroup of P that does not have a direct summand isomorphic to S must have a dual isomorphic to S.

Assuming the continuum hypothesis CH we shall answer this question negatively by proving the theorem below.

To ensure that the group we construct does not have a summand isomorphic to S, we arrange that its endomorphism ring be as small as possible. Among the endomorphisms of any torsion-free abelian group are the scalar multiplications by any $z \in \mathbb{Z}$, so we can identify $\mathbb{Z}$ with a subring of $\mathrm{End}(G)$. In addition, $\mathrm{End}(G)$ contains a two-sided ideal $\mathrm{Fin}(G)$ consisting of endomorphisms of finite rank, i.e., endomorphisms sending all of G into a subgroup of finite rank. We call an endomorphism *almost scalar* if it is the sum of a scalar multiplication and an endomorphism of finite rank, i.e., if it lies in $\mathbb{Z} \oplus \mathrm{Fin}(G)$. We shall arrange that all endomorphisms of our group are almost scalar.

Groups with such a small endomorphism ring are constructed in [6, 9] and inside P in [12, 16, 20]. The new ingredient in the present paper is to ensure the dual of G is uncountable. While an easy form of a "black-box-type argument" as in [6, 9, 12] will force endomorphisms to be almost scalar, on the other hand we must ensure that uncountably many homomorphisms from S into $\mathbb{Z}$ survive while we build G up by a transfinite chain of extensions from S inside P. In order to use topological arguments, we will restrict our source of new elements for a potential group G to the $\mathbb{Z}$-adic closure D of S in P, which is the pre-image of the maximal divisible subgroup D/S in P/S. It will be important that this group D can be used to test slenderness, as observed many years ago by Ti Yen, cf. Fuchs [15, Vol. 2, p. 163, Exercise 5] and extended recently in [20, p. 276, Corollary 2.5]:

$(*)$ *A group X is slender if and only if $Hom(P, X) = Fin(P, X)$ if and only if $Hom(D, X) = Fin(D, X)$. In particular, every homomorphism from D to S has finite rank.*

Choosing suitable generators by induction we will construct the desired group G as in the theorem. CH allows us to perform this in $\aleph_1$ steps so that at every induction step the set chosen so far is countable.

Theorem. *Assuming CH, there is a group G with the properties:*

(a) $S \subset G \subset D$ and G is pure in D (hence also in P).

(b) $End\, G = \mathbb{Z} \oplus Fin\, G$, i.e. every endomorphism of G is almost scalar.

(c) The dual group $G^* = Hom(G, \mathbb{Z})$ is uncountable.

This indeed answers Irwin's question. Part (b) of the theorem implies that G does not have a direct summand isomorphic to S, for such a summand, and therefore G itself, would have many endomorphisms that are not almost scalar. And of course part (c) implies that G^* is not isomorphic to the countable group S.

§2 Preliminary Facts

We begin by proving a simple lemma strengthening, in the infinite case, the well-known fact that a group cannot be covered by two proper subgroups.

Lemma 1. *If A and B are proper subgroups of an infinite group G, then*

$$|G\backslash(A \cup B)| = |G|.$$

Proof. First of all observe that for every proper subgroup C of an infinite group H we have $|H\backslash C| = |H|$, for all cosets of C have the same cardinality which, of course, needs to be applied only if $|C| = |H|$.

Thus the assertion follows in case one of the subgroups contains the other, and the result is trivial if both of them have lesser cardinality than G. So assume without loss of generality that $|A| = |G|$ and that neither of A and B contains the other. Fix an element $b \in B \setminus A$. By the first paragraph of this proof (applied to A and $A \cap B$), $A \setminus B$ has $|A| = |G|$ elements a. Then the $|G|$ sums $a + b$ are in neither A nor B. $\square$

Given a family $\mathcal{F}$ of proper subgroups of an infinite group G which can be partitioned into subfamilies $\mathcal{F}_0$ and $\mathcal{F}_1$ neither of which generates all of G, we see that also $|G\backslash \bigcup \mathcal{F}| = |G|$.

At the end of the proof of the theorem we will apply the following rather special instance of this.

