MAD FAMILIES AND THEIR NEIGHBORS

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Abstract. We study several sorts of maximal almost disjoint families, both on a countable set and on uncountable, regular cardinals. We relate the associated cardinal invariants with bounding and dominating numbers and also with the uniformity of the meager ideal and some of its generalizations.

1. Who Are These Families? A Background Check

Almost disjoint (ad) families have been a topic of interest in set theory since its early days [20, 23]. Once the forcing technique became available, a great deal was learned about their properties, in particular the properties of maximal almost disjoint (mad) families of sets of natural numbers; see for example [14, 16]. A recent major breakthrough is Shelah's proof [19] (see also [5]) that the minimum size of a mad family may be larger than the dominating number.

In this paper, we investigate some properties of mad families, not only of sets but also of functions and of permutations. Although part of our work (Section 5) is concerned with mad families on the set ω of natural numbers, most of what we do is in the context of arbitrary regular cardinals. Indeed, some of our results extend to the uncountable case results already known for ω, while others exhibit differences in the uncountable case from the known facts for ω.

We begin by defining the notation we shall use and recalling some information about mad families and certain other families of sets and functions (the neighbors mentioned in the title). We use standard set-theoretic notation and terminology as in [13, 14].

Convention 1.1. Throughout the paper, κ is an infinite, regular cardinal.

Definition 1.2. Two sets are *κ-almost disjoint* (*κ-ad*) if their intersection has cardinality strictly smaller than κ. A family of sets is κ-ad if every two distinct members of it are κ-ad.

Definition 1.3. A *maximal almost disjoint (mad) family of subsets of* κ is a κ-ad family of unbounded subsets of κ that has size at least κ and that is not properly included in another such family. $\mathfrak{a}(\kappa)$ is the smallest cardinality of any such family.

Observe that the maximality clause in the definition means that every unbounded subset of κ intersects some member of the mad family in an unbounded set.

The requirement that a mad family have size at least κ is needed to exclude trivial examples, like a partition of κ into fewer than κ unbounded pieces. It implies that

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a mad family has size strictly larger than κ. Indeed, given any ad family of κ unbounded subsets of κ, we obtain another unbounded set, almost disjoint from all those in the given family, by an easy diagonalization argument. Thus, $\mathfrak{a}(\kappa) \geq \kappa^+$.

In the definitions that follow, functions are to be identified with their graphs, i.e., with sets of ordered pairs. Thus, two functions f and g are κ-ad if there are fewer than κ values of $\alpha \in \mathrm{Dom}(f) \cap \mathrm{Dom}(g)$ such that $f(\alpha) = g(\alpha)$. When the functions have domain κ, almost disjointness means that they are eventually different, i.e., $f(\alpha) \neq g(\alpha)$ for all sufficiently large $\alpha < \kappa$.

Definition 1.4. A *maximal almost disjoint (mad) family of functions* on κ is a κ-ad family of functions $\kappa \to \kappa$ that is not properly included in another such family. $\mathfrak{a}_e(\kappa)$ is the smallest cardinality of any such family.

Definition 1.5. A *maximal almost disjoint (mad) family of permutations* of κ is a κ-ad family of bijections $\kappa \to \kappa$ that is not properly included in another such family. $\mathfrak{a}_p(\kappa)$ is the smallest cardinality of any such family.

Definition 1.6. A *maximal almost disjoint (mad) group of permutations* of κ is a κ-ad subgroup of the group $\mathrm{Sym}(\kappa)$ of bijections $\kappa \to \kappa$ that is not properly included in another such subgroup. $\mathfrak{a}_g(\kappa)$ is the smallest cardinality of any such family.

In connection with the last of these definitions, observe that two permutations f and g are κ-ad if and only if $f^{-1}g$ leaves fewer than κ points fixed. Thus, a subgroup of $\mathrm{Sym}(\kappa)$ is κ-ad if and only if the only element in the subgroup that fixes κ points is the identity. In the case $\kappa = \omega$, such subgroups are called *cofinitary*; see [9] for a readable presentation of much of their theory.

Easy diagonalization arguments show that $\mathfrak{a}_e(\kappa), \mathfrak{a}_p(\kappa) \geq \kappa^+$. It is also true that $\mathfrak{a}_g(\kappa) \geq \kappa^+$, but the proof requires more work. The case of $\kappa = \omega$ is due independently to Adeleke [1] and Truss [24]. The general case seems to be well known, but we do not have an explicit reference for it. It is an immediate consequence of our Theorem 3.2.

Convention 1.7. For all the cardinals $\mathfrak{a}(\kappa), \mathfrak{a}_e(\kappa), \mathfrak{a}_p(\kappa), \mathfrak{a}_g(\kappa)$ defined above as well as for other cardinals to be defined below, when κ is omitted from the notation, it is understood to be ω.

Among these cardinals for $\kappa = \omega$, $\mathfrak{a}$ has been quite throroughly studied; see for example [3, 4, 5, 6, 7, 10, 17, 19]. Eventually different functions were investigated in [10] and [15]. The cardinal $\mathfrak{a}_e$ was defined in [25] based on a suggestion from B. Veličković. The definition of $\mathfrak{a}_p$ was contributed by S. Thomas (see [25]). Finally, the introduction of $\mathfrak{a}_g$ was based on work of Adeleke [1], Truss [24], and P. Neumann (see [9, 26]), and on the third author's answer to a question of Neumann and Cameron [26, 27]. The relationship between (not necessarily maximal) almost disjoint families of sets and almost disjoint permutation groups was also studied by Galvin [11].

Notice that, although all permutations are functions and all functions are sets, one cannot trivially infer inequalities between the associated cardinals $\mathfrak{a}_p(\kappa)$, $\mathfrak{a}_e(\kappa)$, and $\mathfrak{a}(\kappa)$. For example, the requirements for a mad family of functions are in one respect more restrictive than for a mad family of sets (because the members of the former are required to be functions) and in another respect less restrictive (because maximality is required only among families of functions). Similar comments apply to other pairs of our cardinals. Indeed, the following problems are still open.

Question 1.8. Is it provable that $\mathfrak{a} \leq \mathfrak{a}_e, \mathfrak{a}_p,$ or $\mathfrak{a}_g$?

Question 1.9. Are any strict inequalities among $\mathfrak{a}_e$, $\mathfrak{a}_p$, and $\mathfrak{a}_g$ consistent with ZFC?

Both questions are also open for the cardinals associated to an uncountable κ instead of ω.

It is known that $\mathfrak{a}$ can consistently be strictly smaller than all of $\mathfrak{a}_e$, $\mathfrak{a}_p$, and $\mathfrak{a}_g$. The case of $\mathfrak{a}_e$ is easy (see below); see [8] for the rest.

Now we introduce some neighbors of these mad families.

Definition 1.10. Let $f, g : \kappa \to \kappa$. By $f <^* g$, we mean that there is some $\alpha < \kappa$ such that $f(\beta) < g(\beta)$ for all β in the range $\alpha < \beta < \kappa$. Equivalently (since κ is regular), $|\{\beta < \kappa : g(\beta) \leq f(\beta)\}| < \kappa$.

A family $\mathcal{B}$ of functions $\kappa \to \kappa$ is *unbounded* if for every $g : \kappa \to \kappa$ there is $f \in \mathcal{B}$ with $f \not<^* g$. The cardinal $\mathfrak{b}(\kappa)$ is defined as the smallest cardinality of an unbounded family.

A family $\mathcal{D}$ of functions $\kappa \to \kappa$ is *dominating* if for every $g : \kappa \to \kappa$ there is $f \in \mathcal{D}$ with $g <^* f$. The cardinal $\mathfrak{d}(\kappa)$ is defined as the smallest cardinality of a dominating family.

In accordance with Convention 1.7, $\mathfrak{b}$ and $\mathfrak{d}$ have their customary meanings, as in [4, 10].

It is not difficult to show that $\mathfrak{b} \leq \mathfrak{a}$; see for example [4] or [10]. The consistency of $\mathfrak{b} < \mathfrak{a}$ was first proved by Shelah in [17]. The consistency of $\mathfrak{a} = \mathfrak{a}_e = \mathfrak{a}_p = \mathfrak{a}_g < \mathfrak{d}$ can be proved by adding Cohen reals to a ground model that satisfies ZFC+GCH. See [14, Theorem VIII.2.3] for the proof that $\mathfrak{a}$ remains small in this model; the proofs for the other almost-disjointness numbers are similar. The consistency of $\mathfrak{d} = \omega_1 < \mathfrak{a}_e = \mathfrak{a}_p = \mathfrak{a}_g$ can be proved as a corollary of the main result in [8]. Shelah's recent proof in [19] (see also [5, 7]) of the consistency of $\mathfrak{d} < \mathfrak{a}$ not only solves a long-standing open problem but introduces important new methods, particularly iteration of forcing along a template. Nevertheless, the question whether $\mathfrak{d} = \omega_1 < \mathfrak{a}$ is consistent, asked by Roitman in the 1970's, remains open.

To complete our census of the neighborhood, we need one more cardinal characteristic, but it is one which arises from several kinds of families. The cardinal in question is the uniformity of Baire category, most often written $\mathfrak{non}(\mathcal{M})$. It has several definitions, equivalent for $\kappa = \omega$ but not for general κ. We shall give here its standard definition only for $\kappa = \omega$, postponing a discussion of generalizations to uncountable κ until Section 4.

Definition 1.11. $\mathfrak{non}(\mathcal{M})$ is the smallest cardinality of any non-meager subset of ${}^{\omega}\omega$.

Here the space ${}^{\omega}\omega$ of functions $\omega \to \omega$ is equipped with the product topology arising from the discrete topology on ω. A set is *meager* (or of first Baire category) if it can be covered by countably many closed sets whose interiors are empty. It is well-known that the same cardinal $\mathfrak{non}(\mathcal{M})$ would arise if we replaced ${}^{\omega}\omega$ by ${}^{\omega}2$ or $\mathbb{R}$ or indeed any uncountable, complete, separable, metric space.

Bartoszyński proved in [2] (see also [3, 4, 12]) that $\mathfrak{non}(\mathcal{M})$ equals the cardinal in the following definition when $\kappa = \omega$; we therefore name this cardinal $\mathfrak{nm}$ to suggest the similarity.

Definition 1.12. A family $\mathcal{F}$ of functions $\kappa \to \kappa$ is *cofinally matching* if for each $g : \kappa \to \kappa$ there is some $f \in \mathcal{F}$ such that $|\{\beta < \kappa : f(\beta) = g(\beta)\}| = \kappa$, i.e., f and g are not κ-ad. $\mathfrak{nm}(\kappa)$ is the smallest cardinality of any cofinally matching family.

It is clear that a mad family of functions is cofinally matching; indeed, mad is exactly the conjunction of κ-ad and cofinally matching. It is also clear that a cofinally matching family is unbounded. So we immediately have $\mathfrak{b}(\kappa) \leq \mathfrak{nm}(\kappa) \leq \mathfrak{a}_e(\kappa)$.

The second author [12] proved several interesting theorems about these and related families of functions. In [8], it is proved that $\mathfrak{non}(\mathcal{M}) \leq \mathfrak{a}_e, \mathfrak{a}_p, \mathfrak{a}_g$. As a corollary, one has the consistency of $\mathfrak{d} = \mathfrak{b} = \mathfrak{a} < \mathfrak{non}(\mathcal{M}) = \mathfrak{a}_e = \mathfrak{a}_p = \mathfrak{a}_g$, since the model obtained by adding many random reals to a model of ZFC+GCH has $\mathfrak{d} = \mathfrak{b} = \mathfrak{a} = \omega_1$ and $\mathfrak{non}(\mathcal{M}) = 2^\omega$ (see [4]). Brendle showed in [5] that $\mathfrak{a} < \mathfrak{non}(\mathcal{M}) < \mathfrak{a}_e$ is consistent.

In Section 5, we shall establish the consistency of $\mathfrak{non}(\mathcal{M}) < \mathfrak{a} = \mathfrak{a}_e = \mathfrak{a}_p = \mathfrak{a}_g$. This answers a question from [8].

The following problem, which T. Jech and B. Veličković asked the third author, remains open.

Question 1.13. Let κ be an uncountable, regular cardinal. Is it consistent that $\mathfrak{b}(\kappa) < \mathfrak{a}(\kappa)$? Is it consistent that $\mathfrak{a}(\kappa) < \mathfrak{a}_e(\kappa), \mathfrak{a}_p(\kappa), \mathfrak{a}_g(\kappa)$?

