FREE SUBGROUPS OF THE BAER-SPECKER GROUP

Andreas Blass
Mathematics Department
University of Michigan
Ann Arbor, MI 48109–1109, U.S.A.
ablass@umich.edu
and
John Irwin
9001 Starmount
Las Vegas, NV 89134, U.S.A.
drjmirwin@lvcm.com

Abstract

The Baer-Specker group $\Pi = \mathbb{Z}^{\aleph_0}$ is the product of countably many copies of the additive group of integers. We are concerned with subgroups of Π that are free abelian groups. Among the issues we consider are testing freeness of a subgroup by means of its intersections with other specified subgroups, the relationship between freeness and other "smallness" properties, and constraints on the location of free subgroups within Π.

1 Introduction

We shall prove a number of results concerning free (or close to free) subgroups of the Baer-Specker group $\Pi = \mathbb{Z}^{\aleph_0}$, the direct product of infinitely many copies of the additive group $\mathbb{Z}$ of integers. All groups considered in this paper are abelian, and "free" always means "free abelian." The members of Π, infinite sequences of integers, will be written in the form $x = (x(n) : n \in \mathbb{N})$.

Specker [18] proved that Π is not free, but Baer [1] proved that all its countable subgroups are free. Thus, we shall be concerned with uncountable subgroups of Π.

Among the questions we consider are the following. What subgroups of Π contain uncountable, pure, free subgroups? What subgroups have pure, free subgroups nearly complementary to them? What does the quotient of Π by a large, pure, free subgroup look like? To what extent is a pure, free subgroup F of Π determined by the quotient Π/F? Can one infer freeness of a group from freeness of "many" subgroups? How is freeness related to other smallness properties of subgroups of Π? (Freeness can reasonably be considered a smallness property because subgroups of free groups are free.)

The rest of this introductory section is devoted to fixing notation and recalling some known results that we shall need. The standard reference for the material on abelian groups is [10].

We use the notation Σ for the subgroup of Π consisting of those sequences x in which only finitely many terms are non-zero. Thus, Σ is the free abelian group generated by the "unit vectors," i.e., the sequences e_j in which the j^{th} component is 1 and all the others are 0.

Balcerzyk [2] (see also [10, vol. 1, page 177]) showed that the quotient Π/Σ is the direct sum of a divisible group and an algebraically compact group. Furthermore, the algebraically compact summand is isomorphic to the direct product of countably many copies of the groups of p-adic integers for all primes p. The pre-image in Π of the divisible part of Π/Σ will be denoted by D. Thus, D consists of those sequences $x \in \Pi$ with the following property: Each positive integer d divides all but finitely many of the components $x(n)$ of x. Since D is pure in Π and has the cardinality of the continuum, the quotient D/Σ, which is the divisible part of Π/Σ, is a torsion-free, divisible group of rank $2^{\aleph_0}$.

The algebraically compact summand complementary to D/Σ in Π/Σ is uniquely determined only up to isomorphism. In fact, there are $2^{2^{\aleph_0}}$ such complements (see [3]). We shall refer to any one of them as a *reduced part* of Π/Σ, and we shall refer to their pre-images in Π as the *pre-reduced parts* of Π. Thus, a pre-reduced part of Π is a subgroup $A \subseteq \Pi$ such that $A + D = \Pi$ and $A \cap D = \Sigma$. (It is shown in [3] that, although the reduced parts of Π/Σ are all isomorphic, the pre-reduced parts of Π are not.)

Topological considerations will play a role in some of our results. In this connection, we adopt the following conventions.

- $\mathbb{Z}$ has the discrete topology.
- Π has the product topology, as a product of $\mathbb{Z}$'s.
- Any subset of Π has the subspace topology inherited from Π.

Notice that the product topology on Π is the same as the topology induced by the complete metric in which the distance between x and y is 2^{-n} if the sequences x and y first differ at the n^{th} component. We shall also need the observation that Π is separable in the topological sense, i.e., it has a countable dense subset, for example Σ.

We shall need several well known results about Π, which we list here for reference. The first is a very strong form of the fact that Π is not free.

Theorem 1.1 (Specker [18]) *Every homomorphism $h : \Pi \to \mathbb{Z}$ sends all but finitely many of the unit vectors e_j to 0. Furthermore, h is completely determined by the values $h(e_j)$.*

Corollary 1.2 *Every homomorphism $\Pi \to \mathbb{Z}$ is continuous. Therefore, so is every homomorphism $\Pi \to \Pi$.*

Theorem 1.3 (Chase [7]) *Any countable, dense, pure subgroup of Π can be mapped onto Σ by an automorphism of Π. Therefore, any countable subgroup of Π can be mapped into Σ by an automorphism of Π.*

Theorem 1.4 (Nunke [16]) *Every closed subgroup of Π is isomorphic to a product of copies of $\mathbb{Z}$.*

We denote by B the subgroup of Π consisting of the bounded sequences, i.e., the sequences that have only finitely many distinct components. The main theorem about this group is the following theorem of Nöbeling, for which a shorter proof due to Bergman is presented in [10, vol. 2, section 97].

Theorem 1.5 (Nöbeling [15]) *B is a free abelian group of rank $2^{\aleph_0}$, and Σ is a direct summand of B.*

Notice also that B is pure in Π and that the quotient Π/B is divisible. The latter is because given any $x \in \Pi$ and any positive integer n, we can find $y \in \Pi$ such that all components of y are divisible by n and all components of $x - y$ are bounded by n. These observations and Nöbeling's theorem imply that B is a basic subgroup of Π in the sense of the following definition.

Definition 1.6 A *basic subgroup* of a torsion-free abelian group G is a pure, free subgroup K of G such that the quotient G/K is divisible.

Notice also that $B \cap D = \Sigma$. Indeed, if all terms of a sequence x are bounded by n and all but finitely many of these terms are divisible by $n + 1$, then all but finitely many of the terms are zero.

Finally, we shall need some concepts and results from the descriptive set theory of complete separable metric spaces. They will be applied to Π and to certain closed subspaces. Good references for most of this information are the books of Kechris [11] and Moschovakis [13].

Recall that a set (in a topological space) is said to be *meager* (or *of first (Baire) category*) if it can be covered by countably many nowhere dense closed sets. (For closed sets, "nowhere dense" simply means having empty interior.) Complements of meager sets are called *comeager*; thus a set is comeager if and only if it includes the intersection of countably many dense open sets. The Baire category theorem says that a complete metric space is not meager (in itself). We shall occasionally say "X is meager in Y" when X is not even a subset of Y; we mean by this that $X \cap Y$ is meager in Y.

Recall also that the *Borel* sets constitute the smallest σ-algebra of sets containing the open sets; i.e., they are obtained from open sets by (transfinitely) iterating the operations of countable union and complement. The *analytic* sets are the continuous images of Borel sets.

A set is said to have the *Baire property* if it differs from some Borel set by a meager set. It is equivalent to require that it differs from some open set by a meager set. Thus, if a set with the Baire property is not meager, then it is comeager in some nonempty open set.

A nonempty subset of a complete metric space is called *perfect* if it is closed and has no isolated points. It is well known that such a set must have cardinality at least $2^{\aleph_0}$.

Theorem 1.7 (Lusin and Sierpiński, [12]) *In a complete, separable, metric space, every analytic set has the Baire property.*

Theorem 1.8 (Mycielski, [14]) *Suppose R is a complete, separable, metric space without isolated points and X is a meager subset of the product space $R \times R$. Then R has a perfect subset S such that no two distinct elements $s \neq t$ of S have $(s,t) \in X$.*

2 Countability

Baer [1] showed that every countable subgroup of Π is free. In this brief section, we record for future reference a consequence of Baer's result. It says that anything countable is irrelevant to questions of freeness inside Π and that such questions can be moved into Π/Σ.