Corollary. *If $\mathcal{F}$ is a family of linearly independent proper subgroups of an infinite abelian group G, then $|G\backslash \bigcup \mathcal{F}| = |G|$.*

Proof. If $\mathcal{F}$ is empty, there is nothing to prove. So let $A \in \mathcal{F}$. Then clearly $\mathcal{F}_0 = \{A\}$ and $\mathcal{F}_1 = \mathcal{F}\backslash\{A\}$ meet the requirements of the lemma and the Corollary follows. $\square$

The corollary is used to have enough room to find particular elements in P. It is interesting to observe that [7] also needs a lemma about free space in vector spaces in order to study aspects of P.

A *torsionless*(abelian) group is (up to isomorphism) a subgroup of a product $\mathbb{Z}^\kappa$ or equivalently any group G where G^* separates points in the sense that for any $0 \neq g \in G$ there is a $\sigma \in G^*$ such that $g\sigma \neq 0$; see [14] and [15] for more details about torsionless groups.

For any integer z and any group G, we write z also for the endomorphism of G given by multiplication by z, and we call such an endomorphism *scalar*. We also write $\mathbb{Z}$ for the set of scalar endomorphisms of G. We also use this notation and terminology when z is a rational number and G is torsion-free, but then the homomorphism z maps G into its divisible hull rather than into itself. We call an endomorphism of G *almost scalar* if it differs from a scalar by a homomorphism which has an image of finite rank. The following lemma shows that, when G is torsionless, it doesn't matter whether we take the scalars to be integers or rationals in this definition. (The lemma holds for many groups other than torsionless ones, but we won't need any more generality.)

Lemma 2. *Let G be a torsionless abelian group of infinite rank. Let z be a rational scalar and σ a finite-rank homomorphism of G into its divisible hull. Suppose $z+\sigma$ maps G into itself. Then z is an integer.*

Proof. Because G is torsionless and has infinite rank, there is a non-zero homomorphism $\varphi : G \to \mathbb{Z}$ vanishing on $G \cap \mathrm{Range}(\sigma)$. Without loss of generality, φ maps G onto $\mathbb{Z}$ (just divide by a generator of the range); fix $e \in G$ with $(e)\varphi = 1$. We write φ also for the homomorphic extension of φ mapping the divisible hull of G into $\mathbb{Q}$, which vanishes on $\mathrm{Range}(\sigma)$. Then, as $e(z+\sigma) \in G$, we have

$$z = (e)\varphi \cdot z = (ez)\varphi = (ez + e\sigma)\varphi \in G\varphi = \mathbb{Z}. \quad \square$$

§3 Proof Of The Theorem

G will be obtained as the union of an increasing ω_1-sequence of subgroups G_α ($\alpha < \omega_1$), where $G_0 = S$, $G_\lambda = \bigcup_{\alpha<\lambda} G_\alpha$ for limit λ, and each G_α is a countable, pure subgroup of D. This ensures that part (a) of the theorem holds. The successor stages of the induction are designed to ensure parts (b) and (c).

To deal with (b), we begin by enumerating all homomorphisms $\eta : S \to P$ as an ω_1-sequence $(\eta_\alpha)_{\alpha<\omega_1}$ such that each homomorphism η occurs stationarily often in the sequence, i.e., $\{\alpha \mid \eta_\alpha = \eta\}$ is stationary. Recall that a subset of ω_1 is called stationary if it meets every closed unbounded subset of ω_1. Recall also that ω_1 can be partitioned into $\aleph_1$ stationary subsets; see [21, p. 59, Lemma 7.6]. The number of homomorphisms η to be enumerated is $\aleph_1$, because we are assuming the continuum hypothesis. So we can set up a bijection between the η's and the stationary sets in a partition of ω_1, and we obtain the desired enumeration by letting η_α be the η corresponding to the stationary set that contains α.

With respect to the $\mathbb{Z}$-adic topology, S is dense in D and each η_α is a continuous map from S into P. Hence η_α has a unique largest extension with domain pure in D, say $\overline{\eta_\alpha} : D_\alpha \to P$. This is obtained by first extending η_α to a homomorphism from D, which is the $\mathbb{Z}$-adic closure of S in P, to the $\mathbb{Z}$-adic completion of P, and then restricting to the subgroup D_α of D that is mapped into P.