By a result in [12], the following statement is not provable in ZFC: There is a κ^+-cc poset which does not collapse κ and which forces $\mathfrak{a}(\kappa) = \kappa^+ < \mathfrak{a}_e(\kappa) = \kappa^{++} = 2^\kappa$. We suspect that the same holds with $\mathfrak{a}_p$ and $\mathfrak{a}_g$ in place of $\mathfrak{a}_e$.

2. MAD Families as Small as a Neighbor

Although it remains an open question whether $\mathfrak{d} = \omega_1 < \mathfrak{a}$ is consistent, we obtain in this section a negative answer to the analogous question for $\mathfrak{d}(\kappa)$ and $\mathfrak{a}(\kappa)$ when κ is uncountable (and, as always, regular). We also obtain a similar result for $\mathfrak{a}_e(\kappa)$ when κ is a successor cardinal.

Theorem 2.1. *If κ is an uncountable, regular cardinal and $\mathfrak{d}(\kappa) = \kappa^+$ then $\mathfrak{a}(\kappa) = \kappa^+$.*

Proof. By assumption, there is a dominating family $\{f_i : i < \kappa^+\}$ in ${}^\kappa\kappa$. Replacing each f_i by a possibly larger function, we can assume that each f_i is a strictly increasing function $\kappa \to \kappa$. Furthermore, since an easy diagonalization argument allows us to majorize, in the sense of $<^*$, any κ functions $\kappa \to \kappa$ by a single such function (i.e., since $\mathfrak{b}(\kappa) \geq \kappa^+$), we can assume that $f_i <^* f_j$ whenever $i < j < \kappa^+$.

We intend to build a mad family $\{S_i : i < \kappa^+\}$ of subsets of κ by induction on i. Quite generally, whenever we have a family of subsets A_i of κ, indexed by some set I, the task of producing another set B, κ-ad from all the A_i, can be viewed as follows. For each $i \in I$, the set $A_i \cap B$ must be bounded by some element of κ, which we call $h(i)$. By first choosing a function to serve as h, we obtain a certain B, the largest B for which the chosen h works (i.e., $h(i)$ bounds $A_i \cap B$ for all i), namely

$$B = \{\gamma < \kappa : (\forall i \in I) \text{ if } h(i) < \gamma \text{ then } \gamma \notin A_i\}.$$

This definition of B simply excludes from membership in B any γ that would violate the bounds given by h. Of course, if h is too small, then B might be bounded and so of no use for our purposes. Our objective is to inductively define the sets S_i, using at each step the family of previously defined S_j's and an h large enough to ensure that we get unbounded sets B, at least at enough of the steps to produce a mad family at the end of the construction. The dominating family of functions f_i will play an essential role in getting the required h's.

Two additional ingredients are needed in the production of the h's for the construction outlined above. One is a matter of bookkeeping. At stage i of our construction, the previous S_j's, from which the new set S_i should be κ-ad, are indexed not by κ (the domain of f_i) but by i. So we re-index them by κ. That is, we fix, for each i in the interval $[\kappa, \kappa^+)$, a bijection $G_i : \kappa \to i$. Thus, we can view the family of previous S_j's ($j < i$) as $\{S_{G_i(\alpha)} : \alpha < \kappa\}$, indexed by κ.

The other ingredient is more complicated; its purpose is to circumvent difficulties arising from the arbitrariness of the choice of the G_i's. It is shown in [18, Chapter III] that one can assign, to each ordinal $\alpha \in [\kappa, \kappa^+)$ of cofinality ω, a cofinal subset $C_\alpha \subseteq \alpha$ of order type ω, in such a way that, for every closed unbounded (club) subset C of κ^+, there is an $\alpha \in [\kappa, \kappa^+)$ of cofinality ω with $C_\alpha \subseteq C$. Such an assignment is known as a club-guessing sequence. Fix such a sequence, and define, for each $i \in [\kappa, \kappa^+)$ of cofinality ω,

$$D_i = \{\alpha < \kappa : (\forall j \in C_i)\, \{G_i(\xi) : \xi < \alpha\} \cap j = \{G_j(\xi) : \xi < \alpha\}\}.$$

Because both G_j and the restriction of G_i to $G_i^{-1}[j]$ enumerate the same set j, it is easy to see (and well known) that each D_i is a club subset of κ. The point of introducing the D_i is that they give us certain values of α for which the enumeration G_i agrees, in a certain sense, with the earlier enumerations G_j arising from elements j of C_i. It turns out that this much agreement, even with just those countably many j's, will provide the required control over the construction of the S_i's, to which we now turn.

It will be convenient to use the notation $\operatorname{next}(X, \alpha)$ for the first element of X that is greater than α (provided it exists).

To begin the construction, let the S_i for $i < \kappa$ be any κ-ad unbounded subsets of κ; for example, they could be a partition of κ into κ unbounded pieces. For $i \geq \kappa$, we use only (some) i's of cofinality ω to index our sets S_i; formally, we set $S_i = \varnothing$ for $\operatorname{cf}(i) \neq \omega$, but these empty S_i's will be omitted when we form our mad family. Finally, when $i \in [\kappa, \kappa^+)$ and $\operatorname{cf}(i) = \omega$, we define S_i by applying the general process outlined above. The role of the A_i's is played by the previous S_j's, reindexed via G_i, and the role of h is played by the function

$$\alpha \mapsto f_i(\operatorname{next}(D_i, \alpha)).$$

Thus, S_i consists of those $\gamma < \kappa$ such that, whenever $j < i$ and $f_i(\operatorname{next}(D_i, G_i^{-1}(j))) < \gamma$, then $\gamma \notin S_j$. If the S_i so obtained has cardinality smaller than κ, however, we replace it by $\varnothing$ and omit it from our final mad family. (At this stage of the proof, we don't know that we will ever get an unbounded S_i for $i > \kappa$, but this will, of course, follow when we show that we get a mad family.)

It is clear from the construction that the family of sets S_i is κ-ad. To show that it is mad, suppose, toward a contradiction, that X is an unbounded subset of κ and is κ-ad from every S_i. We define a continuous, increasing sequence of ordinals $\alpha_i \in [\kappa, \kappa^+)$, for $i < \kappa^+$ by the following induction. Start with $\alpha_0 = \kappa$, and at limit ordinals i make the sequence continuous, i.e., $\alpha_i = \bigcup_{j<i} \alpha_j$. For a successor ordinal $i = j + 1$, use the fact that X is κ-almost disjoint from the earlier S_k's to obtain a function h_i, as described at the beginning of this proof, bounding the intersections

of X with the S_k's. More precisely, taking the reindexing by G_{α_j} into account, we define the function $h_i : \kappa \to \kappa$ by

$$h_i(\beta) = \bigcup (X \cap S_{G_{\alpha_j}(\beta)}).$$

Then we define α_i to be any ordinal such that $\alpha_j < \alpha_i < \kappa^+$ and $\mathrm{cf}(\alpha) = \omega$ and $h_i <^* f_{\alpha_i}$.

Since $C = \{\alpha_i : i < \kappa^+\}$ is a club, there is some $\alpha \in (\kappa, \kappa^+)$ such that $\mathrm{cf}(\alpha) = \omega$ and $C_\alpha \subseteq C$. Fix such an α.

We shall shrink X slightly, removing fewer than κ elements. Since $|X| = \kappa$, some elements will remain, and by considering such an element we shall arrive at a contradiction.

The first step in shrinking X is to expel from it all elements of $X \cap S_\alpha$. There are fewer than κ of these elements because X is, by assumption, κ-ad from all the S_i's.

For each i such that $\alpha_i \in C_\alpha$, we have $h_{i+1} <^* f_{\alpha_{i+1}} <^* f_\alpha$, so we can fix an ordinal $\delta_i < \kappa$ beyond which f_α is larger than h_{i+1}. Further, since C_α is countable, we have only countably many of these δ_i's, so their supremum δ is still $< \kappa$. Thus, we have an ordinal $\delta < \kappa$ such that, for all $\eta \geq \delta$ and all i such that $\alpha_i \in C_\alpha$, $h_{i+1}(\eta) < f_\alpha(\eta)$.

Having fixed such a δ, we now expel from X some more elements. For each $\beta \in C_\alpha$, expel those elements of X that also belong to some S_j where $j < \beta$ and $G_\beta^{-1}(j) < \delta$. Notice that

- for any single j, only $< \kappa$ elements are expelled, because X is κ-ad from S_j,
- for any single β, there are only $< \kappa$ values of j, because $\delta < \kappa$, and
- there are only countably many values for β, because C_α is countable.

Thus, the number of newly expelled elements of X is $< \kappa$. So some elements of X remain unexpelled. Let γ be such an element.

Since γ was not expelled, it is not in S_α. So for some $j < \alpha$, we have both $\gamma \in S_j$ and $f_\alpha(\mathrm{next}(D_\alpha, G_\alpha^{-1}(j)) < \gamma$.

Since C_α is cofinal in α and included in C, fix an i such that $j < \alpha_i \in C_\alpha$. Since $j < \alpha_i$ and $\gamma \in S_j$, the fact that γ was not expelled means that $G_{\alpha_i}^{-1}(j) \geq \delta$. By our choice of δ, it follows that

$$h_{i+1}(G_{\alpha_i}^{-1}(j)) < f_\alpha(G_{\alpha_i}^{-1}(j)).$$

Furthermore, by definition of h_{i+1}, the left side of this inequality is greater than or equal to all elements of $X \cap S_j$, in particular γ. So we have

$$\gamma < f_\alpha(G_{\alpha_i}^{-1}(j)).$$

On the other hand, we saw above that, because γ is assumed not to be in S_α,

$$f_\alpha(\mathrm{next}(D_\alpha, G_\alpha^{-1}(j))) < \gamma.$$

(Note that the subscripts of G in the last two inequalities are different; this is why we need the D's.) Comparing the last two inequalities and remembering that f_α is an increasing function, we conclude that

$$\mathrm{next}(D_\alpha, G_\alpha^{-1}(j)) < G_{\alpha_i}^{-1}(j).$$

So the left side of this inequality is an element β of D_α such that j is G_α of an ordinal $G_\alpha^{-1}(j)$ smaller than β but j is also G_{α_i} of an ordinal $G_{\alpha_i}^{-1}(j)$ larger than β. Since $\alpha_i \in C_\alpha$, this contradicts the definition of D_α, and so the proof is complete. □

Theorem 2.2. *If $\kappa = \mu^+$ and $\mathfrak{d}(\kappa) = \kappa^+$ then $\mathfrak{a}_e(\kappa) = \kappa^+$.*

Proof. Since $\mathfrak{a}_e(\kappa) \geq \kappa^+$ in any case, we need only produce a mad family of κ^+ functions $\kappa \to \kappa$.

By hypothesis, we have a dominating family of κ^+ functions $g_i : \kappa \to \kappa$. Choose, for each limit ordinal $i < \kappa^+$, an increasing cofinal sequence $(i_\gamma)_{\gamma<\mathrm{cf}(i)}$ in i. Using these sequences and the g_i's, inductively define functions $f_i : \kappa \to \kappa$ with the following properties.

(1) Each f_i is weakly increasing.
(2) If $i < j < \kappa^+$ then $f_i <^* f_j$.
(3) $f_0(\alpha) = 0$ for all α.
(4) If i is a limit of cofinality $< \kappa$ then for all α

$$f_i(\alpha) = \sup\{f_{i_\gamma}(\alpha) : \gamma < \mathrm{cf}(i)\}.$$

(5) If i is a limit of cofinality κ then for all α

$$f_i(\alpha) = \sup\{f_{i_\gamma}(\alpha) : \gamma < \alpha\}.$$

(6) For all i and α,

$$f_{i+1}(\alpha) \geq \max\{g_i(\alpha), f_i(\alpha) + \mu\}.$$

We record two consequences of this construction. First, by clause (vi), since the g_i's form a dominating family, so do the f_i's.

Second, the two clauses for limit i's ensure the following pressing down property.

Lemma 2.3. *Suppose $h : \kappa \to \kappa$ and i is a limit ordinal such that $\{\beta < \kappa : h(\beta) < f_i(\beta)\}$ is stationary. Then there is $j < i$ such that $\{\beta < \kappa : h(\beta) < f_j(\beta)\}$ is stationary.*

Proof. Write X for the stationary set $\{\beta < \kappa : h(\beta) < f_i(\beta)\}$.