Theorem 2.1 *Let G be a subgroup of Π with $\Sigma \subseteq G$. Then the following are equivalent.*

1. *G is free.*

2. *G is the sum of a free group and a countable group.*

3. *G/Σ is the sum of a free group and a countable group.*

4. *G/Σ is the direct sum of a free group and a countable group.*

Proof (1) implies (4): Given that G is free, fix a basis X for it. If we express all the elements of Σ as $\mathbb{Z}$-linear combinations of elements of X then, as Σ is countable, only a countable subset X_0 of X is involved in these linear combinations. Then G is the direct sum of the subgroup G_0 generated by X_0 and the subgroup G_1 generated by $X - X_0$. Because $\Sigma \subseteq G_0$, we have

$$\frac{G}{\Sigma} = \frac{G_0 \oplus G_1}{\Sigma} = \frac{G_0}{\Sigma} \oplus G_1,$$

where the first summand is countable and the second free.

(4) implies (3) trivially.

(3) implies (2): Assume that $G/\Sigma = F' + C$ with F' free and C countable. Because F' is free, there is a subgroup F of G that maps isomorphically to F' under the projection $p : G \to G/\Sigma$. Then $G = F + p^{-1}(C)$ with F free and $p^{-1}(C)$ countable.

(2) implies (1): Assume that $G = F + C$ with F free and C countable. Fix a basis X for F, and let X_0 be the countable subset of X consisting of all the basis elements involved when the (countably many) elements of $F \cap C$ are expressed as linear combinations of elements of X. Then $F = F_0 \oplus F_1$ where F_0 is generated by X_0 and F_1 by $X - X_0$.

We claim that $F_0 + C$ and F_1 have only the zero element in common. Indeed, if $f_0 + c = f_1$ with $f_i \in F_i$ and $c \in C$ then $c = f_1 - f_0$ is in $F \cap C$ and therefore in F_0 by our choice of X_0. Then, in the equation $f_0 + c = f_1$, the left side is in F_0 and the right side in F_1, so both are 0 because X is independent.

Therefore, $G = F + C = (F_0 + C) + F_1$ is a direct sum, $G = (F_0 + C) \oplus F_1$. Here $F_0 + C$, being countable, is free by Baer's theorem. As F_1 is clearly also free, it follows that G is free. □

Corollary 2.2 *If $F \leq G \leq \Pi$ with F free and G/F countable, then G is free.*

Proof Condition (2) in the theorem is obviously preserved by extensions with countable index. Therefore so is the equivalent condition (1). □

3 Limitations on Pure Free Subgroups

This section is concerned with some limitations on the size of free subgroups and in particular with the non-existence of uncountable, pure, free subgroups in various parts of Π. We begin with a result whose proof is essentially the same as Dugas's and Irwin's proof [9] that all basic subgroups of any single group are isomorphic.

Theorem 3.1 *Let G be an abelian group, p a prime, and κ an infinite cardinal number. If G has a subgroup H such that $|H| \leq \kappa$ and G/H is p-divisible, then all pure, free subgroups of G have cardinality $\leq \kappa$.*

Proof Because G/H is p-divisible, we have $G = H + pG$. For any pure subgroup P of G, we have $pP = P \cap pG$, and therefore

$$\frac{P}{pP} = \frac{P}{P \cap pG} \cong \frac{P + pG}{pG} \subseteq \frac{G}{pG} = \frac{H + pG}{pG} \cong \frac{H}{H \cap pG}.$$

Since the last group here obviously has cardinality at most κ, so does the first. If P is free, then it follows, since κ is infinite, that P also has cardinality at most κ. □

Example 3.2 Fix a prime p and let D_p be the subgroup of Π consisting of the "eventually arbitrarily p-divisible" sequences, i.e., those $x \in \Pi$ such that, for each natural number e, all sufficiently large n have $x(n)$ divisible by p^e. Then D_p/Σ is the p-divisible part of Π/Σ. The theorem just proved implies, since Σ is countable, that D_p has no uncountable, pure, free subgroups. It easily follows that D_p has no basic subgroup.

This example implies, of course, the known result of Yen [10, vol. 2, page 163] that all pure free subgroups of D are countable, since D is a pure subgroup of D_p. It also implies the following constraint on free pure subgroups of Π extending out of D or D_p.

Theorem 3.3 *Π has no pure, free subgroup F satisfying $\Pi = D + F$. The same holds with D_p, for any prime p, in place of D.*

Proof Any counterexample F to the first assertion would also be a counterexample to the second, for all p. So we concentrate on the second assertion. Suppose there were such an F. Then $F \cap D_p$ would be a pure, free subgroup of D_p and therefore countable. As in the proof of Theorem 2.1, we can decompose F as a direct sum $F = F_0 \oplus F_1$ where F_0 is countable and includes $F \cap D_p$. Then

$$\frac{\Pi}{D_p} = \frac{F + D_p}{D_p} \cong \frac{F}{F \cap D_p} = \frac{F_0}{F \cap D_p} \oplus F_1.$$

But Π / D_p admits no non-zero homomorphism to $\mathbb{Z}$ (because by Specker's theorem (1.1) Π/Σ has this property), so its free direct summand F_1 must be 0. But then Π/D_p is isomorphic to a quotient of F_0 and is therefore countable. This is absurd, for Π/D_p includes copies of the p-adic integers. □

Example 3.2 is also covered by the following theorem.

Theorem 3.4 *Let H be a torsion-free group with a subgroup S that is free on $\aleph_0$ generators. Then H has an uncountable, free, pure subgroup if and only if H/S does.*

Proof One direction is easy. For any pure free subgroup F of H/S, its preimage in H is pure and free (being isomorphic to $F \oplus S$) and of rank equal to the rank of F plus $\aleph_0$.

For the other direction, let F be an uncountable, pure, free subgroup of H. Let P be the purification in H of $F + S$.

Lemma 3.5 *F has countable index in P.*

Proof of Lemma For each $s \in S$ and each integer $n \geq 1$, if there is an $f \in F$ such that n divides $s + f$ in H, then let $f_{s,n}$ be one such f. Let C be the set, obviously countable, of all the resulting elements

$$\frac{s + f_{s,n}}{n} \in H.$$

It suffices to show that C and F together generate P.

Any element $p \in P$ has the form $p = (s + f)/n$ for some $s \in S$, $f \in F$, and $n \geq 1$. Since $s + f$ is divisible by n in H, $f_{s,n}$ is defined and $(s + f_{s,n})/n \in C$. Also, since n divides, in H, both $s + f$ and $s + f_{s,n}$, it divides their difference $f - f_{s,n}$, which lies in F. As F is pure in H, there is $g \in F$ with $ng = f - f_{s,n}$. Then

$$p = \frac{s + f}{n} = \frac{s + f_{s,n}}{n} + g \in C + F.$$

□

Let $\bar{P}$ and $\bar{F}$ be the images of P and F in H/S. Since $(H/S)/\bar{P} \cong H/P$ is torsion-free, $\bar{P}$ is pure in H/S. Thus, to complete the proof of the theorem, it suffices to find an uncountable, free, pure subgroup in $\bar{P}$.

Begin by fixing a basis for F and expressing each member of $S \cap F$ in terms of this basis. As S is countable, only countably many elements of the basis occur in these expressions. Let F_0 be the subgroup of F generated by these countably many, and let F_1 be the subgroup of F generated by the rest of the basis. So $F = F_0 \oplus F_1$, F_0 has countable rank, and F_1 has uncountable rank. As the projection from F to $\bar{F}$ has kernel $F \cap S \subseteq F_0$, it maps F_1 isomorphically onto its image $\bar{F}_1$, which is therefore uncountable and free. (We'd be done if we knew that $\bar{F}_1$ is pure in $\bar{P}$, but we don't seem to know that.) As F_1 has countable index in F and therefore in P by the preceding lemma, $\bar{F}_1$ has countable index in $\bar{P}$. Therefore, the following lemma, applied with $\bar{P}$ and $\bar{F}_1$ in the roles of G and K, finishes the proof of the theorem.

