Some Questions Arising from Hindman's Theorem

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Abstract

We present, with some background material, four open questions connected with Neil Hindman's Finite Unions Theorem. Two questions are in set theory and the other two are in computability theory.

1 Introduction

Let me first congratulate Neil Hindman on winning the JAMS International Prize for 2003. I'd also like to thank the JAMS for giving me this opportunity to contribute a brief paper to the celebration of the award to Prof. Hindman.

Although it is customary, in situations like this, to reminisce, I would rather look to the future, by presenting some open problems on which it seems reasonable to hope for progress — perhaps even full solutions — in the near future. The problems that I'll present are closely connected with Prof. Hindman's famous Finite Unions Theorem from [6]:

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Theorem 1 *Suppose the set $\mathbb{F}$ of finite sets of natural numbers is partitioned into finitely many pieces. Then there is an infinite, pairwise unmeshed family $H \subseteq \mathbb{F}$ that is FU-homogeneous in the sense that*

$$FU(H) := \{\bigcup F : \varnothing \neq F \subseteq H \text{ and } F \text{ finite } \}$$

is included in one piece of the given partition.

Here two nonempty finite sets a, b of natural numbers are said to be *unmeshed* if $\max(a) < \min(b)$ or $\max(b) < \min(a)$.

Many authors present Hindman's theorem in the finite sums form, where the set $\mathbb{N}$ of natural numbers is partitioned (instead of $\mathbb{F}$) and the theorem asserts the existence of an infinite FS-homogeneous set, where FS refers to finite sums (instead of the finite unions of FU). The sum and union forms of the theorem are, as was shown in [6], equivalent, and the union form will be slightly more convenient for our present purposes.

2 The Partition Number

It is clear that, for any finitely many partitions of $\mathbb{F}$ into finitely many pieces, there is an infinite, pairwise unmeshed H that is simultaneously FU-homogeneous for all of the given partitions; just apply the finite unions theorem to a common refinement of the given partitions. It is also clear that the analogous statement for countably many partitions is false, as each element of $\mathbb{F}$ could be separated from all the others by one of the partitions. Nevertheless, one can almost handle countably many partitions, in the following sense.

Definition 2 An infinite set $H \subset \mathbb{F}$ is *almost FU-homogeneous* for a partition of $\mathbb{F}$ if there is a finite subset $E \subseteq H$ such that $H - E$ is FU-homogeneous for this partition.

Proposition 3 *Given countably many partitions of $\mathbb{F}$ into finitely many pieces each, there is an infinite, pairwise unmeshed $H \subseteq \mathbb{F}$ that is almost homogeneous for all of the given partitions.*

We omit the easy deduction of this result from Theorem 1; it is included in the argument on page 93 of [2].

Of course, even with FU-homogeneity weakened to almost FU-homogeneity, we cannot hope to obtain an analogous result for $2^{\aleph_0}$ partitions. Indeed, then the given family could consist of all partitions and it is clear that, given any infinite $H \subseteq \mathbb{F}$, one can construct a partition for which it is not almost FU-homogeneous. These observations suggest the following problem.

Question 4 *What is the smallest possible cardinality* $\mathfrak{par}_H$ *for a family of partitions of* $\mathbb{F}$*, into finitely many pieces each, such that no infinite, pairwise unmeshed* $H \subseteq \mathbb{F}$ *is almost FU-homogeneous for all of the given partitions simultaneously?*

The preceding observations show that $\aleph_1 \leq \mathfrak{par}_H \leq 2^{\aleph_0}$, so the continuum hypothesis makes the problem trivial. In the absence of the continuum hypothesis, however, there is a rich theory of cardinal characteristics of the continuum (see, for example, [5], [8], or [3]), many of which are cardinals with definitions qualitatively similar to the $\mathfrak{par}_H$ in the question. It is reasonable to ask for connections between $\mathfrak{par}_H$ and the more familiar characteristics in [3, 5, 8]. The known (to me) connections are only the following.

Proposition 5 $\mathfrak{p} \leq \mathfrak{par}_H \leq \min\{\mathfrak{b}, \mathfrak{s}\}$.