At stage α of the construction, when we have G_α and are defining $G_{\alpha+1}$, we shall do nothing (i.e., set $G_{\alpha+1} = G_\alpha$) if either the domain of $\overline{\eta_\alpha}$ does not include all of G_α, or $\overline{\eta_\alpha}$ maps some element of G_α outside G_α, or the restriction $\overline{\eta_\alpha} \restriction G_\alpha$ is almost scalar. So the only α's that are *active* in the sense that we do something in the step from α to $\alpha+1$ are those for which $\overline{\eta_\alpha} \restriction G_\alpha$ is a (total) function from G_α into itself and is not almost scalar. At such an active stage, we shall ensure that $\overline{\eta_\alpha}$ does not map (all of) $G_{\alpha+1}$ into G.

If we achieve this, then (b) will hold. To see this, suppose e were an endomorphism of G that is not almost scalar. Thus, for each scalar z, the endomorphism $e-z$ of G has infinite rank. Fixing a countable infinity of independent elements in the range of $e-z$ and fixing pre-images under $e-z$ for them in G, we see that these pre-images are all in G_β for some countable β. That is, $(G_\beta)(e-z)$ has infinite rank. Furthermore, since there are only countably many scalars z, we can fix a single β

that works for them all. Now since e maps G into itself, it follows easily (in view of the continuity of the G_α sequence at limit ordinals) that $\{\alpha \geq \beta \mid (G_\alpha)e \subseteq G_\alpha\}$ is closed and unbounded. So it has a non-empty intersection with the stationary set $\{\alpha \mid e \restriction S = \eta_\alpha\}$. Fix some α in this intersection. Since e is an extension of η_α and $\overline{\eta_\alpha}$ is the unique largest extension of η_α, we have that $e = \overline{\eta_\alpha} \restriction G$. In particular, the domain of $\overline{\eta_\alpha}$ includes G and therefore includes G_α. Also, our choice of α ensures that $\overline{\eta_\alpha}$ maps G_α into itself and $\overline{\eta_\alpha} \restriction G_\alpha$ is not almost scalar. So α is an active stage in our construction, and therefore $\overline{\eta_\alpha}$ does not map $G_{\alpha+1}$ into G. That is absurd, as $\overline{\eta_\alpha}$ is an extension of e, which maps all of G into G.

It remains to carry out the construction at active stages. So we have $\overline{\eta_\alpha}$ defined on all of G_α, mapping G_α into itself, and not almost scalar on G_α. In this situation, we shall set $G_{\alpha+1} = \langle G_\alpha \cup \{x\}\rangle_*$, the pure subgroup of D generated by G_α and one new, carefully chosen $x \in D$. If x is in the domain D_α of $\overline{\eta_\alpha}$, then we shall also ensure that $x\overline{\eta_\alpha} \notin G_\beta$ for all β; thus $\overline{\eta_\alpha}$ will not map $G_{\alpha+1}$ into G. Of course, in order to do this, we must choose x so that $x\overline{\eta_\alpha} \notin G_{\alpha+1}$ and, at later stages of the construction, we must be careful not to put $x\overline{\eta_\alpha}$ into any G_β.

Thus, as the construction proceeds, we accumulate a set V of forbidden elements, which must never be put into any G_β. Each stage contributes at most one element $x\overline{\eta_\alpha}$ to V, so V is countable at each stage of the construction. Of course the choice of x at stage α is constrained by the elements already in V from previous stages.

To deal with part (c) of the theorem, we shall have, at each stage α, some homomorphisms $f_\xi^\alpha : G_\alpha \to \mathbb{Z}$, indexed by the ordinals $\xi < \alpha$, such that for $\xi < \alpha < \beta$ we have $f_\xi^\alpha = f_\xi^\beta \restriction G_\alpha$. Then, for each $\xi < \omega_1$, the homomorphisms f_ξ^α for all α (from $\xi + 1$ on) combine to give a homomorphism $f_\xi : G \to \mathbb{Z}$. If we make the f_ξ^α at each stage α distinct (for distinct ξ), then all these f_ξ will be distinct and (c) will hold.