Lemma 3.6 *If a group G has an uncountable free subgroup K of countable index, then it also has an uncountable, free, pure subgroup. In fact, G has an uncountable, free direct summand.*

Proof of Lemma Fix a basis X for K, and fix a countable set C of representatives for all the cosets of K in G. Let A be a countable, pure subgroup of G with $C \subseteq A$ and such that, if $a \in K \cap A$, then all the basis elements $x \in X$ involved in the expansion of a (with respect to X) are in A. Notice that the requirements on A amount to its being closed under countably many functions, so a countable such A exists.

Let $X^- = \{x \in X : x \notin A\}$, and let K^- be the (free) subgroup of K generated by X^-. As A is countable and X is not, K^- is uncountable. We complete the proof by showing that $K^- \oplus A = G$.

That K^- and A generate G is clear, because between them they contain all the generators in X of K and all the coset representatives (modulo K) in C. Now suppose $a \in K^- \cap A$. Being in K^-, a is an integral linear combination of certain x's from X^-. But as $a \in K \cap A$, each of these x's is in A, by the assumed closure property of A, and therefore is not in X^-. So there are no such x's, and $a = 0$. Thus, $G = K^- \oplus A$. □

□

Remark 3.7 The preceding theorem remains true if one replaces $\aleph_0$ with an arbitrary infinite cardinal κ and replaces uncountable with cardinality strictly above κ. The proof is essentially unchanged.

The following theorem gives an additional restriction on where free subgroups can be located in Π. In fact, it applies to a somewhat broader class of groups than just the free ones, namely the fully starred ones, defined as follows.

Definition 3.8 A torsion-free abelian group G is *fully starred* if, for every subgroup K of G, the divisible part of G/K has cardinality no larger than that of K.

It is easy to check that free abelian groups are fully starred, so the following result applies to them.

Theorem 3.9 *Let F be a fully starred, pure subgroup of Π of infinite rank. Then the divisible part of Π/F has rank $2^{\aleph_0}$.*

Proof Recall that Π is equipped with the product topology produced from the discrete topology on $\mathbb{Z}$ and subsets of Π are equipped with the subspace topology.

Let $\bar{F}$ be the topological closure of F in Π. By Nunke's theorem (1.4), $\bar{F}$ is a product of some number of copies of $\mathbb{Z}$. This number must be infinite, because F has infinite rank. But it cannot be uncountable, for it is shown in [3] that $\mathbb{Z}^{\aleph_1}$ cannot be embedded in Π. So the number of factors must be exactly $\aleph_0$, which means we have an isomorphism $\varphi : \Pi \to \bar{F}$. By Specker's theorem (1.1), φ is continuous as a function from Π to Π and therefore also as a function from Π to $\bar{F}$. We shall need the following lemma, whose proof we postpone.

Lemma 3.10 *$\varphi^{-1} : \bar{F} \to \Pi$ is continuous. So φ is a homeomorphism.*

Granting the lemma for the moment, we proceed as follows. Since the topology of Π is second-countable, let S be a countable dense subset of F. By closing S under a few operations, we can arrange that it be a pure subgroup of F (while remaining countable). Since F is pure in Π, S is also pure in Π and in $\bar{F}$. It follows that $\varphi^{-1}(S)$ is a countable, pure, dense subgroup of Π. By Chase's theorem, we may replace φ with its composition with a suitable automorphism of Π and so arrange that $\varphi^{-1}(S) = \Sigma$. Thus, $\bar{F}/S \cong \Pi/\Sigma$. In particular, the divisible part Δ of $\bar{F}/S$ has rank $2^{\aleph_0}$.

As F is pure in Π and therefore in $\bar{F}$, F/S is pure in $\bar{F}/S$. So $(F/S) \cap \Delta$ is divisible. But F is fully starred and S is countable, so the divisible part of F/S is countable. This means that only a countable part of Δ is in F/S, and so the image of Δ in

$$\frac{\bar{F}/S}{F/S} \cong \frac{\bar{F}}{F} \subseteq \frac{\Pi}{F}$$

is a divisible group of rank $2^{\aleph_0}$. □

We now return to the lemma whose proof we postponed.

Proof of Lemma 3.10 It suffices to show that, for each $n \in \mathbb{N}$, the n^{th} component of φ^{-1} is a continuous map $(\varphi^{-1})_n : \bar{F} \to \mathbb{Z}$.

Fix $n \in \mathbb{N}$, and partition Π into the sets

$$V_a = \{x \in \Pi : x(n) = a\}$$

for $a \in \mathbb{Z}$. As each V_a is closed (and open) in Π and φ is continuous, $\varphi(V_a)$ is an analytic set in $\bar{F}$ and therefore has the Baire property there by the Lusin-Sierpiński theorem (1.7). The countably many sets $\varphi(V_a)$ cover the complete metric space $\bar{F}$, so by the Baire category theorem they cannot all be meager. Fix some a such that $\varphi(V_a)$ is not meager; since it has the Baire property, it must be comeager in some basic open set in $\bar{F}$. A basic open set is, by definition of the topology, the intersection of $\bar{F}$ with a translate $x + N$ of a basic neighborhood N of 0 in Π. Such an N has the form

$$N = \{0\}^k \times \mathbb{Z}^{\mathbb{N}-k},$$

i.e., the subgroup of Π consisting of all sequences whose first k terms are zero.

It follows that every element $z \in N \cap \bar{F}$ is the difference of two elements of $\varphi(V_a)$. Indeed, for such a z, translation by z is a homeomorphism of $\bar{F}$ onto itself. As $\varphi(V_a)$ is comeager in $(x + N) \cap \bar{F}$, $z + \varphi(V_a)$ is comeager in $z + ((x + N) \cap \bar{F}) = (x + N) \cap \bar{F}$. Two comeager sets in the same $(x + N) \cap \bar{F}$ must intersect, so there is $y \in \varphi(V_a) \cap (z + \varphi(V_a))$. Then $\varphi(V_a)$ contains both y and $y - z$. So z is the difference of two elements of $\varphi(V_a)$, as claimed.

Since $V_a - V_a = V_0$ and since φ is a homomorphism, the claim just established implies that $N \cap \bar{F} \subseteq \varphi(V_0)$. So we have found a neighborhood of 0 in $\bar{F}$, namely $N \cap \bar{F}$, on which $(\varphi^{-1})_n$ is constant with value 0. In particular, $(\varphi^{-1})_n$ is continuous at 0. Since $(\varphi^{-1})_n$ is a homomorphism, it follows by translation that it is continuous everywhere. □

4 Test Classes

In this section, we consider the problem of determining that a subgroup G of Π is free by looking at suitable subgroups of G. Of course, if G is free then so are all its subgroups. We are interested in converse assertions of the form, "If enough subgroups of G are free then so is G itself." The following definition is intended to make this idea precise.

Definition 4.1 Let $\mathcal{T}$ be a family of subgroups of Π. We call $\mathcal{T}$ a *test class* if the following holds for all pure subgroups G of Π: If $G \cap T$ is free for all $T \in \mathcal{T}$ then G is free.

The reason for the terminology is that we think of testing G for freeness by inspecting its intersections with groups from $\mathcal{T}$.

One could introduce a more restrictive notion of test class by removing the assumption that G is pure in Π. The presence of that assumption will, however, make the theory go somewhat more smoothly, and it will make the main result of this section — which says that a certain class is *not* a test class — stronger.

The following theorem gives an example of a test class.