The cardinals $\mathfrak{p}$, $\mathfrak{b}$, and $\mathfrak{s}$ are defined as follows. $\mathfrak{p}$ is the smallest possible cardinality for a family $\mathcal{F}$ of subsets of $\mathbb{N}$ such that each finite subfamily of $\mathcal{F}$ has infinite intersection but there is no infinite set X almost included in every $A \in \mathcal{F}$ (where "almost included" means that $X - A$ is finite). $\mathfrak{b}$ is the smallest possible cardinality for a family $\mathcal{B}$ of functions from $\mathbb{N}$ to $\mathbb{N}$ such that no single function eventually majorizes each of them. $\mathfrak{s}$ is the smallest possible cardinality for a family $\mathcal{S}$ of subsets of $\mathbb{N}$ such that each infinite subset X of $\mathbb{N}$ is split by at least one member S of $\mathcal{S}$, meaning that both $X \cap S$ and $X - S$ are infinite.

Proof That $\mathfrak{par}_H \leq \mathfrak{s}$ is trivial, since there are $\mathfrak{s}$ partitions of any countable set (such as $\mathbb{F}$) into two pieces, such that no infinite set is even almost homogeneous (let alone almost FU-homogeneous) for all of them.

That $\mathfrak{p} \leq \mathfrak{par}_H$ is essentially proved by the Martin's axiom argument in [2, page 93], together with the observation that the forcing to which one applies Martin's axiom is σ-centered, so that Bell's theorem [1] applies.

To prove that $\mathfrak{par}_H \leq \mathfrak{b}$, let $\mathcal{B}$ be an unbounded family of $\mathfrak{b}$ functions $\mathbb{N} \to \mathbb{N}$, and assume, without loss of generality, that each $g \in \mathcal{B}$ is an increasing function. Associate to each g the partition Π_g of $\mathbb{F}$ into two parts, defined as follows. Put $a \in \mathbb{F}$ into the first piece if $g(\min(a)) < \max(a)$ and into the second piece otherwise. Consider, for the time being, one g and an infinite, pairwise unmeshed H that is FU-homogeneous for Π_g. For any $a \in H$, there is $b \in H$ with $g(\min(a)) < \max b$. Thanks to the unmeshed property of H, we have $\min(a \cup b) = \min(a)$ and $\max(a \cup b) = \max(b)$, and therefore $a \cup b$ is in the first piece of our partition. By homogeneity, so is a. Thus, g is majorized by the function

$$f_H : n \mapsto \max(a) \text{ for the first } a \in H \text{ with } \min(a) \geq n.$$

If H is merely almost FU-homogeneous for Π_g, then g is only eventually majorized by f_H. Thus, if there were an H that is almost FU-homogeneous for all $\mathfrak{b}$ of the partitions Π_g for $g \in \mathcal{B}$, then we would have a function f_H eventually majorizing all elements g of $\mathcal{B}$. That contradicts the choice of $\mathcal{B}$. □

For some other partition theorems, one has considerably more precise information about the corresponding cardinals. For example, the analogous cardinal associated to Ramsey's theorem, i.e., the smallest cardinal of a family of partitions of the set of pairs from $\mathbb{N}$ such that no infinite set is almost homogeneous for all these partitions, is known to be $\min\{\mathfrak{b}, \mathfrak{s}\}$. For some other partition theorems, the analogous cardinal is not known exactly, but one has tighter lower bounds than $\mathfrak{p}$. (For a brief discussion of these matters, see [3, Section 3].) In fact, of all the partition theorems for which I have looked into the matter, Hindman's theorem is the one for which the least is known about this cardinal.

3 Complexity of FU-Homogeneous Sets

Question 6 *Suppose we have an effectively computable partition of $\mathbb{F}$; how complicated might an FU-homogeneous set have to be?*

For example, for which Turing degrees $\mathbf{d}$ can one prove that every computable partition of $\mathbb{F}$ into finitely many pieces has an infinite, pairwise unmeshed, FU-homogeneous set of Turing degree $\leq \mathbf{d}$? For which $\mathbf{d}$ can one disprove that assertion?

One can also ask this question with other measures of simplicity in place of $\leq \mathbf{d}$, for example $\ngeq \mathbf{d}$, or levels of some hierarchy.

The following results in this direction were proved nearly twenty years ago [4, Theorems 2.1, 2.2, and 4.9]; as far as I know they have not yet been improved.