Suppose first that $\mathrm{cf}(i) < \kappa$. For each $\beta \in X$, we have, by clause (iv) in the construction of f_i, some $j < i$, namely an appropriate i_γ, such that $h(\beta) \leq f_j(\beta)$. Since there are only $\mathrm{cf}(i) < \kappa$ possible values for γ and thus for j, one of them must occur for a stationary set of β's, and so the proof is complete in this case.

There remains the case that $\mathrm{cf}(i) = \kappa$. As before, we get for each $\beta \in X$ some j such that $h(\beta) < f_j(\beta)$, where $j = i_\gamma$ for some $\gamma < \beta$. Thus, the function that maps each $\beta \in X$ to some suitable γ is regressive. By Fodor's theorem, it is constant on a stationary set of β's, and so the proof is complete. □

For each $i < \kappa^+$, $\gamma < \kappa$, and $\alpha < \mu$, define $F_i^\alpha(\gamma)$ so that, for each fixed i and γ, as α varies over the elements of μ, $F_i^\alpha(\gamma)$ enumerates the interval $[f_i(\gamma), f_{i+1}(\gamma))$ in a one-to-one manner. Thanks to clause (vi) of the definition of the f_i's (and the fact that they all map into $\kappa = \mu^+$), the interval $[f_i(\gamma), f_{i+1}(\gamma))$ has cardinality μ and so the required enumerations exist.

We shall show that $\{F_i^\alpha : i < \kappa^+, \alpha < \mu\}$, whose cardinality is κ^+, is a mad family of functions.

For $(i, \alpha) \neq (j, \beta)$, the functions F_i^α and F_j^β are eventually different. Indeed, if $i = j$ and (therefore) $\alpha \neq \beta$, then these functions disagree at all γ because our enumerations of the intervals $[f_i(\gamma), f_{i+1}(\gamma))$ were one-to-one. If, on the other hand, $i \neq j$, then clause (ii) in the construction of the f_i's ensures that the intervals $[f_i(\gamma), f_{i+1}(\gamma))$ and $[f_j(\gamma), f_{j+1}(\gamma))$ are disjoint once γ is sufficiently large.

To complete the proof, we consider an arbitrary function $h : \kappa \to \kappa$ and we find some F_i^α that agrees with h on an unbounded subset of κ. We observed above that the family of f_i's is dominating, so in particular, there is some ordinal $j < \kappa^+$ such that $\{\beta < \kappa : h(\beta) < f_j(\beta)\}$ is stationary. Take the smallest such ordinal j. By the lemma, it cannot be a limit, and by clause (iii) of the construction of the f_i's it cannot be 0. So $j = i+1$ for some i. By the minimality of j, there is an unbounded (in fact stationary) set X of ordinals $\beta < \kappa$ such that $f_i(\beta) \leq h(\beta) < f_{i+1}(\beta)$. According to the definition of the F's, there is, for each $\beta \in X$, some $\alpha < \mu$ such that $F_i^\alpha(\beta) = h(\beta)$. Furthermore, since there are only μ possible values for α but $\kappa = \mu^+$ values for β, the same α must work for unboundedly many $\beta \in X$. This means that F_i^α agrees with h on an unbounded set of β's, as required. □

3. MAD Families Are Never Smaller Than This Neighbor

It is immediate by inspection of the definitions that $\mathfrak{b}(\kappa) \leq \mathfrak{nm}(\kappa) \leq \mathfrak{a}_e(\kappa)$. Furthermore, Solomon's proof [21] that $\mathfrak{b} \leq \mathfrak{a}$ works equally well at larger cardinals to show $\mathfrak{b}(\kappa) \leq \mathfrak{a}(\kappa)$. In this section, we complete the picture of the relations between $\mathfrak{b}(\kappa)$ and the mad cardinals by proving $\mathfrak{b}(\kappa) \leq \mathfrak{a}_p(\kappa), \mathfrak{a}_g(\kappa)$. For $\kappa = \omega$, these inequalities were proved in [8] in the stronger form $\mathfrak{nm} \leq \mathfrak{a}_p, \mathfrak{a}_g$. We do not know whether the analog of this stronger form is valid for uncountable κ.

Theorem 3.1. *$\mathfrak{b}(\kappa) \leq \mathfrak{a}_p(\kappa)$ for all regular cardinals κ.*

Proof. Let a family $\mathcal{F}$ of fewer than $\mathfrak{b}(\kappa)$ permutations of κ be given. We must produce another permutation almost disjoint from all members of $\mathcal{F}$. Since $|\mathcal{F}| < \mathfrak{b}(\kappa)$, there is a function $b : \kappa \to \kappa$ such that $f <^* b$ and $f^{-1} <^* b$ for all $f \in \mathcal{F}$. Increasing b if necessary, we can assume that it is a strictly increasing function. Define a permutation p of κ such that, for each $\alpha < \kappa$,

$$\max\{\alpha, p(\alpha)\} > b(\min\{\alpha, p(\alpha)\}),$$

and notice that p^{-1} enjoys the same property. The definition of p is an induction of length κ, defining $p(\alpha)$ for two values of α per step. At each step, first find the smallest α not yet in $\mathrm{Dom}(p)$ and assign it an image $p(\alpha) > b(\alpha)$, and then find the smallest α not yet in $\mathrm{Range}(p)$ and assign it a pre-image $p^{-1}(\alpha) > b(\alpha)$. We complete the proof by showing that p is almost disjoint from all members of $\mathcal{F}$. Consider any $f \in \mathcal{F}$. We must show that there are fewer than κ ordinals α such that $p(\alpha) = f(\alpha)$.

We consider separately the cases $\alpha < p(\alpha)$ and $\alpha > p(\alpha)$. In the first case, we have, by our construction of p, that $b(\alpha) < p(\alpha) = f(\alpha)$. Since $f <^* b$, there are fewer than κ such α's. In the second case, we can apply the same argument with p, f, and α replaced by p^{-1}, f^{-1}, and $p(\alpha)$, to conclude that there are fewer than κ possibilities for $p(\alpha)$ and therefore also for α. □

Theorem 3.2. *$\mathfrak{b}(\kappa) \leq \mathfrak{a}_g(\kappa)$ for all regular cardinals κ.*

Proof. Let $\mathcal{G}$ be a κ-ad group of permutations of κ with $|\mathcal{G}| < \mathfrak{b}(\kappa)$. We shall produce a permutation $p \notin \mathcal{G}$ such that the group $\langle \mathcal{G}, p\rangle$ generated by $\mathcal{G}$ and p is κ-ad. As in the preceding proof, let $b : \kappa \to \kappa$ be a strictly increasing function that is $>^* f$ for all $f \in \mathcal{G}$. Call two ordinals $\alpha, \beta < \kappa$ *near* if

$$\max\{\alpha, \beta\} \leq b(\min\{\alpha, \beta)\}),$$

and *far* otherwise. As in the preceding proof, we shall construct our permutation p so that $p(\alpha)$ is always far from α. In addition, we shall make p an involution, i.e., p^2 will be the identity function. Finally, we shall arrange that, whenever α and β are distinct ordinals, each smaller than its p-image, then these two images are far apart. Such a p is easily constructed in an induction of length κ, defining, at each stage, $p(\alpha)$ for some value of α and thus also defining $p(p(\alpha))$ since p is required to be an involution. At any stage, find the smallest α not yet in the domain (equivalently the range) of p, and assign a value for $p(\alpha)$ that is larger than $b(\alpha)$ and larger than both β and $b(\beta)$ for all β that were already in the domain of p. Such a value for $p(\alpha)$ always exists because fewer than κ ordinals have been put into the domain of p in the fewer than κ previous stages of the induction (and because κ is regular, so these ordinals have an upper bound in κ).

We shall show that every non-identity element of the group $\langle G, p\rangle$ fixes fewer than κ points. Notice first that any non-identity element of this group is a product of factors that are either p or non-identity elements of G. We may assume that the factors from G alternate with the p's, since two adjacent factors from G could be multiplied together to make a single factor from G and since two adjacent p's would cancel. We may also assume that that there is at least one factor p, so that the product is not merely an element of G, and there is at least one factor from G, so that the product is not just p. This is because we already know that any element of G has fewer than κ fixed points and that p has no fixed points.

We may further assume that the first factor in the product is from G. To see this, recall that a permutation has the same number of fixed points as any of its conjugates; indeed, the fixed points of qrq^{-1} are the images under q of the fixed points of r. So if we are dealing with a product that starts with p, we can work instead with its conjugate by p, in which the initial factor p has been moved to the end.

We may also assume that the last factor in our product is p. If the last factor was from G, we could conjugate by it, moving that factor to the beginning, where it would combine with the first factor. Thus, our product has the form

$$g_1 p g_2 p \cdots g_n p$$

for some non-identity elements g_i of G.

Consider now an ordinal α that is fixed by this product. We define the *trail* of α to be the sequence of ordinals obtained when we apply this product of permutations to α, one factor at a time. That is, the k^{th} element of the trail is obtained by applying to α the rightmost k factors in the product; the trail begins $\alpha, p(\alpha), g_n p(\alpha), p g_n p(\alpha), \dots$ and ends, since α is a fixed point, with $\dots, p(g_1^{-1}(\alpha)), g_1^{-1}(\alpha), \alpha$. Thus, the trail forms a cycle of ordinals, beginning and ending with α.

For any given ordinal γ and any k, there is at most one α whose trail has γ in the k^{th} position. Since trails have only $2n$ elements, each ordinal γ appears in at most $2n$ trails. Call an ordinal γ *bad* if

- it is fixed by some factor g_i in our product, or
- for some factor g_i in our product, $g_i(\gamma)$ is far from γ, or
- it is fixed by our product and its trail contains an ordinal that is bad by virtue of the preceding two clauses.

Fewer than κ ordinals are bad by the first clause, since G is a κ-ad group. Fewer than κ are bad by the second clause, since $b >^* g_i, g_i{}^{-1}$ for each i. And fewer than κ are bad by the third clause, since trails have finite length.

We shall complete the proof by showing that any fixed point α of our product $g_1 p g_2 p \cdots g_n p$ is necessarily bad.

Let ξ be the largest ordinal in the trail of α. One of the neighbors (in the cyclic sense) of ξ in the trail of α is $p(\xi)$, which is far from ξ by the construction of p. The other neighbor of ξ is an ordinal η obtained by applying to ξ either one of the factors g_i in our product or the inverse of such a g_i. If this η is far from ξ, then either ξ or η is bad by the second clause, and so α is bad by the third clause, as desired.

So we may assume from now on that η is near ξ.

Temporarily suppose $\eta < p(\eta)$. Since $p(\eta)$ is far from η while ξ is near η, it follows that $\xi < p(\eta)$, contrary to our choice of ξ. Thus, our temporary supposition is untenable, and we must have $p(\eta) < \eta$.

Since we also have $p(\xi) < \xi$ by our choice of ξ, and since p is an involution, we have that $p(\xi)$ and $p(\eta)$ are both mapped by p to larger ordinals, namely ξ and η, that are near each other. By our construction of p, this can happen only if $p(\xi) = p(\eta)$ and therefore $\xi = \eta$. But then ξ is fixed by a factor g_i in our product, so it is bad (by the first clause in the definition of bad). Then α is bad by the third clause, and so the proof is complete. □

4. Meagerness and Uniformity

In this section and the next, we discuss the relationship between the various almost-disjointness cardinals and the uniformity of category, the smallest cardinality of a non-meager set. Unfortunately, much remains unclear about this relationship in the case of uncountable cardinals. The present section is devoted to clarifying some basic issues involved in the very definition of meagerness when the familiar space ${}^{\omega}\omega$ is replaced with ${}^{\kappa}\kappa$. Then, in the next section, we shall prove a result for the case $\kappa = \omega$, in a way that offers some hope for a generalization to larger κ.

As always, let κ be an infinite regular cardinal. We consider two equivalent definitions of Baire category on ${}^{\omega}2$ and generalize them in a natural way to ${}^{\kappa}2$. It turns out that the generalizations are no longer equivalent, and we discuss the connection between them. Then, we discuss connections between the two resulting versions of the uniformity of category $\mathfrak{non}(\mathcal{M})(\kappa)$ and the cardinal $\mathfrak{nm}(\kappa)$. Finally, the difference between the two definitions of category suggests a third sort of Baire category on ${}^{\kappa}2$, and we also discuss its relations with the first two.