Theorem 4.2 *Let $\mathcal{T}$ consist of the subgroup D of Π and all the pre-reduced parts of Π. Then this $\mathcal{T}$ is a test class.*

Proof Let G be a pure subgroup of Π such that $G \cap D$ and all the groups $G \cap A$ are free, where A ranges over the pre-reduced parts of Π. We must show that G is free.

Without loss of generality, we can assume that $\Sigma \subseteq G$. To see this, let G^+ be the purification in Π of $G + \Sigma$. As in Lemma 3.5, we have that G has countable index in G^+. By Corollary 2.2 it follows that $G^+ \cap D$ and all the intersections $G^+ \cap A$ with pre-reduced parts are free. So, if we knew the desired result for groups that include Σ, then we could apply it to conclude that G^+ is free, and therefore so is its subgroup G.

Thus we assume, from now on, that $\Sigma \subseteq G$. Let $K = G \cap D$. Since K is free and pure in D, it is countable, by Example 3.2. Fix one reduced part A_0/Σ of Π/Σ. So we have $\Pi/\Sigma = (A_0/\Sigma) \oplus (D/\Sigma)$.

Consider the image of G under the projection from Π to Π/Σ. Its intersection with D/Σ is K/Σ, so if we project further, from $\Pi/\Sigma = (A_0/\Sigma) \oplus (D/\Sigma)$ to $(A_0/\Sigma) \oplus (D/K)$, then the image of G in this last group is a subgroup meeting the second summand only at 0. In other words, it is the graph of a homomorphism h from a subgroup of A_0/Σ into D/K.

As D/K is divisible, we can extend h to a homomorphism $\bar{h}$ from all of A_0/Σ into D/K. Furthermore, the exact sequence

$$0 \to \frac{K}{\Sigma} \to \frac{D}{\Sigma} \to \frac{D}{K} \to 0$$

splits, because K/Σ is divisible (because K is pure in D). Therefore, $\bar{h}$ can be lifted to a homomorphism $h' : A_0/\Sigma \to D/\Sigma$. The graph of this homomorphism h' is another complement of D/Σ in Π/Σ, i.e., another reduced part of Π/Σ; call it A_1/Σ.

Since $A_1 \in \mathcal{T}$, its intersection with G is free, by assumption. We shall show that this intersection has countable index in G. By Corollary 2.2, this will suffice to show that G is free, thus completing the proof of the theorem. In fact, we will show that $G \subseteq A_1 + K$. Since $K \subseteq G$, this implies that $G = (A_1 \cap G) + K$, and since K is countable the proof will be complete.

So consider an arbitrary $g \in G$. In view of the decomposition $\Pi/\Sigma = (A_0/\Sigma) \oplus (D/\Sigma)$, we can write g as $a + d$ with $a \in A_0$ and $d \in D$. (The a and d here are not unique; one can add any member of Σ to one and subtract it from the other.) By definition of h, it has $[a]_\Sigma$ in its domain and sends it to $[d]_K$. (We use square brackets to denote cosets with respect to the subscripted group.) Therefore, h' sends $[a]_\Sigma$ to something in D/Σ whose image in D/K is the same $[d]_K$. That is, $h'([a]_\Sigma) = [d+k]_\Sigma$ for some $k \in K$. Thus, $([a]_\Sigma, [d+k]_\Sigma)$ is in the graph A_1/Σ of this homomorphism. That is, $g + k = a + d + k \in A_1$. We have shown that g is the sum of a member of A_1 (namely $g + k$) and a member of K (namely $-k$), so the proof is complete. □

The following definition and theorem provide far simpler examples of test classes, going only slightly beyond the trivial observation that the one-element class $\{\Pi\}$ is a test class. Their real purpose is to point out why the hypothesis of the subsequent theorem needs to exclude certain groups.

Definition 4.3 A subgroup H of Π is *immense* if it includes a subgroup of the form $n \cdot (\{0\}^k \times \mathbb{Z}^{\mathbb{N}-k})$, for some positive integers n and k.

Theorem 4.4 *If H is immense, then the one-element class $\{H\}$ is a test class.*

Proof Let n and k witness the immensity of H, and abbreviate $\{0\}^k \times \mathbb{Z}^{\mathbb{N}-k}$ as Π_k. So $n\Pi_k \subseteq H$. Consider any subgroup G of Π such that $G \cap H$ is free; we must show that G is free.

Notice first that $(nG) \cap \Pi_k \subseteq G \cap (n\Pi_k) \subseteq G \cap H$, so $(nG) \cap \Pi_k$ is free. The quotient $\dfrac{nG}{(nG) \cap \Pi_k}$ is also free, because

$$\frac{nG}{(nG) \cap \Pi_k} \cong \frac{(nG) + \Pi_k}{\Pi_k} \leq \frac{\Pi}{\Pi_k} \cong \mathbb{Z}^k.$$

Therefore, nG is free. But $G \cong nG$ by multiplication by n, so G is free. □

Notice that the preceding proof never used that G is pure. So $\{H\}$ is a test class even in the stronger sense mentioned after Definition 4.1.

Theorem 4.5 *Assume the continuum hypothesis. The class of all analytic, non-immense subgroups of Π is not a test class.*

The conclusion of this theorem says intuitively that, except for the trivial test classes given by Theorem 4.4, any test class must contain some rather complicated groups. As a fairly typical example, notice that the test class given by Theorem 4.2 contains the pre-reduced parts of Π, whose existence is guaranteed only by the axiom of choice. (Without the axiom of choice, divisible subgroups need not be direct summands.)

Proof We must construct a pure, non-free subgroup G of Π such that $G \cap T$ is free for all analytic, non-immense subgroups T of Π.

The number of analytic sets in Π is the cardinality of the continuum, and we have assumed the continuum hypothesis. Therefore we can enumerate all the analytic, non-immense subgroups of Π in a sequence $(T_\alpha : \alpha < \omega_1)$, indexed by ω_1, the first ordinal that has uncountably many predecessors.

We shall obtain the desired G as the union of a strictly increasing ω_1-sequence of countable, pure subgroups G_α of Π, satisfying:

1. $G_0 = \Sigma$.
2. $G_\lambda = \bigcup_{\alpha<\lambda} G_\alpha$ for limit ordinals λ.
3. If $\xi \leq \alpha \leq \beta < \omega_1$ then $T_\xi \cap G_\alpha = T_\xi \cap G_\beta$.

This will ensure that G is an uncountable pure subgroup of Π and that, for each $\xi < \omega_1$, $G \cap T_\xi = G_\xi \cap T_\xi \subseteq G_\xi$. So $G \cap T_\xi$ is countable and therefore free for all ξ. We must, of course, also take steps to ensure that G is not free.

To this end, we shall define (simultaneously with the groups G_α) infinite sets P_α of prime numbers such that:

4. P_0 consists of all the primes.
5. If $\alpha < \beta$ then $P_\beta - P_\alpha$ is finite.
6. If $x \in G_\alpha$ then, for all but finitely many $p \in P_\alpha$, there is $k \in \mathbb{N}$ (depending on x and p) such that p divides $x(n)$ for all $n \geq k$.

We shall say that a set X is *almost included* in another set Y, and we shall write $X \subseteq^* Y$, to mean that $X - Y$ is finite. Thus, we would express condition (5) by saying that the sequence of P_α's is almost decreasing.

Requirement (6) can be rephrased in terms of the image $[x]_\Sigma$ of x in Π/Σ as follows. If $x \in G_\alpha$ then $[x]_\Sigma \in G_\alpha/\Sigma$ is divisible (once) by almost every prime $p \in P_\alpha$. This implies that in G/Σ every element is divisible by infinitely many primes and therefore G/Σ has no non-zero free direct summands. In particular, since G/Σ is uncountable, it is not the direct sum of a countable group and a free one. By Theorem 1, this ensures that G is not free.

We now proceed to the inductive construction of the groups G_α and the sets P_α. Requirements (1) and (4) tell us what to do at stage 0. Requirements (2) and (5) are vacuous at this stage, and requirements (3) and (6) are clearly satisfied.