Thus, what needs to be done at stage α of the construction is the following. We are given

(1) a countable pure subgroup G_α of D,
(2) countably many homomorphisms $f_\xi^\alpha : G_\alpha \to \mathbb{Z}$, indexed by $\xi < \alpha$,
(3) a countable set V_α of forbidden elements with $V_\alpha \cap G_\alpha = \emptyset$ and
(4) a homomorphism $\overline{\eta_\alpha} : D_\alpha \to P$ whose domain includes G_α, which maps G_α into G_α, and which is not almost scalar on G_α.

We seek an element $x \in D$ such that

(5) $\langle G_\alpha \cup \{x\}\rangle_*$ contains neither $x\overline{\eta_\alpha}$ nor any element of V_α, and
(6) each f_ξ^α extends to $f_\xi^{\alpha+1} : \langle G_\alpha \cup \{x\}\rangle_* \to \mathbb{Z}$.

In addition, we need one new homomorphism $f_\alpha^{\alpha+1} : \langle G_\alpha \cup \{x\}\rangle_* \to \mathbb{Z}$, distinct from all $f_\xi^{\alpha+1}$ for $\xi < \alpha$.

Getting $f_\alpha^{\alpha+1}$ is easy, as $\langle G_\alpha \cup \{x\}\rangle_*$ is a countable subgroup of P containing S, hence is free of infinite rank by (†) of the introduction, and hence has uncountably many homomorphisms to $\mathbb{Z}$. So our efforts from now on will be directed toward finding an $x \in D$ subject to (5) and (6). Once we have such an x, we can set $G_{\alpha+1} = \langle G_\alpha \cup \{x\}\rangle_*$ and $V_{\alpha+1} = V_\alpha \cup \{x\overline{\eta_\alpha}\}$, and the proof will be complete.

Notice that G_α, being a countable, pure subgroup of P and dense in the product topology of P, can be mapped onto S by an automorphism of P, according to a theorem of Chase [4, p. 605, Corollary 3.3]. Furthermore, since $P/S = (D/S) \oplus R$ where R is reduced and D/S is divisible, and since $S \subseteq G_\alpha \subseteq D$, we have $P/G_\alpha = (D/G_\alpha) \oplus R$ with D/G_α divisible (being a quotient of D/S). So D is the pre-image in P of the divisible part of P/G_α, as well as the pre-image in P of the divisible part of P/S. Therefore, an automorphism of P that maps G_α onto S necessarily maps D onto D. Applying such an automorphism, we obtain the following description of what needs to be done at stage α (numbered to match the earlier description). We are given

(2) countably many homomorphisms $f_i : S \to \mathbb{Z}$ $(i \in \omega)$, where we have re-indexed by ω in place of α,
(3) a countable set $V \subseteq P - S$,
(4) a homomorphism $\bar{\eta} : E \to P$, whose domain E includes S, such that $\bar{\eta} \restriction S$ maps S into itself and is not almost scalar.

We seek an $x \in D$ such that

(5) $\langle S \cup \{x\}\rangle_*$ contains neither $x\bar{\eta}$ nor any member of V, and
(6) each f_i extends to a homomorphism $\langle S \cup \{x\}\rangle_* \to \mathbb{Z}$.

The V part of (5) is easy to handle. We must exclude any x such that

$$s + xm = vn \tag{7}$$

for some $s \in S$, $m \in \mathbb{Z}$, $n \in \mathbb{Z} - \{0\}$, and $v \in V$. We need not worry about $m = 0$, as $S \cap V = \emptyset$ (and S is pure). Then, for each s, m, n, and v, there is at most one x satisfying (7) (as P is torsion-free). So only countably many x are excluded. Therefore, it suffices to find uncountably many $x \in D$ satisfying (6) and

$$x\bar{\eta} \notin \langle S \cup \{x\}\rangle_*. \tag{5'}$$

Here (5′) is regarded as vacuously true if $x \notin E$.