Our first definition directly generalizes the usual topological approach to Baire category.

Definition 4.1. Give ${}^{\kappa}2$ the *κ-box topology.* The basic open sets for this topology are of the form $\{x \in {}^{\kappa}2 : s \subseteq x\}$ for $s \in {}^{<\kappa}2$. Call a subset of ${}^{\kappa}2$ *topologically comeager* if it includes the intersection of some κ dense open sets for this topology.

The essential point of this definition is that it allows a direct generalization of the Baire Category Theorem; every comeager set is dense. In fact, the usual proof of the Baire Category Theorem shows a bit more in the present context, so we record it for future reference.

Proposition 4.2. *Let X be topologically comeager and let $s \in {}^{<\kappa}2$. There exists an $x \in X$ such that $s \subseteq x$ and such that, if $\kappa > \omega$, then $x(\alpha) = 1$ for all α in some closed unbounded subset of κ.*

Proof. As X is topologically comeager, it includes $\bigcap_{\alpha\in\kappa} U_\alpha$ for some dense open sets $U_\alpha \subseteq {}^{\kappa}2$. We define inductively initial segments $s_\alpha \in {}^{<\kappa}2$ of the desired x, for all $\alpha < \kappa$, as follows. Start with the given s as s_0. At limit stages, take unions. At a successor stage $\alpha + 1$, if s_α is already defined and has length λ_α (which is $< \kappa$ because κ is regular), then first extend it by giving it the value 1 at λ_α. Then extend the resulting sequence (of length $\lambda_\alpha + 1$) to a sequence $s_{\alpha+1} \in {}^{<\kappa}2$ such that all of its extensions in ${}^{\kappa}2$ are in U_α. This last extension is possible because U_α is dense and open in the κ-box topology. Define x to be the union of all the s_α; its length is κ because at each successor step we properly increased the length of the initial segment. It is in X because, for each α, it extends $s_{\alpha+1}$ and therefore lies in U_α. Also, the sequence $\langle\lambda_\alpha\rangle_{\alpha<\kappa}$ is continuous and increasing, so $\{\lambda_\alpha : \alpha < \kappa\}$ is a closed unbounded set. Since x was defined to map all λ_α to 1, the proof is complete. □

Remark 4.3. The use of the κ-box topology is designed to make the preceding proof work in complete analogy with the usual proof of the Baire Category Theorem (except for the minor modification to get x to be 1 on a club). Had we used the ordinary product topology, the proof would get stuck at stage ω, so we would get just the usual Baire Category Theorem, with comeager defined as including the intersection of countably many, rather than κ, dense open sets.

The proposition might still hold, with the product topology and with intersections of κ dense open sets, for some small uncountable κ. For example, it would hold if Martin's Axiom for κ holds. But it would certainly fail once $\kappa \geq 2^{\aleph_0}$. Indeed, for each $s \in {}^{\omega}2$, the set $\{x \in {}^{\kappa}2 : x \restriction \omega \neq s\}$ is dense and open in the product topology, and the intersection of these sets, over all s, is empty.

In the case $\kappa = \omega$, there is an alternative, purely combinatorial definition of Baire category. Here is its generalization to general regular κ.

Definition 4.4. An *interval partition* of κ is a partition Π of κ into intervals $[\alpha, \beta)$. A *chopped κ-sequence* is a pair (x, Π) where $x \in {}^{\kappa}2$ and Π is an interval partition of κ. (We think of the κ-sequence x as being chopped into the pieces $x \restriction I$ for $I \in \Pi$.) We say that $y \in {}^{\kappa}2$ *matches* a chopped κ-sequence (x, Π) if there are κ intervals $I \in \Pi$ such that $y \restriction I = x \restriction I$. A subset X of ${}^{\kappa}2$ is *combinatorially comeager* if there is a chopped κ-sequence (x, Π) such that every y that matches (x, Π) is in X.

Convention 4.5. If Π is an interval partition of κ, we denote the intervals in it, in increasing order of their first elements, as I_α. We write i_α for the first element of I_α. Thus, $I_\alpha = [i_\alpha, i_{\alpha+1})$. Notice that the sequence $\langle i_\alpha\rangle_{\alpha<\kappa}$ is increasing and continuous and that it begins with $i_0 = 0$.

Proposition 4.6. *Every combinatorially comeager set is topologically comeager.*

Proof. For any chopped κ-sequence (x, Π), the set of y that match it is the intersection of the κ sets

$$U_\alpha = \{y \in {}^{\kappa}2 : (\exists\beta > \alpha)\, y \restriction I_\beta = x \restriction I_\beta\}.$$

Each of these sets is dense and open, so any superset of their intersection is topologically comeager. □

The converse of this proposition holds for $\kappa = \omega$; the earliest reference we know for this fact is Talagrand's paper [22]. The proof generalizes easily to all strongly inaccessible κ, so we give the generalization only sketchily.

Proposition 4.7. *If κ is strongly inaccessible, then every topologically comeager subset of ${}^\kappa 2$ is combinatorially comeager.*

Proof. It suffices to consider topologically comeager sets of the form $\bigcap_{\alpha<\kappa} U_\alpha$, where $\langle U_\alpha\rangle$ is a decreasing κ-sequence of dense open sets. (In the box topology, the intersection of any strictly fewer than κ dense open sets is dense and open, so if the U_α were not decreasing we could intersect each one with all its predecessors.) Produce a chopped κ-sequence (x, Π) by inductively defining the intervals I_α and the restrictions of x to these intervals. Suppose this has been done for all α below a certain β. The first ordinal not in the union of these I_α's will be the first point i_β of the interval I_β that we must construct next.

List all the functions $u \in {}^{i_\beta}2$ in a sequence $\langle u_\xi\rangle_{\xi<\lambda}$, where $\lambda < \kappa$ by strong inaccessibility. Properly extend the already defined part $x \restriction i_\beta$ in λ substeps, making sure at substep ξ that any function y that agrees with the extension up to this substep, beyond i_β, and that agrees with u_ξ below i_β is in U_β. This can be done because U_β is dense and open. After λ substeps, let $i_{\beta+1}$ be the domain of the part of x defined so far. This completes the inductive step for β.

The step for β ensures that any y that agrees with x on I_β is in U_β. So any y that matches (x, Π) is in U_β for cofinally many β. Since the sequence $\langle U_\alpha\rangle$ is decreasing, this implies $y \in X$. □

In contrast to the preceding proposition, the implication from topological to combinatorial comeagerness fails for all accessible cardinals κ.

Proposition 4.8. *If κ is an uncountable, regular cardinal but is not strongly inaccessible, then ${}^\kappa 2$ has a topologically comeager set that is not combinatorially comeager.*

Proof. The hypothesis on κ gives us a cardinal μ such that $\mu < \kappa \leq 2^\mu$. Fix such a μ. Let us say that $y \in {}^\kappa 2$ *repeats* at an ordinal $\alpha < \kappa$ if, for all $\xi < \alpha$, $y(\xi) = y(\alpha+\xi)$. In other words, the initial segment of y of length α is repeated immediately in the next α values of y. Recall that an ordinal number is *indecomposable* if it is not the sum of two strictly smaller ordinal numbers; equivalently (except for 0), it has the form ω^α for some α. Now define X to be the set of those $y \in {}^\kappa 2$ that repeat at some indecomposable ordinal $\alpha \in [\mu, \kappa)$. This X is easily seen to be dense and open, since the indecomposable ordinals are cofinal in any uncountable cardinal. So X is topologically comeager. To show that it is not combinatorially comeager, we consider an arbitrary chopped κ-sequence (x, Π) and we produce a y that matches it yet is not in X.

Notice that, if we replace Π by a coarser interval partition Π' then any y that matches (x, Π') also matches (x, Π). As usual, write $\Pi = \{I_\alpha : \alpha < \kappa\}$ with $I_\alpha = [i_\alpha, i_{\alpha+1})$, and recall that the set $\{i_\alpha : \alpha < \kappa\}$ is a closed unbounded set. Since the set of indecomposable ordinals is also a closed unbounded subset of κ and since any intersection of two closed unbounded sets is again such a set, we can, by coarsening Π, assume without loss of generality that all i_α are indecomposable and $> \mu$ except for $i_0 = 0$.

We construct the desired y, matching (x, Π) but not in X, by induction, defining at each step its restriction to two consecutive intervals of Π. That is, in step β we define $y \upharpoonright [i_{2\beta}, i_{2\beta+2})$. To start, in step 0, define $y \upharpoonright i_2$ to be any function $i_2 \to 2$ that does not repeat at any indecomposable ordinal $< i_2$. For example, let $y \upharpoonright i_2$ be identically zero except that $y(0) = 1$. For the induction step, suppose we have already defined $y \upharpoonright i_{2\beta}$ for a certain $\beta \geq 1$; we next define $y \upharpoonright [i_{2\beta}, i_{2\beta+2})$ as follows. First, define y to agree with x on $[i_{2\beta+1}, i_{2\beta+2})$; by doing this at each step, we ensure that y will match (x, Π). The rest of this step, defining y on $[i_{2\beta}, i_{2\beta+1})$, is devoted to ensuring that y does not repeat at any indecomposable ordinal $\alpha \in [i_{2\beta}, i_{2\beta+2})$. If we achieve this, then the proof will be complete, for the intervals $[i_{2\beta}, i_{2\beta+2})$, from the various steps, cover all of κ, and so y will not repeat at any indecomposable α.

It is very easy to prevent y from repeating at indecomposable $\alpha \in [i_{2\beta}, i_{2\beta+1})$, because we can simply set $y(\alpha) \neq y(0)$ for all such α. Notice that this still leaves y entirely undefined on the interval $(i_{2\beta}, i_{2\beta}+\mu)$, because any indecomposable $\alpha > i_{2\beta}$ is also $> i_{2\beta}+\mu$, by definition of indecomposability and our assumption that $i_\alpha > \mu$ for all $\alpha \neq 0$.

For each indecomposable $\alpha \in [i_{2\beta+1}, i_{2\beta+2})$, let f_α be the function from μ to 2 defined by

$$f_\alpha(\xi) = y(\alpha + i_{2\beta} + 1 + \xi).$$

The right side of this equation is defined, because we have already defined y at arguments between $i_{2\beta+1}$ and $i_{2\beta+2}$, and because the indecomposability of $i_{2\beta+2}$ ensures that $\alpha + i_{2\beta} + 1 + \xi < i_{2\beta+2}$. Since the number of these f_α's is at most $|i_{2\beta+2}| < \kappa \leq 2^\mu$, there is a function $g : \mu \to 2$ that is distinct from all the f_α's. Define y on $(i_{2\beta}, i_{2\beta} + \mu)$ by setting

$$y(i_{2\beta} + 1 + \xi) = g(\xi) \quad \text{for all } \xi < \mu,$$

and then extend y arbitrarily to those points in $[i_{2\beta}, i_{2\beta+2})$ where it is not yet defined.

For each indecomposable $\alpha \in [i_{2\beta+1}, i_{2\beta+2})$, we have, since $g \neq f_\alpha$, some $\xi < \mu$ such that $g(\xi) \neq f_\alpha(\xi)$ and therefore

$$y(i_{2\beta} + 1 + \xi) = g(\xi) \neq f_\alpha(\xi) = y(\alpha + i_{2\beta} + 1 + \xi).$$

Since $i_{2\beta} + 1 + \xi < i_{2\beta+1} \leq \alpha$, we conclude that y does not repeat at α. This completes the proof of the proposition. □

Remark 4.9. If there exists μ such that $\mu < \kappa < 2^\mu$, then the preceding proposition admits a simpler proof. Let

$$X = \{y \in {}^\kappa 2 : (\exists \alpha \in [\mu, \kappa))(\forall \xi < \mu)\, y(\xi) = y(\alpha + \xi)\}$$

In other words, $y \in X$ if and only if the initial segment $y \upharpoonright \mu$ recurs as a segment of y after μ. This X is easily seen to be dense and open, and therefore topologically comeager. We show that it is not combinatorially comeager by finding, for any given chopped κ-sequence (x, Π), a y that matches it but is not in X. To do this, first define y on the final segment $[\mu, \kappa)$ to agree with x there. This ensures that y matches (x, Π). Then look at all the functions

$$f_\alpha : \mu \to 2 : \xi \mapsto y(\alpha + \xi) \quad \text{for } \alpha \in [\mu, \kappa).$$

The number of such functions is at most $\kappa < 2^\mu$. So we can define $y \upharpoonright \mu$ to differ from all the f_α's. This ensures that $y \notin X$.