At a limit stage λ, G_λ is determined by (2). To satisfy (5), we take P_λ to be any infinite set of primes almost included in all the earlier P_α's. Since there are only countably many earlier P_α's, the existence of the desired P_λ follows from a well-known result, which we isolate as a lemma because we shall need it again later.

Lemma 4.6 *Let $\mathcal{F}$ be a countable family of sets, every finitely many of which have an infinite intersection. Then there is an infinite set X almost included in every member of $\mathcal{F}$.*

Proof As $\mathcal{F}$ is countable, enumerate it in an ordinary sequence $(F_n : n \in \mathbb{N})$. The hypothesis about intersections allows us to choose distinct elements $x_0 \in F_0$, $x_1 \in F_0 \cap F_1$, ... , $x_n \in F_0 \cap F_1 \cap \cdots \cap F_n$, ... Then let X be the set of all these x_n's. $\square$

Notice that the lemma applies in particular if $\mathcal{F}$ is given by an almost decreasing sequence of any ordinal length $< \omega_1$.

Having chosen G_λ and P_λ to satisfy requirements (2) and (5), we easily verify that requirement (6) for λ follows from the same requirement for the earlier α's. And the remaining three requirements are vacuous or trivial.

It remains to handle the inductive step from α to $\alpha + 1$. From now on, α is fixed, we assume the inductive construction has been carried out through stage α, satisfying all our requirements, and we wish to carry out stage $\alpha + 1$. Clearly, we need not worry about requirements (1), (2), and (4). To satisfy the others, we must define a countable pure subgroup $G_{\alpha+1}$ of Π with $G_\alpha \subsetneqq G_{\alpha+1}$, and we must define an infinite set $P_{\alpha+1} \subseteq P_\alpha$ (actually, $\subseteq^*$ would suffice, but we'll get $\subseteq$), so that requirements (3) and (6) continue to hold.

To do this, we shall produce an infinite $P_{\alpha+1} \subseteq P_\alpha$ and a certain $z \in \Pi$, and we shall define $G_{\alpha+1}$ to be $\langle G_\alpha \cup \{z\}\rangle_*$, the smallest pure subgroup of Π that includes G_α and contains z. To satisfy all the requirements above, we shall arrange the following.

7. $z \notin G_\alpha$.

8. For each $p \in P_{\alpha+1}$, all sufficiently large n have $z(n)$ divisible by p.

9. For each $\xi \leq \alpha$, $T_\xi \cap \langle G_\alpha \cup \{z\}\rangle_* \subseteq G_\alpha$.

Here requirement (7) ensures that $G_\alpha \subsetneqq G_{\alpha+1}$, and (9) ensures that (3) continues to hold. The purpose of (8) is to ensure that (6) continues to hold. To see that it fulfills this purpose, note first that what (8) asserts of z is also true of each $x \in G_\alpha$, except for finitely many primes p, because (6) is true for α and $P_{\alpha+1} \subseteq P_\alpha$. The same statement therefore holds of all elements of $\langle G_\alpha \cup \{z\}\rangle$. Finally, it is preserved by purification and therefore holds of all $x \in G_{\alpha+1}$. Indeed, for any such x, we have some multiple $qx \in \langle G_\alpha \cup \{z\}\rangle$, and in the passage from qx to x, only finitely many prime divisors can be lost, namely the factors of q.

The construction of $P_{\alpha+1}$ and z as above requires some definitions and lemmas. As in the proof of Theorem 4.4, let us write Π_k for $\{0\}^k \times \mathbb{Z}^{\mathbb{N}-k}$, the subgroup of Π consisting of the sequences whose first k terms are zero.

Lemma 4.7 *If U is a non-meager, analytic subgroup of Π, then there is k such that $\Pi_k \subseteq U$.*

Proof This is established in the proof of [5, Theorem 2.1], so we give only a quick summary here. Being analytic, U has the Baire property. So, being non-meager, it is comeager in some basic open set, i.e., in some coset of Π_k for some k. But then, being a subgroup, it includes Π_k by the same subtraction argument as in the proof of Lemma 3.10 above. □

For any infinite set P of primes, let

$$\begin{aligned}\Pi(P) &= \{x \in \Pi : (\forall p \in P)\,(\forall n \geq p)\, p \text{ divides } x(n)\} \\ &= \prod_{n\in\mathbb{N}} \left(\left(\prod_{p\in P,\ p\leq n} p\right)\mathbb{Z}\right).\end{aligned}$$

Notice that $\Pi(P)$ is isomorphic and homeomorphicto Π via the map $\Pi \to \Pi(P)$ that multiplies the n^{th} component of any sequence by $\prod_{p\in P,\ p\leq n} p$ for each n. Also, if $P \subseteq P'$ then $\Pi(P) \supseteq \Pi(P')$. More generally, we have the following easy lemma concerning almost inclusions between P's.

Lemma 4.8 *If $P \subseteq^* P'$ then there is a positive integer r with $\Pi(P) \supseteq r \cdot \Pi(P')$.*

Proof Let r be the product of the finitely many primes in $P - P'$. Consider an arbitrary element of $r \cdot \Pi(P')$, say rx with $x \in \Pi(P')$, and consider arbitrary $p \in P$ and $n \geq p$. If $p \in P'$ then p divides $x(n)$, while if $p \notin P'$ then p divides r. So in either case p divides $rx(n)$. □

The next lemma is a key step in the construction of $P_{\alpha+1}$. It is where the hypothesis of non-immensity is used.

Lemma 4.9 *If P is an infinite set of primes and T is an analytic, non-immense subgroup of Π, then there is an infinite $Q \subseteq P$ such that, for all positive integers n, $\frac{1}{n}T$ is meager in $\Pi(Q)$.*

Here the notation $\frac{1}{n}T$ means $\{x \in \Pi : nx \in T\}$.

Proof Split P into two infinite subsets, P' and P''. We shall show that one of these can serve as the required Q.

Suppose not. This means that there exist positive integers n' and n'' such that $\dfrac{1}{n'}T$ is non-meager in $\Pi(P')$ and similarly with double-primes. Since T is analytic in Π, $\dfrac{1}{n'}T \cap \Pi(P')$ is analytic in $\Pi(P')$. By Lemma 4.7 applied in $\Pi(P')$, $\dfrac{1}{n'}T \cap \Pi(P')$ includes $\Pi_{k'}(P') = \Pi_{k'} \cap \Pi(P')$ for some k'. Therefore $n'\Pi_{k'}(P') \subseteq T$. Similarly, $n''\Pi_{k''}(P'') \subseteq T$ for some k''.

Then, letting n be a common multiple of n' and n'' and letting k be the larger of k' and k'', we have both $n\Pi_k(P')$ and $n\Pi_k(P'')$ included in T. So $n\left(\Pi_k(P') + \Pi_k(P'')\right) \subseteq T$. But by the Chinese remainder theorem, $\Pi_k(P') + \Pi_k(P'') = \Pi_k$. Therefore, T is immense, contrary to hypothesis. □

Rearrange the countably many analytic, non-immense subgroups T_ξ ($\xi \leq \alpha$) in an ordinary sequence $\tilde{T}_n$, and iteratively apply Lemma 4.9 to get a sequence

$$P_\alpha = Q_0 \supseteq Q_1 \supseteq Q_2 \supseteq \dots .$$

Here Q_{n+1} is the Q obtained by applying Lemma 4.9 with Q_n in the role of P and $\tilde{T}_n$ in the role of T. This ensures that

$$(\forall \xi \leq \alpha)(\exists m)(\forall n > 0)\left(\frac{1}{n}T \text{ is meager in } \Pi(Q_m)\right).$$

By Lemma 4.6, there is an infinite set of primes almost included in all the Q_m's. Let $P_{\alpha+1}$ be such a set. Its crucial property is the following consequence of its construction.