Proposition 7 *There is a computable partition of $\mathbb{F}$ into finitely many pieces such that every infinite, pairwise unmeshed, FU-homogeneous set has Turing degree $> \mathbf{0}'$.*

Proposition 8 *Every computable partition of $\mathbb{F}$ into finitely many pieces has an infinite, pairwise unmeshed, FU-homogeneous set of Turing degree $\leq \mathbf{0}^{(\omega+1)}$.*

The gap between the single Turing jump in Proposition 7 and the $\omega+1$-fold Turing jump in Proposition 8 is embarrassingly large. Can it be tightened?

Remark 9 Prof. Hindman's original proof of Theorem 1 is used in an essential way in the proof of Proposition 8. The various simplified proofs of Theorem 1 that were found afterward give far weaker upper bounds for the complexity of the homogeneous set.

4 Weaker Forms of Hindman's Theorem

For any $H \subseteq \mathbb{F}$ and any $k \in \mathbb{N}$, let

$$kU(H) := \{\bigcup F : \varnothing \neq F \subseteq H \text{ and } |F| \leq k\}.$$

Since $kU(H) \subseteq FU(H)$, Hindman's theorem trivially implies the analogous theorem with kU-homogeneity in place of FU-homogeneity. Similarly, we define $\mathfrak{par}_{kH}$ just like $\mathfrak{par}_H$ except that kU-homogeneity replaces FU-homogeneity. Since the property of kU-homogeneity becomes stronger as k increases, we have the chain of inequalities

$$\mathfrak{par}_H \leq \cdots \leq \mathfrak{par}_{kH} \leq \mathfrak{par}_{k-1,H} \leq \cdots \leq \mathfrak{par}_{2H} \leq \mathfrak{par}_{1H} = \mathfrak{s},$$

where the last equality holds by the definitions of $\mathfrak{par}_{1H}$ and $\mathfrak{s}$. Inspection of the proof of $\mathfrak{par}_H \leq \mathfrak{b}$ in Proposition 5 reveals that only 2U-homogeneity was used there. Thus, that theorem remains true with any $\mathfrak{par}_{kH}$ in place of $\mathfrak{par}_H$ as long as $k \geq 2$.

Question 10 *Is it provable in ZFC that $\mathfrak{par}_H = \mathfrak{par}_{2H}$? If not, then which of the inequalities in the chain above can consistently be strict?*

Note that the last inequality in the chain can certainly be strict, since, as noted above, $\mathfrak{par}_{2H} \leq \mathfrak{b}$, whereas it was proved in [7] that $\mathfrak{b}$ can consistently be strictly smaller than $\mathfrak{s}$.

Inspection of the proofs of Theorems 2.1 and 2.2 of [4], which we have combined into Proposition 7, shows that again only 2U-homogeneity was used. So Proposition 7 remains true with 2U in place of FU. Of course the same applies trivially to Proposition 8. In view of this observation and of the available space between the bounds $\mathbf{0}'$ and $\mathbf{0}^{(\omega+1)}$, one might conjecture that, as k increases, kU-homogeneous sets for recursive partitions require greater and greater complexity. Let me formulate a specific question, but with the warning that the proposed bounds are chosen more for notational elegance than for any mathematical reason.

Question 11 *Does every computable partition of $\mathbb{F}$ admit an infinite, pairwise unmeshed, kU-homogeneous set of Turing degree $\mathbf{0}^{(k)}$? And is that degree optimal, at least among degrees of the form $\mathbf{0}^{(n)}$?*

Remark 12 Just as we weakened FU-homogeneity to kU-homogeneity, one can, of course, weaken FS-homogeneity (for partitions of $\mathbb{N}$) to kS-homogeneity. It is, as was shown in [6, Corollary 3.3], not difficult to pass from FS-homogeneous sets to FU-homogeneous sets and vice versa. In particular, the results and the question in Section 3 are unchanged if we work with FS instead of FU. But I do not see how to pass effectively from kS-homogeneous sets to kU homogeneous sets. (The reverse direction is trivial.) So Question 11 may turn into a different question if we replace kU with kS; it may be possible to get kS-homogeneous sets of lower degrees than kU-homogeneous sets.

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