The next proposition is an easy one, asserting that combinatorial comeagerness is preserved by intersections of κ sets; the corresponding fact for topological comeagerness is immediate from the definition.

Proposition 4.10. *The intersection of any κ combinatorially comeager sets is combinatorially comeager.*

Proof. It suffices to produce, for any κ chopped κ-sequences, say (x^ξ,Π^ξ) for $\xi<\kappa$, a single chopped κ-sequence (x,Π) such that any y matching the latter also matches all of the former. To do this, first define the intervals I_α of Π inductively, making sure that I_α includes disjoint intervals, say J^ξ, from all the partitions Π^ξ for $\xi<\alpha$. Since we are concerned, at any particular stage α, with fewer than κ partitions Π^ξ, and since κ is regular, we can easily find such an I_α, of length $<\kappa$. Then define x on I_α to agree with x^ξ on J^ξ for each $\xi<\alpha$. These requirements on $x\restriction I_\alpha$ are consistent because the J^ξ are disjoint.

If y matches (x,Π), then for any ξ we can find κ intervals I_α of Π such that y agrees with x on each of them and each of them has $\alpha>\xi$. But then y also agrees with x^ξ on intervals from Π^ξ included in each of these I_α's. So y matches (x^ξ,Π^ξ). □

We turn next to the uniformity cardinals associated with our two sorts of comeagerness.

Definition 4.11. $\mathfrak{non}(\mathcal{M})_{\rm top}(\kappa)$ is the smallest cardinality of any subset of ${}^\kappa 2$ that intersects every topologically comeager set. $\mathfrak{non}(\mathcal{M})_{\rm comb}(\kappa)$ is the smallest cardinality of any subset of ${}^\kappa 2$ that intersects every combinatorially comeager set.

Equivalently, if we define "meager" to mean having comeager complement, these are the smallest cardinalities of non-meager subsets of ${}^\kappa 2$ in the topological and combinatorial senses.

Corollary 4.12. $\mathfrak{non}(\mathcal{M})_{comb}(\kappa)\leq\mathfrak{non}(\mathcal{M})_{top}(\kappa)$. *If κ is strongly inaccessible, then equality holds.*

Proof. Immediate from Propositions 4.6 and 4.7. □

We do not know whether the assumption of inaccessibility is needed in the second part of the corollary. That is, we do not know whether the difference between combinatorial and topological comeagerness, established in Proposition 4.8, can be strengthened to give a difference between the uniformity cardinals in some models of set theory.

As mentioned earlier, Bartoszyński [2] has given, in the case $\kappa=\omega$, another characterization of $\mathfrak{non}(\mathcal{M})$, namely $\mathfrak{non}(\mathcal{M})=\mathfrak{nm}$. For uncountable κ, we have the following partial generalization.

Proposition 4.13. $\mathfrak{nm}(\kappa)\leq\mathfrak{non}(\mathcal{M})_{comb}(\kappa)$. *If κ is strongly inaccessible, then equality holds.*

Proof. Fix a set $X\subseteq{}^\kappa 2$ of size $\mathfrak{non}(\mathcal{M})_{\rm comb}(\kappa)$ that meets every combinatorially comeager set. To each $x\in X$ that is not eventually constant with value 1, associate a function $\varphi_x:\kappa\to\kappa$ by defining $\varphi_x(\alpha)$ to be the length of the string of consecutive 1's in x immediately preceding the $\alpha^{\rm th}$ zero. (We begin counting with zero, so the position of the $\alpha^{\rm th}$ zero is at least α.) We shall find, for each function $f:\kappa\to\kappa$, an $x\in X$ such that f and φ_x agree at κ places. Thus, $\{\varphi_x:x\in X\}$ witnesses that $\mathfrak{nm}(\kappa)\leq|X|=\mathfrak{non}(\mathcal{M})_{\rm comb}(\kappa)$ as required.

So let an arbitrary $f:\kappa\to\kappa$ be given. Define a chopped κ-sequence (y,Π) so that the $\alpha^{\rm th}$ block, $y\restriction I_\alpha$, consists of $i_\alpha+1$ zeros, each preceded by exactly $f(i_\alpha)$

1's. (Recall Convention 4.5, according to which I_α is the α^{th} interval in Π and i_α is its first element.) As X meets every combinatorially comeager set, fix an $x \in X$ that matches (y, Π). Consider any one of the κ values of α such that x and y agree on I_α. The ${i_\alpha}^{\text{th}}$ zero in x occurs somewhere in I_α. (It cannot be earlier, as the position of the ${i_\alpha}^{\text{th}}$ zero is at least i_α; it cannot be later, as x agrees with y on I_α and y has more than i_α zeros there.) So, using again the agreement between x and y on I_α, we see that, in x, the ${i_\alpha}^{\text{th}}$ zero is immediately preceded by exactly $f(i_\alpha)$ 1's. This means that $\varphi_x(i_\alpha) = f(i_\alpha)$. Since this happens for κ values of α, the proof of the proposition's first assertion is complete.

We omit the proof of the reverse inequality for inaccessible κ, because it is essentially the same as Bartoszyński's proof for the case $\kappa = \omega$. □

Corollary 4.14. $\mathfrak{b}(\kappa) \leq \mathfrak{nm}(\kappa) \leq \mathfrak{non}(\mathcal{M})_{comb}(\kappa) \leq \mathfrak{non}(\mathcal{M})_{top}(\kappa)$.

Proof. This just summarizes Proposition 4.13, Corollary 4.12, and a trivial consequence of the definitions of $\mathfrak{b}(\kappa)$ and $\mathfrak{nm}(\kappa)$. □

There is another, easier lower bound for $\mathfrak{non}(\mathcal{M})_{\text{comb}}(\kappa)$ (and therefore also for $\mathfrak{non}(\mathcal{M})_{\text{top}}(\kappa)$).

Proposition 4.15. $\mathfrak{non}(\mathcal{M})_{comb}(\kappa) \geq 2^{<\kappa}$

Proof. Since $\mathfrak{non}(\mathcal{M})_{\text{comb}}(\kappa) \geq \mathfrak{b}(\kappa) > \kappa$, the result is clear if $2^{<\kappa} = \kappa$ (or even if $2^{<\kappa} = \kappa^+$), so we assume from now on that $2^{<\kappa} > \kappa$.

Let X be a set of size $\mathfrak{non}(\mathcal{M})_{\text{comb}}(\kappa)$ that meets every combinatorially comeager subset of ${}^{\kappa}2$. For each cardinal $\lambda < \kappa$ and each function $r : \lambda \to 2$, define a chopped κ-sequence (y_r, Π_r) as follows. Π_r is the partition of κ into intervals of length λ, and, on each of these intervals, y_r is a copy of r; that is, $y_r(i_\alpha + \xi) = r(\xi)$ for all $\alpha < \kappa$ and $\xi < \lambda$. By assumption, X contains an element x_r that matches (y_r, Π_r). In particular, there is $\beta < \kappa$ such that, for all $\xi < \lambda$, we have $x(\beta+\xi) = r(\xi)$. In this situation, we can recover r if we know x_r, β, and λ. Since there are $2^{<\kappa}$ possible r's, $|X|$ possible x_r's, and κ possible β's and λ's, we must have $2^{<\kappa} \leq |X| \cdot \kappa$. Since we have assumed $2^{<\kappa} > \kappa$, it follows that $2^{<\kappa} \leq |X|$. □

Using this bound on $\mathfrak{non}(\mathcal{M})_{\text{comb}}(\kappa)$, we show that it can consistently be strictly larger than $\mathfrak{nm}(\kappa)$.

Proposition 4.16. *It is consistent with ZFC to have* $\mathfrak{nm}(\aleph_1) = \aleph_2 \ll \mathfrak{non}(\mathcal{M})_{comb}(\aleph_1)$.

Proof. Begin with a model of GCH and adjoin a large number of Cohen reals. By the preceding proposition, $\mathfrak{non}(\mathcal{M})_{\text{comb}}(\aleph_1)$ will be large in the forcing extension, so we need only check that $\mathfrak{nm}(\aleph_1) = \aleph_2$ there. For this it suffices to show that every $f : \aleph_1 \to \aleph_1$ in the extension agrees in $\aleph_1$ places with some function from the ground model. This is undoubtedly well known, but we give the proof because it is easier to give than to look up.

Since the forcing satisfies the countable chain condition, there is, in the ground model, a function $h : \aleph_1 \to \aleph_1$ that majorizes f everywhere. In the ground model, fix, for each nonzero, countable ordinal α, a surjection $e_\alpha : \omega \to \alpha$, and define functions $g_n : \aleph_1 \to \aleph_1$ by $g_n(\xi) = e_{h(\xi)}(n)$. For each $\xi < \aleph_1$, we have, since $f(\xi) < h(\xi)$, some n such that $f(\xi) = g_n(\xi)$. Since there are uncountably many ξ's and only countably many n's, some single n must work for uncountably many ξ's. Then g_n is a function in the ground model that agrees with f uncountably often. □

Remark 4.17. We record for future reference that the same idea as in the last paragraph of this proof establishes the observation in [12] that $\mathfrak{b}(\kappa) = \mathfrak{nm}(\kappa)$ for successor cardinals κ.

To close this section, we return to the argument used, in the proof of Proposition 4.8, to separate topological and combinatorial comeagerness. To evade the counterexample used there, we introduce the following strengthening of the notion of matching and thus a weakening of the notion of combinatorially comeager.

Definition 4.18. We say that $y \in {}^{\kappa}2$ *stationarily matches* or *stat-matches* a chopped κ-sequence (x, Π) if $\{\alpha < \kappa : y \restriction I_\alpha = x \restriction I_\alpha\}$ is a stationary subset of κ. A set X is *stat-comeager* if there is a chopped κ-sequence (x, Π) such that every y that stat-matches (x, Π) is in X.

We remark that the definition of stat-matching could be equivalently formulated to require that $\{i_\alpha : y \restriction I_\alpha = x \restriction I_\alpha\}$ be stationary, because the function $\alpha \mapsto i_\alpha$ is continuous and increasing.

For the next proposition, we need the following $\diamondsuit$-like principle. See [14, Section II.7] for more information about such variants of $\diamondsuit$.

Definition 4.19. $\diamondsuit^*_\kappa$ is the assertion that one can assign to each $\alpha < \kappa$ a family $\mathcal{A}_\alpha$ of $< \kappa$ subsets of α in such a way that, for every $X \subseteq \kappa$, the set $\{\alpha < \kappa : X \cap \alpha \in \mathcal{A}_\alpha\}$ includes a closed unbounded set.

Proposition 4.20. *Assume $\diamondsuit^*_\kappa$. Then every topologically comeager set is stat-comeager.*

Proof. Fix families $\mathcal{A}_\alpha$ as in the definition of $\diamondsuit^*_\kappa$. Then we proceed exactly as in the proof of Proposition 4.7 except that, at step β, instead of listing all the functions $u \in {}^{i_\beta}2$, we only list the characteristic functions of the sets in $\mathcal{A}_{i_\beta}$. Thus, the list has length $< \kappa$ without any need for the assumption of strong inaccessibility used in Proposition 4.7. When the induction is complete, we have (x, Π) with the property that, if

- y agrees with x on an interval $[i_\beta, i_{\beta+1})$ of Π and
- $y \restriction i_\beta$ is the characteristic function of a set in $\mathcal{A}_\beta$,

then $y \in U_\beta$. If y stat-matches (x, Π), then the first of these conditions is satisfied for a stationary set of β's. The second condition is satisfied for a closed unbounded set of β's, by our choice of the $\mathcal{A}_\alpha$'s in accordance with $\diamondsuit^*_\kappa$. So both conditions are satisfied simultaneously for an unbounded (in fact stationary) set of β's. As in the proof of Proposition 4.7, this suffices to complete the proof. □

Unfortunately, the converse of this proposition fails, whether or not any diamond principles are assumed.