Lemma 4.10 *For all $\xi \leq \alpha$ and all positive integers n, $\dfrac{1}{n}T_\xi$ is meager in $\Pi(P_{\alpha+1})$.*

Proof Suppose ξ and n were a countererxample. Because $\dfrac{1}{n}T_\xi$ is analytic, Lemma 4.7 provides a k such that $\Pi_k(P_{\alpha+1}) \subseteq \dfrac{1}{n}T_\xi$. By our choice of the Q_m's, we can fix m so that $\frac{1}{n'}T_\xi$ is meager in $\Pi(Q_m)$ for all n'. Since $P_{\alpha+1} \subseteq^* Q_m$, Lemma 4.8 gives us a positive integer r such that $r\Pi(Q_m) \subseteq \Pi(P_{\alpha+1})$. Thus,

$$nr\Pi_k(Q_m) \subseteq n\Pi_k(P_{\alpha+1}) \subseteq T_\xi.$$

In other words, $\Pi_k(Q_m) \subseteq \frac{1}{nr}T_\xi$. But this is absurd, because $\frac{1}{nr}T_\xi$ is meager in $\Pi(Q_m)$ while $\Pi_k(Q_m)$ is not. □

Corollary 4.11 $\bigcup_{\xi \le \alpha} \bigcup_{n>0} \frac{1}{n}T_\xi$ *is meager in* $\Pi(P_{\alpha+1})$.

Proof The union of countably many meager sets is meager. □

We now begin to work toward the construction of an element z with properties (7), (8), and (9), which will complete the proof.

Let X be the meager set in the corollary. Then

$$\{(x, y) \in (\Pi(P_{\alpha+1}))^2 : x - y \in X\}$$

is a meager subset of the product space $(\Pi(P_{\alpha+1}))^2$. By Mycielski's theorem (1.8), there is a perfect set $S \subseteq \Pi(P_{\alpha+1})$ such that no two distinct $x, y \in S$ satisfy $x - y \in X$. Fix such a set S and note that, being perfect, it is uncountable.

To finish the proof, we show that some $z \in S$ satisfies all three of (7), (8), and (9). Since $S \subseteq \Pi(P_{\alpha+1})$, (8) is satisfied by every $z \in S$. Since G_α is countable, requirement (7) excludes only countably many members of S. Suppose, toward a contradiction, that (9) excludes all of the remaining, uncountably many members of S.

Then, for each of these uncountably many z's, there would be some $\xi \le \alpha$ and some $t \in T_\xi$ such that $t \notin G_\alpha$ but

$$qt = g + nz$$

for some positive integer q, some $g \in G_\alpha$, and some integer n. Since G_α is pure in Π, $qt \notin G_\alpha$ and so $n \neq 0$.

So we have associated, to each of uncountably many $z \in S$, some $\xi \le \alpha$, some $g \in G_\alpha$, and some $n \in \mathbb{Z} - \{0\}$, such that $g + nz \in T_\xi$. But there are only countably many choices for ξ, g, and n. So there must be two (in fact uncountably many, but we need only two) z's giving the same ξ, g, and n. Thus, we have some $z \neq z'$, both in S, with $g + nz$ and $g + nz'$ both in T_ξ. But then $n \cdot (z - z') \in T_\xi$, so

$$z - z' \in \frac{1}{n}T_\xi \subseteq X,$$

contrary to the choice of S. □

Remark 4.12 The construction in the proof is flexible enough to allow additional requirements to be imposed on G. For example, we could arrange that $G \cap D = \Sigma$, i.e., that G is included in a pre-reduced part of Π. To do this, just choose the enumeration of the non-immense analytic subgroups so that $T_0 = D$. (Note that D is indeed analytic and not immense.) Then requirement (3) in the construction ensures that

$$G \cap D = G \cap T_0 = G_0 \cap T_0 = \Sigma \cap D = \Sigma.$$

Remark 4.13 Theorem 4.5 generalizes the result of Dugas and Irwin [9, Theorem 16] that, assuming the continuum hypothesis, the so-called special summands of Π do not constitute a test class. A special summand is a subgroup M of Π such that both M and Π/M are isomorphic to Π. As a continuous (by Specker's theorem (1.1)) image of Π, such an M is analytic, and the size of Π/M ensures that M isn't immense. So the class of special summands of Π is a subclass of the non-test class of Theorem 4.5.

Remark 4.14 Under suitable set-theoretic hypotheses, Theorem 4.5 can be extended to far broader classes of subgroups than the analytic ones. All we really needed about analytic sets are two facts. First, the inverse image of an analytic set under a continuous function is again analytic. Second, all analytic sets have the Baire property. It is consistent relative to ZFC (by a theorem of Shelah [17]) and it follows from projective determinacy, a commonly assumed hypotheses in descriptive set theory (see [13]), that all projective sets have the Baire property. (Projective sets are the smallest class of sets containing the Borel sets and closed under complements and continuous images.) Thus, in Shelah's model from [17] or in models of projective determinacy, Theorem 4.5 remains true if "analytic" is weakened to "projective."

5 Examples

In this section, we present several examples intended to elucidate the connection (or lack of connection) between freeness and other notions of smallness — like meager or fully starred — for subgroups of Π.

To compare freeness and meagerness, we first observe that D, though not free, is meager. Indeed, even the much larger subgroup

$$\{x \in \Pi : x(n) \text{ is even for all but finitely many } n\} = \Sigma + 2\Pi$$

is meager, for it is

$$\bigcup_{k\in\mathbb{N}}\bigcap_{n\geq k}\{x\in\Pi : x(n) \text{ is even}\},$$

the union of countably many sets

$$\bigcap_{n\geq k}\{x\in\Pi : x(n) \text{ is even}\},$$

each of which is closed and nowhere dense.

For the converse non-implication, i.e., for a non-meager but free subgroup of Π, the most obvious example of a "large" free subgroup, namely B, will not do. It is easily seen to be meager by writing its definition as

$$\bigcup_{k\in\mathbb{N}}\bigcap_{n\in\mathbb{N}}\{x\in\Pi : |x(n)|\leq k\},$$

a countable union of nowhere dense, closed sets. Nevertheless there is a counterexample, as the next theorem shows. We do not know whether the continuum hypothesis can be eliminated from the theorem's second assertion.

Theorem 5.1 *There is a free subgroup of Π that is not meager. If the continuum hypothesis holds, then there is a free, pure subgroup of Π that is not meager.*

Proof We first prove the second assertion, assuming the continuum hypothesis; afterward, we indicate the simpler proof of the first assertion.

Call an element x in a group G *indivisible* if it is not a torsion element and it is not of the form ny for any integer $n>1$ and any $y\in G$. We need three easy lemmas, one about indivisibility in general and two about indivisibility in certain quotients of Π.

Lemma 5.2 *Suppose x is indivisible in a torsion-free group G and $nx=ky$ for some integers $n\neq 0$ and k and some $y\in G$. Then k divides n.*

Proof Cancelling common factors of k and n, we can assume that they are relatively prime. So there are integers a,b with $an+bk=1$. Then

$$x=anx+bkx=aky+bkx=k\cdot(ay+bx).$$

As x is indivisible, $k=\pm 1$, and so k divides n. □

In what follows, we shall again use square brackets and subscripts to denote cosets with respect to the subscripted group.