Fix a sequence $(r_n)_{n\in\omega}$ of rational numbers in which each rational number occurs infinitely often. We shall construct a sequence of elements $b_n \in S - \{0\}$ with the following properties. We write $\operatorname{supp}(x) = \{i \in \omega : x_i \neq 0\}$ for the set of indices where an element $x = \sum_{i\in\omega} e_i x_i \in P$ has non-zero components x_i; for elements of S, this is a finite set, so the maximum and minimum mentioned in (8) below make sense.

(8) If $m < n$ then $\max\operatorname{supp}(b_m) < \min\operatorname{supp}(b_n)$.
(9) If $i \leq n$ then $b_n f_i = 0$.
(10) $\max\operatorname{supp}(b_n\bar{\eta} - b_n r_n) > n$.

In (10), we multiplied by a rational scalar, so we should work in the divisible hull of P. Alternatively, by clearing the denominators, we can reformulate such statements (here and below) to involve only integer scalars.

The b_n are defined inductively. Suppose $b_0, \dots, b_{n-1}$ are given, and let $q \in \omega$ be larger than all elements of their supports. We shall choose b_n in the subgroup

$S_q = \bigoplus_{i \geq q} e_i\mathbb{Z}$ of S consisting of elements of S whose supports have minimum $\geq q$. This will ensure that (8) holds. Since S_q has finite corank q in S, the range of $(\bar{\eta} - r_n) \restriction S_q$ has infinite rank. (Otherwise, $\bar{\eta} \restriction S$ would be the scalar r_n plus an endomorphism of finite rank, contrary to (4).) So we can find a finitely generated subgroup F of S_q whose image under $\bar{\eta} - r_n$ has rank $\geq 2n+3$. The subgroup F' of F consisting of elements b with $(\forall i \leq n)\, bf_i = 0$, i.e., the subgroup of elements that could serve as b_n and satisfy (9), has corank $\leq n+1$ in F (being the intersection of the kernels of $n+1$ homomorphisms f_i to $\mathbb{Z}$). So the image of F' under $\bar{\eta} - r_n$ has rank at least $(2n+3) - (n+1) > n+1$. This image must therefore contain an element with a non-zero component past position n. That is, we can choose $b_n \in F'$ (thereby satisfying (9)) in such a way that (10) also holds. This completes the construction of the sequence (b_n).

Define an endomorphism β of P taking e_i to b_i, that is

$$\beta : P \to P : \sum_{i=0}^{\infty} e_i x_i \mapsto \sum_{i=0}^{\infty} b_i x_i.$$

The infinite sum in this definition makes sense because the b_i have disjoint supports by (8). It is trivial to check that β maps S into S and D into D. (The latter also follows from the former by $\mathbb{Z}$-adic continuity.) Since all the b_i are non-zero, β is one-to-one.

Also define, for each $i \in \omega$, a partial homomorphism

$$g_i : P \rightharpoonup \mathbb{Z} : \sum_{j=0}^{\infty} e_j x_j \mapsto \lim_{n\to\infty} (\sum_{j=0}^{q_n} e_j x_j) f_i,$$

where $q_n = \max \operatorname{supp}(b_n)$, and where the limit means the value for all sufficiently large n provided this value is eventually independent of n. (If the value is not eventually independent of n, the limit is undefined; that's why g_i is only a partial homomorphism.) The domain of g_i is clearly a pure subgroup of P that includes S, and $g_i \restriction S = f_i$. Furthermore, if $\sum_{i=0}^{\infty} e_i x_i = (\sum_{i=0}^{\infty} e_i y_i)\beta$, then

$$(\sum_{j=0}^{q_{n+1}} e_j x_j) f_i - (\sum_{j=0}^{q_n} e_j x_j) f_i = (\sum_{j=q_n+1}^{q_{n+1}} e_j x_j) f_i = (b_{n+1} y_{n+1}) f_i = 0$$

by (9) provided $n \geq i$. Thus $P\beta \subseteq \operatorname{domain}(g_i)$. We shall ensure (6) by choosing x to be $y\beta$ for some $y \in D$; then $\langle S \cup \{x\} \rangle_* \subseteq \operatorname{domain}(g_i)$, so we can extend f_i to $g_i \restriction \langle S \cup \{x\} \rangle_*$.