Example 4.21. For any uncountable regular κ, there is a stat-comeager set that is not topologically comeager. Probably the simplest example is the set of those $y \in {}^{\kappa}2$ such that $y(\alpha) = 0$ for a stationary set of α's. This is stat-comeager because it is the set of all y that stat-match (x, Π) when x is the identically 0 function and Π is the partition of κ into singletons. But it is not topologically comeager, by Proposition 4.2.

5. All MAD families May Be Larger Than This Neighbor

This section is motivated by the hope of constructing a model where all of the almost-disjointness numbers $\mathfrak{a}(\omega_1)$, $\mathfrak{a}_e(\omega_1)$,$\mathfrak{a}_p(\omega_1)$,$\mathfrak{a}_g(\omega_1)$ are larger than $\mathfrak{b}(\omega_1)$, or the analog for larger cardinals κ. Although we are unable to realize this hope, we prove a related result for $\kappa = \omega$ by an argument that offers some hope of generalizing to larger κ. We begin, however, with a brief discussion of the problem for uncountable κ. There are several noticeable differences between the case $\kappa = \omega$ and the case of uncountable κ, especially successor κ.

For $\kappa = \omega$, we have Bartoszyński's theorem that $\mathfrak{non}(\mathcal{M}) = \mathfrak{nm}$, and so one easily obtains models of $\mathfrak{b} < \mathfrak{nm}$, for example the random real model (see [3, Model 7.6.8] or the table in [4, Section 11]). In contrast, as noted in [12] and in Remark 4.17 above, $\mathfrak{b}(\kappa) = \mathfrak{nm}(\kappa)$ for all successor cardinals κ.

ZFC + CH implies that there is a ccc poset forcing $\mathfrak{b} < \mathfrak{a}_e = 2^{\aleph_0} = \aleph_2$; see [15] or [25]. In contrast, ZFC + CH + $2^{\aleph_1}$=$\aleph_2$ does not imply that there is a poset that preserves $\aleph_1$, satisfies the $\aleph_2$-chain condition, and forces $\mathfrak{b}(\aleph_1) < \mathfrak{a}_e(\aleph_1) = 2^{\aleph_1} = \aleph_3$; see [12].

For $\kappa = \omega$ it is known to be consistent that $\mathfrak{d} = \aleph_1 < \mathfrak{a}_e$, and it remains an open problem whether $\mathfrak{d} = \aleph_1 < \mathfrak{a}$ is consistent. These facts contrast with our Theorems 2.1 and 2.2.

Let us look next at some constructions making $\mathfrak{b}$ smaller than some almost-disjointness cardinals (for $\kappa = \omega$), in the hope of extending them to uncountable κ.

Miller [15] gave a ccc forcing construction of a model of $\mathfrak{b} < \mathfrak{a}_e$. Results from [12], however, imply that this sort of construction, with the natural generalization of the chain condition, cannot succeed for $\kappa = \omega_1$. A closer inspection of Miller's argument reveals that several steps fail to generalize. The most significant of these is that, for $\kappa = \omega$, the natural forcing to add eventually different reals does not add dominating reals. For $\kappa = \omega_1$, the analogous poset does add a dominating function $\omega_1 \to \omega_1$.

The consistency of $\mathfrak{b} < \mathfrak{a}$ was proved in [17] by a proper forcing construction. (The argument is necessarily more complicated than that for $\mathfrak{b} < \mathfrak{a}_e$ because, according to a result in [12], ZFC+CH does not imply the existence of a ccc poset forcing $\mathfrak{b} < \mathfrak{a} = 2^{\aleph_0} = \aleph_2$.) The construction from [17] does not generalize to larger κ because there is no suitable generalization of proper forcing.

A complicated ccc construction of a model of $\mathfrak{b} < \mathfrak{a}$, based on ideas from [17], is given in [6]. Several aspects of this argument fail to generalize to $\kappa = \omega_1$. For example, the easy case of the proof uses an argument, similar to that in [15] mentioned above, to show that one step of the iteration does not add dominating reals. Also, the proof depends on special properties of finite-support iteration and on the compactness of ω.

The method of iteration along a template is introduced in [19] to prove the consistency of $\mathfrak{d} < \mathfrak{a}$. It is used in [7] to obtain the stronger result that the cofinality of the measure-zero ideal can consistently be $< \mathfrak{a} = \mathfrak{a}_e$. This method uses heavily the fact that the supports in the iteration are finite; it involves induction on the cardinalities of the supports. As a result, it is not suitable for the generalization to uncountable κ.

We shall now present a consistency proof of a statement, $\mathfrak{non}(\mathcal{M}) < \mathfrak{a} = \mathfrak{a}_e = \mathfrak{a}_p = \mathfrak{a}_g = 2^{\aleph_0}$, that is stronger than those just quoted from [15, 17, 6] (because it uses $\mathfrak{non}(\mathcal{M})$ instead of $\mathfrak{b}$) and incomparable with those quoted from [19, 7] (because $\mathfrak{non}(\mathcal{M})$ is incomparable with $\mathfrak{d}$ and because our result applies to $\mathfrak{a}_p$ and $\mathfrak{a}_g$). Furthermore, the construction is relatively easy, relying largely on general facts about forcing, not on difficult preservation theorems. Finally, generalizing it to uncountable κ does not look quite so hopeless as generalizing the arguments mentioned above; there is more room for modifications. Part of our argument is based on ideas from [19].

Theorem 5.1. *Assume that GCH holds and there exists a measurable cardinal. Then there is a notion of forcing that satisfies the countable chain condition and forces* $\mathfrak{non}(\mathcal{M}) < \mathfrak{a} = \mathfrak{a}_e = \mathfrak{a}_p = \mathfrak{a}_g = 2^{\aleph_0}$.

The proof will involve a rather non-traditional iterated forcing and even comparisons between several such forcings. It will therefore be useful to begin by reviewing some general information about iterations and comparisons. Some of this information is quite standard and can be found, for example, in [14, Chapters VII and VIII]. By a forcing notion, we shall always mean a separative partial order with a top element 1.

A *complete embedding* of one notion of forcing, $\mathbb{P}$, into another, $\mathbb{Q}$, is a function $i : \mathbb{P} \to \mathbb{Q}$ that

- respects the ordering: $p \leq_{\mathbb{P}} p'$ if and only if $i(p) \leq_{\mathbb{Q}} i(p')$,
- respects incompatibility: $p \perp_{\mathbb{P}} p'$ if and only if $i(p) \perp_{\mathbb{Q}} i(p')$, and
- has approximations in $\mathbb{P}$ for conditions in $\mathbb{Q}$: for each $q \in \mathbb{Q}$ there exists some $p \in \mathbb{P}$ such that whenever $p' \leq_{\mathbb{P}} p$ then $i(p')$ is compatible with q.

This definition admits several equivalent reformulations. In the first place, we can replace "if and only if" in the first two requirements by the implications from left to right; the converses are then deducible. More importantly, the third condition, about approximations, is equivalent to requiring that, whenever A is a maximal antichain in $\mathbb{P}$, then the antichain $i(A)$ in $\mathbb{Q}$ is also maximal. Finally, the third condition trivially implies the apparently stronger statement obtained by weakening the hypothesis $p' \leq_{\mathbb{P}} p$ to require only that p and p' are compatible in $\mathbb{P}$.

We shall need a stronger notion of embedding, to ensure good behavior at limit stages of comparisons between two iterations. A *superb embedding* from $\mathbb{P}$ to $\mathbb{Q}$ is a complete embedding with the additional property that the approximation p in the third requirement can be chosen so that $i(p) \geq_{\mathbb{Q}} q$. With this additional property, p is uniquely determined, and we denote it by $t(q)$. We call t the *truncation function* associated to the superb embedding i.

A familiar example of a superb embedding is given by two-step iteration of forcing. If $\mathbb{P}$ is a notion of forcing and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a notion of forcing, then there is a superb embedding from $\mathbb{P}$ to $\mathbb{P} * \dot{\mathbb{Q}}$. It is defined by $i(p) = (p, \dot{1})$ (where $\dot{1}$ is a $\mathbb{P}$-name for the top element of $\dot{\mathbb{Q}}$) and $t(p, \dot{q}) = p$. This example is the reason for the terminology "truncation function".

Remark 5.2. A complete embedding betwen notions of forcing extends to a complete embedding (in the Boolean algebra sense — preserving arbitrary suprema and infima) between the associated complete Boolean algebras of regular open sets. Whenever $\mathbb{A}$ is a complete subalgebra of a complete Boolean algebra $\mathbb{B}$, there is a so-called *reflection* map of $\mathbb{B}$ into $\mathbb{A}$, taking any $b \in \mathbb{B}$ to the infimum of all elements of $\mathbb{A}$ that are $\geq_{\mathbb{B}} b$. (If we regard Boolean algebras as posets and thus as categories, then this is a reflection in the category-theoretic sense, a functor that is left adjoint to an inclusion.) A complete embedding of complete Boolean algebras is always a superb embedding because the reflection serves as the truncation map. A complete embedding $i : \mathbb{P} \to \mathbb{Q}$ of posets is superb if and only if the resulting reflection map, from the regular open algebra of $\mathbb{Q}$ to that of $\mathbb{P}$, maps $\mathbb{Q}$ into $\mathbb{P}$. In that case, the restriction to $\mathbb{Q}$ of the reflection map serves as the truncation map.

We shall be working with finite support iterations of forcing, in a context somewhat more general than the traditional one. The iteration will be, as usual, an ordinal-indexed sequence of forcing notions $\mathbb{P}_\alpha$, with a coherent system of complete embeddings from each $\mathbb{P}_\alpha$ to each later $\mathbb{P}_\beta$. "Finite support" means that, for limit ordinals λ, $\mathbb{P}_\lambda$ is the direct limit of the earlier $\mathbb{P}_\alpha$'s with respect to the given complete embeddings. In contrast to the traditional set-up, these embeddings need not be superb, because $\mathbb{P}_{\alpha+1}$ will not in general be of the form $\mathbb{P}_\alpha * \dot{\mathbb{Q}}$. Nevertheless, superb embeddings will play an important role in our argument, partly because of the following lemma.

Lemma 5.3. *Let $\mathbb{P}$ and $\mathbb{Q}$ be the direct limits of systems of forcing notions $\langle \mathbb{P}_\alpha : \alpha < \lambda\rangle$ and $\langle \mathbb{Q}_\alpha : \alpha < \lambda\rangle$, respectively. Suppose that we have, for each $\alpha < \lambda$, a superb embedding $i_\alpha : \mathbb{P}_\alpha \to \mathbb{Q}_\alpha$, such that these embeddings and also the associated truncation functions t_α commute with the complete embeddings $\mathbb{P}_\alpha \to \mathbb{P}_\beta$ and $\mathbb{Q}_\alpha \to \mathbb{Q}_\beta$ in the direct systems. Then there is a superb embedding $i : \mathbb{P} \to \mathbb{Q}$ such that it and its associated truncation function commute with the embeddings of $\mathbb{P}_\alpha$ and $\mathbb{Q}_\alpha$ into $\mathbb{P}$ and $\mathbb{Q}$, respectively.*

If, in this lemma, we had only assumed that the i_α are complete embeddings (not necessarily superb) commuting with the embeddings in the two direct systems, it would not follow that they induce a complete embedding (or even that there exists a complete embedding) $\mathbb{P} \to \mathbb{Q}$.

We shall need the notion of nice names for reals. A *nice name*, with respect to a forcing notion $\mathbb{P}$, for a function $\omega \to \omega$ is a family $\dot{f} = \langle A_n, f_n\rangle_{n\in\omega}$ of maximal antichains A_n of $\mathbb{P}$ and functions $f_n : A_n \to \omega$. In the presence of a generic subset G of $\mathbb{P}$, such an $\dot{f}$ determines a map $\omega \to \omega$, namely the map sending each $n \in \omega$ to $f_n(a)$, where a is the unique element of $A_n \cap G$. Another way to say this is that $\dot{f}$ determines a name (in the usual, not nice, sense) consisting of all the pairs $(p, [n, f_n(p)])$, where $n \in \omega$, $p \in A_n$, and $[x, y]$ means the canonical name for the ordered pair whose components are the natural numbers x and y. Then the function described above is just the value, with respect to G, of this name. From now on, we follow the common practice of neglecting the distinction between a nice name and its associated name; thus, nice names will be treated as names. It is well known that, if $\dot{g}$ is a name forced, by some $p \in \mathbb{P}$, to denote a function $\omega \to \omega$, then there is a nice name $\dot{f}$ that is forced by p to equal $\dot{g}$. In less formal terms, there is no loss of generality in pretending that all names of reals are nice. Note, for future reference, that $p \Vdash_{\mathbb{P}} \dot{f}(n) > m$ if and only if every extension of p in $\mathbb{P}$ is compatible with some element $a \in A_n$ for which $f_n(a) > m$.