Lemma 5.3 *The set of $x \in \Pi$ such that $[x]_\Sigma$ is indivisible in Π/Σ is a comeager subset of Π.*

Proof Since Π/Σ is torsion-free, for $[x]_\Sigma$ to be indivisible there means that no prime p divides all but finitely many components of x. So the set in the lemma is

$$\bigcap_{p \text{ prime}} \bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} \{x \in \Pi : p \text{ doesn't divide } x(k)\}.$$

For each fixed p and n, the union over k is obviously open and dense in Π. $\square$

Lemma 5.4 *For any countable subgroup H of Π, the set of $x \in \Pi$ such that $[x]_H$ is indivisible in Π/H is comeager in Π.*

Proof By Chase's theorem (1.3), we may assume without loss of generality that $H \subseteq \Sigma$. (Note that Specker's theorem (1.1) ensures that the automorphism given by Chase's theorem is also a homeomorphism and thus preserves comeagerness.) But if $[x]_\Sigma$ is indivisible in Π/Σ, a quotient of Π/H, then $[x]_H$ is certainly indivisible in Π/H. So the set in the present lemma includes the comeager set from the preceding lemma. $\square$

Consider all the meager F_σ subsets of Π, i.e., the sets expressible as the unions of countably many nowhere dense, closed sets. There are $2^{\aleph_0}$ of them, so, since we are assuming the continuum hypothesis, let them be enumerated in a sequence $(C_\alpha : \alpha < \omega_1)$ of length ω_1. Thus, each α has only countably many predecessors.

By induction on $\alpha < \omega_1$ choose $x_\alpha \in \Pi$ so that

- $[x_\alpha]_{H_\alpha}$ is indivisible in Π/H_α, where H_α is the subgroup of Π generated by $\{x_\beta : \beta < \alpha\}$, and
- $x_\alpha \notin C_\alpha$.

Notice that H_α is countable, so Lemma 5.4 ensures that a comeager set of x's satisfy the first of these two requirements. Since C_α is meager, there are plenty of choices for x_α to satisfy both requirements.

The subgroup generated by all the x_α's cannot be meager, for it is not included in any C_α. To complete the proof of this part of the theorem, it suffices to check that the x_α's are free generators of a pure subgroup of Π. For this it suffices to show, by induction on $\beta < \omega_1$, that $\{x_\alpha : \alpha < \beta\}$ is a system of free generators for H_β and that H_β is pure in Π. The only non-trivial case in the induction is when β is a successor ordinal, say $\gamma + 1$. By induction hypothesis, H_γ is pure and is freely generated by $\{x_\alpha : \alpha < \gamma\}$.

If $\{x_\alpha : \alpha < \beta\}$ failed to freely generate, then there would be a linear relation among its elements, necessarily involving the newly adjoined element x_γ. That would mean that $nx_\gamma \in H_\gamma$ for some non-zero integer n, contrary to the fact that x_γ was chosen so that $[x]_{H_\gamma}$ is indivisible, hence not a torsion element. (As H_γ is pure, "not torsion" here just means non-zero.) So H_β is freely generated by $\{x_\alpha : \alpha < \beta\}$.

To see that it's pure in Π, suppose some element of H_β is divisible in Π by a positive integer k. Say $nx_\gamma + h = ky$ with $h \in H_\gamma$, $n \in \mathbb{Z}$, and $y \in \Pi$. We want to show that $nx_\gamma + h$ is divisible by k in H_β. If $n = 0$, we're done because H_γ is pure. So assume $n \neq 0$. In Π/H_γ, we have $n[x_\gamma]_{H_\gamma} = k[y]_{H_\gamma}$. Since $[x_\gamma]_{H_\gamma}$ is indivisible here, k divides n by Lemma 5.2; say $n = km$. So we have $kmx_\gamma + h = ky$. As H_γ is pure, $h = kh'$ for some $h' \in H_\gamma$. Therefore, $nx_\gamma + h = k \cdot (mx_\gamma + h')$, as desired.

This completes the proof of the second assertion of the theorem. The proof of the first is similar but easier. Since we don't demand purity, we don't need indivisibility. It suffices to have each $[x_\alpha]_{H_\alpha}$ non-torsion in Π/H_α and, of course, $x_\alpha \notin C_\alpha$. As there are $2^{\aleph_0}$ meager F_σ sets, the length of the transfinite induction is (the initial ordinal of cardinality) $2^{\aleph_0}$. Thus, we know at each stage of the induction that H_α and therefore its purification in Π have cardinality strictly smaller than $2^{\aleph_0}$. A set that small cannot be comeager, so we can always find an x_α outside it and outside the meager set C_α. □

Theorem 5.5 *There is a fully starred pure subgroup of* Π *that is not free.*

Proof Choose a prime p and let $\mathcal{P}$ be the group of P-adic integers; the group of ordinary integers $\mathbb{Z}$ is identified canonically with a subgroup of $\mathcal{P}$. We write $\mathbb{Z}_*$ for the purification of $\mathbb{Z}$ in $\mathcal{P}$. This group is isomorphic to the additive group of the ring $\mathbb{Q}_p$ of rational numbers with denominators not divisible by p.

The structure of Π/Σ, as described in the introduction, implies that $\mathcal{P}^{\aleph_0}$ is (isomorphic to) a factor of Π/Σ, and we work for a while within this factor.

Write $\tilde{B}$ for the subgroup of $\mathcal{P}^{\aleph_0}$ consisting of those sequences $x : \mathbb{N} \to \mathcal{P}$ such that

- all components $x(n)$ are in $\mathbb{Z}_*$ and
- only finitely many of the $x(n)$ are distinct.

For any such x, there is a common denominator for all the components $x(n)$, so we can write $x = \dfrac{1}{q}y$, with q an integer not divisible by p and with $y \in$

$\tilde{B} \cap \mathbb{Z}^{\aleph_0} = B$. From this it follows easily that

$$\tilde{B} = \mathbb{Q}_p \otimes_{\mathbb{Z}} B.$$

Since B is a free abelian group by Nöbeling's theorem (1.5), it follows that $\tilde{B}$ is a free $\mathbb{Q}_p$-module.

Notice that $\tilde{B}$ is clearly pure in $\mathcal{P}^{\aleph_0}$ and therefore in Π/Σ. Let H be its pre-image in Π. Thus, H is a pure subgroup of Π. We shall show that it satisfies the requirements of the theorem.

Since $H/\Sigma = \tilde{B}$ is uncountable and divisible by all primes except p, it is certainly not the direct sum of a countable group and a free group. By Theorem 2.1, H is not free.

It remains to prove that H is fully starred. Suppose K is a subgroup of H of infinite cardinality κ; we must show that the divisible part of H/K is no bigger than κ. We may assume, replacing K by $K + \Sigma$ if necessary, that $\Sigma \subseteq K$. Then $\bar{K} = K/\Sigma$ has cardinality at most κ, and $H/K \cong \tilde{B}/\bar{K}$, so it suffices to show that $\tilde{B}/\bar{K}$ has divisible part no larger than κ. But $\tilde{B}$ is a free $\mathbb{Q}_p$-module and, since $\bar{K}$ is small, we can write $\tilde{B}$ as the direct sum of two free $\mathbb{Q}_p$-modules, one of which includes $\bar{K}$ and has cardinality at most κ. (See the proof of Theorem 2.1 for the details of an analogous argument.) Then $\tilde{B}/\bar{K}$ is the direct sum of a subgroup of size at most κ and a free $\mathbb{Q}_p$-module. As the p-divisible part of $\mathbb{Q}_p$ is 0, it follows that the first of these two summands must include the whole divisible part of $\tilde{B}/\bar{K}$. So this divisible part has cardinality at most κ, as required. □

Theorem 5.6 *There is a free, pure subgroup of* Π *that is maximal among subgroups disjoint from* B.

Of course, "disjoint" means that the intersection is $\{0\}$, not $\emptyset$ which is impossible for subgroups.

Proof Theorem 2 of [6] provides a pure, free subgroup F of Π with $\Pi = B + F$. By Zorn's lemma, let H be a subgroup of F that is maximal among all subgroups of F disjoint from B. Then H is free because F is, and H is pure in F by maximality, so H is also pure in Π. We shall show that H is also maximal among subgroups of Π disjoint from B.