To obtain uncountably many $x \in D$ satisfying (5′) and (6), which is all we need to finish the proof, it now suffices, as β is one-to-one, to find uncountably many $y \in D$ such that

$$y\beta\bar{\eta} \notin \langle S \cup \{y\beta\} \rangle_*. \tag{11}$$

There is an easy case, namely if $D\beta \not\subseteq \operatorname{domain}(\bar\eta)$, so $\beta\bar\eta$ is not defined on all of D. Then the domain $E\beta^{-1}$ of $\beta\bar\eta$ is a proper subgroup of the uncountable group D. By Lemma 1, D has uncountably many elements outside this subgroup, and any of them can serve as y since they satisfy (11) vacuously. This finishes the easy case.

Henceforth, assume that $\beta\bar\eta$ is defined on all of D. To find uncountably many $y \in D$ satisfying (11), we consider how some y could fail to satisfy (11). That would mean that

$$y\beta\bar\eta \cdot q = y\beta \cdot p + s$$

for some $s \in S$ and some integers p, q with $q \neq 0$. Write r for the rational number p/q. Then $y\beta(\bar\eta - r) = s(1/q)$ has finite support.

For each rational number r, let

$$W_r = \{y \in D \mid y\beta\bar\eta - y\beta \cdot r \text{ has finite support}\}.$$

The preceding discussion shows that any $y \in D$ violating (11) is in some W_r. So, to complete the proof, we need to find uncountably many elements of D that are in no W_r.

Since both $\bar\eta$ and β map S into itself, we clearly have $S \subseteq W_r$. Also, W_r is a pure subgroup of D. Therefore, W_r/S is a pure subgroup of the rational vector space D/S. We shall prove that these subgroups are proper, linearly independent, pure subgroups of D/S.

Once this is done, the corollary of Lemma 1 will produce uncountably many elements of D/S not contained in any W_r; the pre-images in D of these elements will be the desired y's, and the proof will be complete.

First, we show that W_r/S is a proper subgroup of D/S. Suppose not, i.e., suppose $W_r = D$ for a certain $r = p/q$, where p and q are integers. In view of the definition of W_r,

$$y \mapsto (y\beta\bar\eta - y\beta \cdot r) \cdot q = y\beta\bar\eta \cdot q - y\beta \cdot p$$

is a homomorphism from $W_r = D$ into S. Such a homomorphism has finite rank, by (*) in the introduction. So $D\beta(\bar\eta - r)$ would have finite rank. But this group contains, by definition of β, the elements $b_n(\bar\eta - r)$ for all n, including the infinitely many n for which $r_n = r$. And those elements generate a group of infinite rank, by property (10). This contradiction shows that W_r/S is a proper subgroup of D/S.

It remains to show that the subgroups W_r/S are linearly independent. This proof is a minor variation of the standard proof that eigenspaces of a linear operator corresponding to different eigenvalues are linearly independent. Suppose we had a linear dependence between these subgroups of D/S. Back in D, this would mean

$$\sum_{i=1}^{k} w_i c_i \in S, \tag{12}$$

where $c_i \in \mathbb{Q} - \{0\}$, $k \geq 1$, $w_i \in W_{r_i} - S$, and all r_i are distinct. (As before, we adopt the convention that we either work with divisible hulls and in particular that

S in (12) really means its divisible hull, so that the rational scalars make sense, or interpret statements like (12) as meaning the result of clearing denominators, so that only integer scalars occur.) Take a relation of the form (12) with k as small as possible. Note that $k \geq 2$, for $w_1 c_1 \in S$ would imply $w_1 \in S$, a contradiction. Apply $\beta\bar{\eta} - \beta \cdot r_1$ to (12), remembering that S is closed under $\bar{\eta}$ and β and that $\beta\bar{\eta}$ is defined on all of D so that this makes sense. The result is, in view of the definition of W_{r_i},

$$\sum_{i=1}^{k} [w_i \beta(r_i - r_1) + \text{ elements of } S] c_i \in S.$$

Therefore,

$$\left(\sum_i w_i (r_i - r_1) c_i\right) \beta = \sum_i w_i \beta (r_i - r_1)(c_i) \in S.$$

