On a problem of H. N. Gupta

ANDREAS BLASS and VICTOR PAMBUCCIAN

Abstract. It is shown that the axiom "For any points x, y, z such that y is between x and z, there is a right triangle having x and z as endpoints of the hypotenuse and y as foot of the altitude to the hypotenuse", when added to 3-dimensional Euclidean geometry over arbitrary ordered fields, is weaker than the axiom "Every line which passes through the interior of a sphere intersects that sphere".

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In [4] H. N. Gupta has provided an elementary axiomatization of finite-dimensional Cartesian spaces coordinatized by arbitrary ordered fields (cf. also [9]). After introducing in [5] the axioms (B) — stating that for any points x, y, z such that y is between x and z, there is a right triangle having x and z as endpoints of the hypotenuse and y as foot of the altitude to the hypotenuse — and (E) — which implies that the coordinate field is Euclidean (i. e. every positive element is a square) — he asked the following question:

Are the axioms (E) and (B) equivalent when added to Euclidean geometry of arbitrary dimension $n \geq 2$ over arbitrary ordered fields? It is easily seen that (E) and (B) are equivalent for $n = 2$. On the other hand, it has been shown by W. Schwabhäuser [8] that the answer to Gupta's question is negative for $n \geq 5$ (with $F = \mathbb{Q}$). We shall fill in the gap between these results by showing in the present note that the answer to Gupta's question is negative for $n \geq 3$.

Axiom (B) holds in the n-dimensional Cartesian space $\mathfrak{C}_n(F)$ over an ordered field F

$(n \geq 2)$ if and only if for all $a_1, \ldots a_n, l \in F$ with $l \geq 0$, the system

$$a_1 x_1 + \ldots + a_n x_n = 0 \text{ and } x_1^2 + \ldots + x_n^2 = l \cdot (a_1^2 + \ldots + a_n^2)$$

has a solution with $x_1, \ldots, x_n$ in F.

It was shown in [8] that (B) holds in $\mathfrak{C}_n(F)$ if and only if every positive element of F is in the range of every diagonalized positive definite quadratic form in $n-1$ variables.

We shall prove that there are ordered fields $(F, \leq)$, which are not Euclidean, that is $F_{\geq 0} \neq F^2$, but for which

$$F_{\geq 0} = aF^2 + bF^2 \text{ for all } a > 0, b > 0 \text{ in } F. \tag{1}$$

This will show that (B) does not imply (E) for $n = 3$ (and hence for all $n \geq 3$).

In what follows we shall point out two different examples of such ordered fields. The first one is nonconstructive, as it depends on the Axiom of Choice; the second one is a primitive-recursive model.

First Example. Let $(F, \leq)$ be a pseudo-real-closed (prc) ordered field (for definitions and axiomatizations see [1], [2] and [7]). In [2, Lemma 2.3] it is shown that any prc ordered field $(F, \leq)$ satisfies (1) with $a = b = 1$. The same proof can be used to show $(F, \leq)$ satisfies (1).

In [2, Corollary 2.7] it is shown that there exists a prc ordered field $(F, \leq)$ in which 2 is not a square; that field is not Euclidean and satisfies (1).

Second Example. Let $(F, \leq)$ be a maximal subfield of the field of real algebraic numbers that does not contain $\sqrt{2}$. F is clearly not Euclidean, but it does satisfy (1).

To see this, let $a > 0$ and $b > 0$ be two elements of F, and let x be any element in $F_{\geq 0}$. If F does not contain $\sqrt{x}$, then, by the maximality of F, $F(\sqrt{x})$ must contain $\sqrt{2}$, i. e. $\sqrt{2} = u + v\sqrt{x}$, for some $u, v \in F$. Squaring, we get $2uv\sqrt{x} = 2 - u^2 - v^2 x$. Since the right hand side is in F and $\sqrt{x}$ is not in F, we must have $uv = 0$. Since $\sqrt{2}$ is not in F, $v \neq 0$; therefore $u = 0$ and $\sqrt{\frac{x}{2}} = \frac{1}{v}$. This shows that

$$\text{for all } x \in F_{\geq 0} \text{ either } \sqrt{x} \in F \text{ or } \sqrt{\frac{x}{2}} \in F. \tag{2}$$

If $\alpha = \sqrt{\frac{x}{a}}$ or $\beta = \sqrt{\frac{x}{b}}$ is in F, then $x = a \cdot \alpha^2 + b \cdot 0 \in aF^2 + bF^2$ or $x = a \cdot 0 + b \cdot \beta^2 \in aF^2 + bF^2$. If $\alpha \notin F$ and $\beta \notin F$, then, by (2), $\frac{\alpha}{\sqrt{2}} \in F$ and $\frac{\beta}{\sqrt{2}} \in F$, hence $x = a \cdot (\frac{\alpha}{\sqrt{2}})^2 + b \cdot (\frac{\beta}{\sqrt{2}})^2 \in aF^2 + bF^2$.

F can be constructed as follows: Let $r_1, r_2, \cdots, r_n, \cdots$ be an enumeration of all the irrational real algebraic numbers. Start with $F_0 = \mathbb{Q}$. At step $i \geq 1$ in the construction, ask if $T_i := F_{i-1}(r_i)$ contains $\sqrt{2}$ or not (algorithms to decide this are given in [3, Section 4.5]; to find minimal polynomials for a given algebraic number one uses [6]). If so, then let $F_i := F_{i-1}$ and proceed to step $i+1$; if not, then let $F_i := T_i$ and proceed to step $i+1$. Let $F := \cup_{i \geq 0} F_i$.

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Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1003

e-mail: ablass@umich.edu

Department of Integrative Studies, Arizona State University West, Phoenix, AZ 85069-7100

e-mail: pamb@math.west.asu.edu