As a first step toward this goal, we show that the purification in Π of $B+H$ is all of Π. It suffices to show that this purification includes F, for it obviously includes B and $B + F = \Pi$. So consider an arbitrary element $f \in F$; we must find a non-zero multiple of it in $B + H$. This is trivial if $f \in H$, so assume $f \notin H$. By the maximality of H (among subgroups of F disjoint from B), the group generated by $H \cup \{f\}$ contains a non-zero element of B. Say $b = h+nf$

with $b \in B - \{0\}$, $h \in H$, and $n \in \mathbb{Z}$. As $b \neq 0$ and $B \cap H = \{0\}$, we cannot have $n = 0$. So nf is a non-zero multiple of f and, being equal to $b - h$, it is in $B + H$. This completes the proof that Π is the purification of $B + H$.

Now to complete the proof that H is maximal among subgroups of Π disjoint from B, suppose K were a strictly larger such group, and consider any element $x \in K - H$. By the preceding paragraph, we have $nx = b + h$ for some $n \in \mathbb{Z} - \{0\}$, $b \in B$, and $h \in H$. Then $b = nx - h \in K$, so, since K is disjoint from B, we have $b = 0$. Thus, $nx = h \in H$. As $x \notin H$, this contradicts the fact that H is pure in Π. □

Remark 5.7 The only properties of Π and B needed in the preceding proof are that Π is torsion-free and B is a pure, free subgroup of Π of the same rank as Π itself.

Recall that a basic subgroup of Π is a pure, free subgroup such that the quotient is divisible. Dugas and Irwin proved in [8] that every pure, free subgroup of Π is a subgroup of a basic subgroup of Π. We show next that "is a subgroup of" cannot be improved to "is a direct summand of."

Theorem 5.8 *There is a pure, free subgroup F of Π that is not a direct summand of any basic subgroup of Π.*

Proof Using Nöbeling's theorem (1.5), write B as a direct sum $B = \Sigma \oplus X \oplus Y$, where X and Y are free and the rank of Y is $\aleph_0$. Since the group $\mathbb{Q}_2$ of rational numbers with odd denominators is countable, it is isomorphic to a quotient of Y, say $\mathbb{Q}_2 \cong Y/Y'$. Let $F = \Sigma \oplus X \oplus Y'$. As a subgroup of B, F is free. Since $B/F \cong \mathbb{Q}_2$ is torsion-free, F is pure in B and therefore in Π. It remains to prove that F is not a direct summand of a basic subgroup of Π.

As a first step, we check that F is not itself basic, i.e., that Π/F is not divisible. We have an exact sequence

$$0 \to B/F \to \Pi/F \to \Pi/B \to 0.$$

Since Π/B is torsion-free, B/F is a pure subgroup of Π/F. But $B/F \cong \mathbb{Q}_2$ has elements not divisible by 2, and therefore so does Π/F.

On the other hand, all elements of $B/F \cong \mathbb{Q}_2$ are divisible by all odd primes, and Π/B is divisible, so the same exact sequence shows that Π/F is p-divisible for all odd p. In particular, Π/F has no pure subgroup isomorphic to $\mathbb{Z}$.

This implies that, in Π, F is not a direct summand of any strictly larger, pure, free subgroup. Indeed, if it were such a summand, then the complementary summand would project isomorphically to a non-trivial, pure, free subgroup of Π/F, contrary to the preceding paragraph. □

Finally, for the sake of completeness, we record a self-contained proof for a result announced in [8] as Theorem 18.

Theorem 5.9 (Dugas and Irwin) *There are $2^{2^{\aleph_0}}$ basic subgroups in Π.*

Notice that the cardinal number here is as large as it could possibly be, for it is also the number of subsets of Π.

Proof We first observe that $B + D$ is pure in Π. To see this, suppose $x \in \Pi$ and q is a positive integer with $qx \in B + D$, say $qx = b + d$ with $b \in B$ and $d \in D$. Since D/Σ is divisible, we have $d = qd' + s$ for some $d' \in D$ and $s \in \Sigma \subseteq B$. Then from $qx = b + qd' + s$ we obtain $q \cdot (x - d') = b + s \in B$. Since B is pure, $x - d' \in B$ and so $x \in B + D$. This completes the verification that $B + D$ is pure.

Recall also that $B \cap D = \Sigma$, so B/Σ and D/Σ are disjoint in Π/Σ. We shall work with their internal direct sum $(B + D)/\Sigma = (B/\Sigma) \oplus (D/\Sigma)$.

Consider an arbitrary homomorphism $h : B/\Sigma \to D/\Sigma$, and let

$$G_h \subseteq (B/\Sigma) \oplus (D/\Sigma) = (B + D)/\Sigma \subseteq \Pi/\Sigma$$

be its graph. Let $\bar{G}_h$ be the pre-image of G_h in Π, i.e., $\Sigma \subseteq \bar{G}_h \subseteq \Pi$ and $\bar{G}_h/\Sigma = G_h$. We shall show that $\bar{G}_h$ is basic in Π.

G_h is isomorphic to B/Σ via the first projection from $(B/\Sigma) \oplus (D/\Sigma)$ to B/Σ, and the latter is free, by Nöbeling's theorem (1.5). So G_h is free. Therefore $\bar{G}_h \cong G_h \oplus \Sigma$ is also free.

We show next that the quotient $(B+D)/\bar{G}_h$ is a torsion-free divisible group. In view of the definition of $\bar{G}_h$, we can view this quotient as $((B + D)/\Sigma)/G_h$. Quite generally, when the direct sum of two groups $X \oplus Y$ is divided by the graph of a homomorphism from X to Y, the quotient is isomorphic to Y. In the present situation, this means that

$$\frac{B + D}{\bar{G}_h} \cong \frac{\frac{B}{\Sigma} \oplus \frac{D}{\Sigma}}{G_h} \cong \frac{D}{\Sigma},$$

and this is divisible and torsion-free, as claimed.

$\Pi/(B + D)$ is divisible, being a quotient of the divisible group Π/B. It is also torsion-free, since we showed at the beginning of this proof that $B + D$ is pure in Π.

The preceding two paragraphs show that both ends of the exact sequence

$$0 \to \frac{B + D}{\bar{G}_h} \to \frac{\Pi}{\bar{G}_h} \to \frac{\Pi}{B + D} \to 0$$

are torsion-free divisible groups. Therefore so is the middle term. This shows that $\bar{G}_h$ is pure in Π with divisible quotient. In combination with our earlier verification that $\bar{G}_h$ is free, it completes the verification that $\bar{G}_h$ is a basic subgroup of Π.

Distinct homomorphisms h lead to distinct groups $\bar{G}_h$, since we can recover h from the group: the graph of h is $\bar{G}_h/\Sigma$. Thus, we have as many basic subgroups of Π as there are homomorphisms $h : B/\Sigma \to D/\Sigma$.

Since B/Σ is free of rank $2^{\aleph_0}$, the number of such homomorphisms h is the number of functions from a set of size $2^{\aleph_0}$ (the free generating set of B) into a set of size $2^{\aleph_0}$ (the group D/Σ). By elementary cardinal arithmetic, the number of such functions is $2^{2^{\aleph_0}}$. □

The following corollary, which is also from [8], contrasts with Theorem 2.1 of [4], which says that every two basic subgroups of D are related by an automorphism of D, in fact by an automorphism of Π that maps D onto itself.

Corollary 5.10 (Dugas and Irwin) *Not all basic subgroups of* Π *can be mapped to each other by automorphisms of* Π.

Proof By Specker's theorem (1.1), an automorphism (or just an endomorphism) of Π is completely determined by its restriction to Σ. So Π has only $2^{\aleph_0}$ automorphisms, too few to connect all its $2^{2^{\aleph_0}}$ basic subgroups. □

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