Because a complete embedding $i : \mathbb{P} \to \mathbb{Q}$ preserves maximal antichains, it sends every nice $\mathbb{P}$-name $\dot{f}$ for a real to a nice $\mathbb{Q}$-name $i_*(\dot{f})$ for a real. In detail, if $\dot{f} = \langle A_n, f_n\rangle_{n\in\omega}$ then $i_*(\dot{f}) = \langle i(A_n), f_n \circ i^{-1}\rangle_{n\in\omega}$. We say that a nice $\mathbb{Q}$-name *comes from* $\mathbb{P}$ (via the embedding i) if it is $i_*(\dot{f})$ for some $\mathbb{P}$-name $\dot{f}$.

This concludes our review of the general facts that we shall need about embeddings of forcing notions and nice names. The next step in proving Theorem 5.1 is to define the desired notion of forcing.

Assume that GCH holds and that μ is a measurable cardinal. Fix a non-principal μ-complete ultrafilter D on μ, and fix two regular cardinals κ and λ such that $\mu < \kappa < \lambda$. We shall obtain the required notion of forcing by a finite-support iteration of length λ. That is, we shall construct a $(\lambda+1)$-sequence of forcing notions $\mathbb{P}_\alpha$, with complete embeddings $e_\alpha^\beta : \mathbb{P}_\alpha \to \mathbb{P}_\beta$ for $\alpha < \beta$, coherent in the sense that, when $\alpha < \beta < \gamma$ then $e_\alpha^\gamma = e_\beta^\gamma \circ e_\alpha^\beta$. What is somewhat unusual about the construction is that we do not insist on superb embeddings. Thus, $\mathbb{P}_{\alpha+1}$ is not always of the form $\mathbb{P}_\alpha * \dot{\mathbb{Q}}$.

We define $\mathbb{P}_\alpha$ for all $\alpha \leq \lambda$ by induction on α.

We begin the construction by letting $\mathbb{P}_0$ be the notion of forcing for adding κ Cohen reals. In detail, a condition is a function $p : \kappa \to {}^{<\omega}\omega$ such that $p(\xi) = \varnothing$ for all but finitely many ξ. The ordering puts $p' \leq p$ if and only if $p'(\xi)$ is an extension of $p(\xi)$ for all $\xi < \kappa$.

If α is even, then we define $\mathbb{P}_{\alpha+1}$ to be the ultrapower ${\mathbb{P}_\alpha}^\mu/D$. The complete embedding $e_\alpha^{\alpha+1} : \mathbb{P}_\alpha \to \mathbb{P}_{\alpha+1}$ is the canonical embedding of the poset into its ultrapower; that is, $p \in \mathbb{P}_\alpha$ is sent to the equivalence class, in the ultrapower, of the constant function $\mu \to \mathbb{P}_\alpha : \xi \mapsto p$. We must verify that this is a complete embedding (Lemma 5.4 below), but first we give the definition of the remaining stages of the iteration.

If α is odd, then we choose (in a way to be specified later) a set $\mathcal{F}_\alpha$ of fewer than μ nice $\mathbb{P}_\alpha$-names of functions $\omega \to \omega$. Roughly speaking, $\mathbb{P}_{\alpha+1}$ will be the iteration $\mathbb{P}_\alpha * \mathbb{Q}_\alpha$, where $\mathbb{Q}_\alpha$ is a Hechler-like forcing, adjoining a function $\omega \to \omega$ that dominates the functions denoted by the names in $\mathcal{F}_\alpha$. It will be convenient, however, to officially define $\mathbb{P}_{\alpha+1}$ as a certain dense subset of this iteration. Specifically, we define a condition in $\mathbb{P}_{\alpha+1}$ to be a triple (p, s, A), where $p \in \mathbb{P}_\alpha$, $s \in {}^{<\omega}\omega$ and A is a finite subset of $\mathcal{F}_\alpha$. An extension of (p, s, A) is any condition (p', s', A') where $p' \leq_{\mathbb{P}_\alpha} p$, $s' \supseteq s$, $A' \supseteq A$, and, for each $k \in \mathrm{Dom}(s') - \mathrm{Dom}(s)$ and each $\dot{f} \in A$,

$$p' \Vdash_{\mathbb{P}_\alpha} \dot{f}(k) < s(k).$$

The complete embedding $e_\alpha^{\alpha+1} : \mathbb{P}_\alpha \to \mathbb{P}_{\alpha+1}$ is defined by $p \mapsto (p, \varnothing, \varnothing)$. In fact, this embedding is superb, with truncation map $(p, s, A) \mapsto p$.

For both even and odd α, we have defined the embedding $e_\alpha^{\alpha+1} : \mathbb{P}_\alpha \to \mathbb{P}_{\alpha+1}$. The embeddings $e_\gamma^{\alpha+1} : \mathbb{P}_\gamma \to \mathbb{P}_{\alpha+1}$ for $\gamma < \alpha$ are, of course, defined by composition — the only possible definition to produce a coherent system of embeddings.

Finally, as indicated by our calling the process a finite support iteration, we define $\mathbb{P}_\beta$ for limit β to be the direct limit of the earlier $\mathbb{P}_\alpha$'s, with the obvious complete embeddings.

Lemma 5.4. *Each $\mathbb{P}_\beta$ satisfies the countable chain condition. The embeddings $e_\alpha^\beta : \mathbb{P}_\alpha \to \mathbb{P}_\beta$ for $\alpha < \beta \leq \lambda$ are complete embeddings of forcing notions.*

Proof. We prove both statements simultaneously by induction on β. For $\beta = 0$, the first statement — that Cohen forcing is ccc — is well known and the second is vacuously true.

For limit β, both statements are well known properties of finite-support iterations of ccc forcing; see [13, Section 23] or [14, Section VIII.5].

When β is a successor ordinal, the second statement will follow for all $\alpha < \beta$ if we prove it for the immediate predecessor of β, for then we can invoke the induction hypothesis and the fact that composition of complete embeddings produces complete embeddings. So we may assume that $\beta = \alpha + 1$ and $\mathbb{P}_\alpha$ satisfies the ccc. Our goal is to prove the ccc for $\mathbb{P}_{\alpha+1}$ and the completeness of the embedding $\mathbb{P}_\alpha \to \mathbb{P}_{\alpha+1}$.

Consider first the case of even α, so $\mathbb{P}_{\alpha+1}$ is the ultrapower with respect to D of $\mathbb{P}_\alpha$. Since the countable chain condition is expressible by a sentence in the infinitary logic L_{ω_2,ω_2} (saying that, of any $\aleph_1$ elements, some two are compatible) and since D is $\aleph_2$-complete (and more), the ultrapower preserves the ccc.

The first two requirements for a complete embedding, respecting the order and incompatibility, are clearly satisfied by the ultrapower embedding (indeed by any elementary embedding), so it remains to check the third requirement, which is equivalent, as we mentioned above, to the fact that, for every maximal antichain A in $\mathbb{P}_\alpha$, its image under the embedding is maximal in the ultrapower. By elementarity of ultrapowers, A^μ/D is certainly a maximal antichain in $\mathbb{P}_{\alpha+1}$. But, by induction hypothesis, A is countable. Since D is countably complete, it follows that A^μ/D is exactly the image of A under the ultrapower embedding. This completes the induction step in the case that α is even.

Now suppose α is odd. Define $\dot{\mathbb{Q}}$ to be the $\mathbb{P}_\alpha$-name for the Hechler-like notion of forcing that adjoins a real dominating the reals named by the elements of $\mathcal{F}_\alpha$. That is, $\mathbb{P}$ forces "$\dot{\mathbb{Q}}$ is the poset of pairs (s, A) where $s \in {}^{<\omega}\omega$, and A is a finite subset of $\mathcal{G}$, ordered as in Hechler forcing." (Here $\mathcal{G}$ is a $\mathbb{P}_\alpha$-name for the set whose members are denoted by the elements of $\mathcal{F}_\alpha$. Intuitively, $\mathcal{G}$ "is" $\mathcal{F}_\alpha$; formally, we can take $\mathcal{G} = \mathbb{P}_\alpha \times \mathcal{F}_\alpha$.) Now $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}$ is a ccc (indeed σ-centered) notion of forcing. Therefore $\mathbb{P}_\alpha * \dot{\mathbb{Q}}$ is a ccc notion of forcing, with a superb embedding from $\mathbb{P}_\alpha$ into it. Our $\mathbb{P}_{\alpha+1}$ is isomorphic to a dense subset of this $\mathbb{P}_\alpha * \dot{\mathbb{Q}}$. Specifically, there is a dense subset of $\mathbb{P}_\alpha * \dot{\mathbb{Q}}$ consisting of conditions $(p, (\dot{s}, \dot{A}))$ where p forces $\dot{s}$ to denote a specific $s \in {}^{<\omega}\omega$ and forces $\dot{A}$ to denote a specific finite $A \subseteq \mathcal{F}_\alpha$. Density is proved by observing that one can always extend p to make the required decisions. Assigning to such conditions the corresponding triples (p, s, A), we obtain an isomorphism to $\mathbb{P}_{\alpha+1}$. Furthermore, the canonical embedding of $\mathbb{P}_\alpha$ into $\mathbb{P}_\alpha * \dot{\mathbb{Q}}$ maps into our dense subset and thus gives rise to our complete embedding $e_\alpha^{\alpha+1}$ of $\mathbb{P}_\alpha$ into $\mathbb{P}_{\alpha+1}$. □

Lemma 5.5.

- *For each $\alpha \leq \lambda$, the cardinality of $\mathbb{P}_\alpha$ is $\max\{\kappa, |\alpha|\}$ if this is regular; otherwise, $|\mathbb{P}_\alpha|$ is either $\max\{\kappa, |\alpha|\}$ or its successor cardinal.*
- *For each $\alpha \leq \lambda$, the number of nice $\mathbb{P}_\alpha$-names of reals is $\max\{\kappa, |\alpha|\}$ if this is regular; otherwise, the number of these names is either $\max\{\kappa, |\alpha|\}$ or its successor cardinal.*
- *$\mathbb{P}_\lambda$ has cardinality λ, and there are λ nice $\mathbb{P}_\lambda$-names for reals.*
- *$\mathbb{P}_\lambda$ forces that the cardinality of the continuum is λ.*

Proof. The first assertion is proved by induction on α. It is clear that the set $\mathbb{P}_0$ of conditions for adding κ Cohen reals has cardinality κ. Limit stages of the induction are also clear, since there we have $|\mathbb{P}_\alpha| = \sup\{|\mathbb{P}_\beta| : \beta < \alpha\}$.

For odd α, we have

$$|\mathbb{P}_{\alpha+1}| = |\mathbb{P}_\alpha| \cdot \aleph_0 \cdot |\mathcal{F}_\alpha|^{<\aleph_0}.$$

Since μ is measurable and $|\mathcal{F}_\alpha| < \mu$, the last two factors on the right side are $< \mu < \kappa \leq |\mathbb{P}_\alpha|$. Thus, $|\mathbb{P}_{\alpha+1}| = |\mathbb{P}_\alpha|$, and the result is proved in this case.

Finally, for even α, we have, thanks to GCH and the fact that $|\mathbb{P}_\alpha| \geq \kappa > \mu$,

$$|\mathbb{P}_{\alpha+1}| = |{\mathbb{P}_\alpha}^\mu / D| \leq |\mathbb{P}_\alpha|^\mu \leq |\mathbb{P}_\alpha|^+.$$

The last inequality can be improved to $|\mathbb{P}_\alpha|^\mu = |\mathbb{P}_\alpha|$ provided $|\mathbb{P}_\alpha|$ has cofinality greater than μ, which includes the case that $|\mathbb{P}_\alpha|$ is regular. Since we certainly have $|\mathbb{P}_{\alpha+1}| \geq |\mathbb{P}_\alpha|$, this suffices to complete the proof of the first assertion of the lemma.