It is immediate from the definition of β (since no b_i is 0) that $S\beta^{-1} = S$. So we have

$$\sum_i w_i (r_i - r_1) c_i \in S.$$

The $i = 1$ term here contains $r_1 - r_1$, so it vanishes, but the other terms have non-zero coefficients. So we have a relation of the form (12) with fewer summands. This contradicts the minimality of k, and this contradiction completes the proof. □

References

1. A. Blass, *Cardinal characteristics and the product of countably many infinite cyclic groups*, J. Algebra **169** (1994), 512–540.
2. A. Blass, *Near coherence of filters, II: Applications to operator ideals, the Stone-Čech remainder of a half-line, order ideals of sequences, and slenderness of groups*, Trans. Amer. Math. Soc. **300** (1987), 557–581.
3. A. Blass and C. Laflamme, *Consistency results about filters and the number of inequivalent growth types*, J. Symbolic Logic **54** (1989), 50–56.
4. S. U. Chase, *Function topologies on abelian groups*, Illinois J. Math. **7** (1963), 593–608.
5. A. L. S. Corner, *A class of pure subgroups of the Baer-Specker group*, unpublished talk given at Montpellier Conference on Abelian Groups (1967).
6. A. L. S. Corner and R. Göbel, *Prescribing endomorphism algebras, a unified treatment*, Proc. London Math. Soc. **50** (1985), 447–479.
7. A. L. S. Corner and B. Goldsmith, *On endomorphisms and automorphisms of some pure subgroups of the Baer-Specker group*, Abelian Group Theory (R. Göbel, P. Hill, and W. Liebert, eds.), Contemporary Mathematics, vol.171, pp. 69–78.
8. M. Dugas and R. Göbel, *Die Struktur kartesischer Produkte der ganzen Zahlen modulo kartesische Produkte ganzer Zahlen*, Math. Z. **168** (1979), 15–21.
9. M. Dugas and R. Göbel, *Endomorphism rings of separable torsion-free abelian groups*, Houston J. Math **11** (1985), 471–483.
10. M. Dugas and J. Irwin, *On basic subgroups of* ΠZ, Comm. Algebra **19** (1991), 2907–2921.
11. M. Dugas and J. Irwin, *On pure subgroups of cartesian products of integers*, Resultate Math. **15** (1989), 35–52.
12. M. Dugas, J. Irwin, and S. Khabbaz, *Countable rings as endomorphism rings*, Quart. J. Math. Oxford **39** (1988), 201–211.

13. K. Eda, *A note on subgroups of* $\mathbb{Z}^{\mathbb{N}}$, Abelian Group Theory (R. Göbel, L. Lady, and A. Mader, eds.), Lecture Notes in Mathematics 1006, Springer-Verlag, 1983, pp. 371–374.
14. P. Eklof and A. Mekler, *Almost Free Modules, Set-theoretic Methods*, North-Holland, 1990.
15. L. Fuchs, *Abelian Groups, Vol. I and II*, Academic Press, 1970 and 1973.
16. R. Göbel and B. Goldsmith, *On separable torsion-free modules of countable density character*, J. Algebra **144** (1991), 79–87.
17. R. Göbel and B. Wald, *Wachstumstypen und schlanke Gruppen*, Symposia Math. **23** (1979), 201–239.
18. R. Göbel and B. Wald, *Martin's axiom implies the existence of certain slender groups*, Math. Z. **172** (1980), 107–121.
19. R. Göbel, B. Wald, and P. Westphal, *Groups of integer-valued functions*, Abelian Group Theory (R. Göbel and E. Walker, eds.), Lecture Notes in Mathematics 874, Springer-Verlag, 1981, pp. 161–178.
20. R. Göbel and B. Wald, *Separable torsion-free modules of small type*, Houston J. Math. **16** (1990), 271–288.
21. T. Jech, *Set Theory*, Academic Press, 1978.
22. E. Specker, *Additive Gruppen von Folgen ganzer Zahlen*, Portugal. Math. **9** (1950), 131–140.

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