For the second assertion, we use the ccc to estimate the number of nice names. The number of maximal antichains in $\mathbb{P}_\alpha$ is $|\mathbb{P}_\alpha|^{\aleph_0}$ because the antichains are countable. Each antichain admits only $2^{\aleph_0}$ functions to ω. So the number of nice names, i.e., ω-sequences of pairs consisting of a maximal antichain and a map of it to ω, is

$$\left(|\mathbb{P}_\alpha|^{\aleph_0} \cdot 2^{\aleph_0}\right)^{\aleph_0} = |\mathbb{P}_\alpha|^{\aleph_0},$$

which is $|\mathbb{P}_\alpha|$ if this has uncountable cofinality and $|\mathbb{P}_\alpha|^+$ otherwise. This establishes the second assertion of the lemma.

The third assertion is a special case of the first two, since λ is regular.

Finally, the fourth assertion follows from the third and the fact that new reals are added at λ stages of the iteration — most obviously in the steps from odd to even α where we are adjoining a Hechler-like real. □

Because $\mathbb{P}_\lambda$ satisfies the ccc (and because λ does not have countable cofinality), every nice $\mathbb{P}_\lambda$-name for a real comes from an earlier $\mathbb{P}_\alpha$. Furthermore, since λ is regular and larger than μ, every set of fewer than μ nice $\mathbb{P}_\lambda$-names of reals comes from some earlier $\mathbb{P}_\alpha$. Thus we can, by routine bookkeeping, make sure that every such set occurs as $\mathcal{F}_\alpha$ at some stage of the forcing. More precisely, we arrange that every set of fewer than μ nice $\mathbb{P}_\lambda$-names of reals is $(e^\lambda_\alpha)_*(\mathcal{F}_\alpha)$ for some odd $\alpha < \lambda$. With this additional specification of our forcing construction, we immediately get the following (weak) lower bounds for some cardinal characteristics in the forcing extension.

Lemma 5.6. *$\mathbb{P}_\lambda$ forces that $\mathfrak{b} \geq \mu$. Therefore, it also forces all of $\mathfrak{a}, \mathfrak{a}_e, \mathfrak{a}_p$, and $\mathfrak{a}_g$ to be at least μ.*

Proof. If the $\mathbb{P}_\lambda$-forcing extension had $\mathfrak{b} < \mu$, then we would have a set $\mathcal{F}$ of fewer than μ nice $\mathbb{P}_\lambda$-names of reals and we would have a condition $p \in \mathbb{P}_\lambda$ forcing that the reals named by elements of $\mathcal{F}$ form an unbounded family. By the bookkeeping that preceded the lemma, we have an odd $\alpha < \lambda$ such that $\mathcal{F} = (e^\lambda_\alpha)_*(\mathcal{F}_\alpha)$. At the next step of the forcing, $\mathbb{P}_{\alpha+1}$ adjoins a generic real g that dominates $(e^{\alpha+1}_\alpha)_*(\dot{f})$ for all $\dot{f} \in \mathcal{F}_\alpha$; the proof of this is exactly like the proof that Hechler forcing adjoins a dominating real. But then g continues to dominate these functions from $\mathcal{F}_\alpha$ in the final forcing extension by $\mathbb{P}_\lambda$. (More formally: At stage $\alpha + 1$, we have a name $\dot{g}$ for a real that is forced, by all of $\mathbb{P}_{\alpha+1}$ to dominate $(e^{\alpha+1}_\alpha)_*(\dot{f})$ for all $\dot{f} \in \mathcal{F}_\alpha$. Then $(e^\lambda_{\alpha+1})_*(\dot{g})$ is forced, by all of $\mathbb{P}_\lambda$, to dominate $(e^\lambda_\alpha)_*(\dot{f})$ for all $\dot{f} \in \mathcal{F}_\alpha$, i.e., to dominate everything in $\mathcal{F}$.) This completes the proof of the first assertion of the lemma. The second follows, since $\mathfrak{a}$ and all the variants listed are $\geq \mathfrak{b}$; see [8] or our Theorems 3.1 and 3.2 for $\mathfrak{a}_p$ and $\mathfrak{a}_g$, and recall that $\mathfrak{b} \leq \mathfrak{a}_e$ is trivial. □

Lemma 5.7. *$\mathbb{P}_\lambda$ forces that $\mathfrak{a} = \mathfrak{a}_e = \mathfrak{a}_p = \mathfrak{a}_g = \lambda$.*

Proof. We give the proof for $\mathfrak{a}_g$; the other three cases are similar but easier. Suppose, toward a contradiction, that some condition $p_0 \in \mathbb{P}_\lambda$ forces that there is a maximal cofinitary group of cardinality $< \lambda$. Extending p_0 if necessary, we can assume that it decides a specific value θ for this cardinality and that it forces a certain θ-sequence of nice $\mathbb{P}_\lambda$-names $\langle \dot{g}_\xi \rangle_{\xi<\theta}$ to be a one-to-one enumeration of such a group. For notational convenience, we assume from now on that $p_0 = 1$, so that all of $\mathbb{P}_\lambda$ forces that $\langle g_\xi \rangle$ is a one-to-one enumeration of a maximal cofinitary group. Note that $\theta \geq \mu$ by Lemma 5.6.

Because λ is regular and greater than θ, there is an $\alpha < \lambda$ such that all the names $\dot{g}_\xi$ come from $\mathbb{P}_\alpha$. That is, there are nice $\mathbb{P}_\alpha$-names $\dot{h}_\xi$ such that $\dot{g}_\xi = (e^\lambda_\alpha)_*(\dot{h}_\xi)$ for all $\xi < \theta$. Increasing α if necessary, we assume in addition that α is even, so that the next stage of the forcing is given by the ultrapower with respect to D. We intend to apply the elementarity property of ultrapowers to a suitably chosen structure, which we now describe.

The structure $\mathcal{S}$ is three-sorted, the domains of the three sorts being θ, ω, and $\mathbb{P}_\alpha$. We use $p, q, \dots$ as variables over $\mathbb{P}_\alpha$; k, m, n, N as variables over ω; and ξ as a variable over θ. The relations in our three-sorted structure are the ordering relations of ω and $\mathbb{P}_\alpha$ and the four-place relation Φ that encodes the sequence of nice names $\langle \dot{h}_\xi \rangle$ as follows. Recall that a nice name $\dot{h}_\xi$ is an ω-sequence of pairs $A_{\xi,n}, h_{\xi,n}$ where $A_{\xi,n}$ is a maximal antichain in $\mathbb{P}_\alpha$ and $h_{\xi,n}$ is a function from $A_{\xi,n}$ into ω. Then $\Phi(\xi, n, p, m)$ is defined to hold in our structure if and only if $p \in A_{\xi,n}$ and $h_{\xi,n}(p) = m$.

In this structure $\mathcal{S}$, we can express all the following properties and facts by means of first-order formulas.

(1) $p \in A_{\xi,n}$, expressed by $\exists m\, \Phi(\xi, n, p, m)$.

(2) $p \Vdash \dot{h}_\xi(n) = m$, expressed by

$$(\forall q \leq p)(\exists r \leq q)\exists s\,(r \leq s \wedge \Phi(\xi, n, s, m)).$$

(3) $A_{\xi,n}$ is a maximal antichain and, for each of its elements p, there is a unique m such that $\Phi(\xi, n, p, m)$.

(4) The set of values of the $\dot{h}_\xi$'s is closed under composition, i.e., for each ξ_1, ξ_2 and p, there exist $q \leq p$ and ξ such that, whenever an $r \leq q$ forces $\dot{h}_{\xi_1}(n) = k$ and $\dot{h}_{\xi_2}(k) = m$ then r also forces $h_\xi(n) = m$.

(5) The set of values of the $\dot{h}_\xi$'s is closed under inversion.

(6) For each $\xi_1 \neq \xi_2$ and p, there exists N such that, for all $q \leq p$ and all $n \geq N$, there is no m such that q forces both $\dot{h}_{\xi_1}(n) = m$ and $\dot{h}_{\xi_2}(n) = m$.

Notice that the last four items in this list are true statements. Item (iii) says that the $\dot{h}_\xi$ are nice names; items (iv) and (v) say that their values form a group; and item (vi) says that this group is cofinitary.

Consider now the ultrapower $\mathcal{S}^\mu/D$ and the elementary embedding of $\mathcal{S}$ into it. Because D is countably complete, the sort ω is unchanged. On the other hand, because D is a non-principal ultrafilter on μ and because $\theta \geq \mu$, the sort θ acquires new elements. Finally, the sort $\mathbb{P}_\alpha$ becomes its ultrapower, $\mathbb{P}_{\alpha+1}$. Since items (iii) through (vi) are first-order statements true in $\mathcal{S}$, they remain true in $\mathcal{S}^\mu/D$. Interpreting these statements in the ultrpower, we find that $\mathbb{P}_{\alpha+1}$ forces the values of the $\dot{h}_\xi$ for $\xi \in \theta^\mu/D$ to form a cofinitary group. Because θ has new elements in the ultrapower, these values properly include the values of $(e^{\alpha+1}_\alpha)_*(\dot{h}_\xi)$ for $\xi \in \theta$. Applying $e^\lambda_{\alpha+1}$, we obtain a cofinitary group in the $\mathbb{P}_\lambda$ forcing extension properly containing the group of values of the $\dot{g}_\xi$. This contradicts the maximality of the latter group and so completes the proof of the lemma. □

To finish the proof of Theorem 5.1, it remains to show that $\mathbb{P}_\lambda$ forces $\mathfrak{nm} < \lambda$. (Here we use Bartoszyński's theorem in [2] that $\mathfrak{non}(\mathcal{M}) = \mathfrak{nm}$.) The idea of the proof is to use the κ Cohen reals adjoined by $\mathbb{P}_0$ to witness that $\mathfrak{nm} \le \kappa$. Roughly speaking, we shall show that every real x in the $\mathbb{P}_\lambda$ extension "involves" fewer than κ of these Cohen reals and that therefore the rest of the Cohen reals remain Cohen over a model containing x. To make this precise, we must specify the notion "involved", we must define the relevant model containing a given x, and we must verify all the necessary properties. Because of the unusual nature of our forcing iteration, we shall give more details than would otherwise be needed, but we still leave some routine verifications to the reader.

We begin by defining the relevant submodels. Temporarily fix some ordinal $\xi < \kappa$. We define a forcing iteration $\langle \mathbb{Q}_\alpha \rangle_{\alpha<\lambda}$ similar to the $\mathbb{P}_\alpha$ sequence defined above, but using only the first ξ of the κ Cohen reals. Along with the forcing notions $\mathbb{Q}_\alpha$, we shall define superb embeddings $i_\alpha : \mathbb{Q}_\alpha \to \mathbb{P}_\alpha$ with truncation functions $t_\alpha : \mathbb{P}_\alpha \to \mathbb{Q}_\alpha$. These embeddings and truncation functions will commute with the embeddings e^β_α between the $\mathbb{P}$'s and the corresponding embeddings between the $\mathbb{Q}$'s.

At stage 0, we define $\mathbb{Q}_0$ to be the forcing that adds ξ Cohen reals, i.e., the subset of $\mathbb{P}_0$ consisting of those conditions p that satisfy $p(\eta) = \varnothing$ for all $\eta \ge \xi$. The embedding i_0 is the inclusion map, and the associated truncation sends each $p \in \mathbb{P}_0$ to the function that agrees with p below ξ and is constant with value $\varnothing$ above ξ.

For the induction step, it is useful to observe that the requirements defining the notions of superb embedding and associated truncation function are first-order and in fact $\forall\exists$ statements about the structure consisting of the two posets, with their order and incompatibility relations, and the two functions. Because of "first-order", our induction step from an even α to $\alpha+1$, the ultrapower step, is trivial. We define $\mathbb{Q}_{\alpha+1}$ to be ${\mathbb{Q}_\alpha}^\mu/D$ and we obtain $i_{\alpha+1}$ and $t_{\alpha+1}$ from i_α and t_α by the ultrapower operation. It is trivial to check that these commute with the embeddings of the two iterations, where of course the embedding $\mathbb{Q}_\alpha \to \mathbb{Q}_{\alpha+1}$ is the canonical embedding into the ultrapower.

Furthermore, because of "$\forall\exists$", limit stages β of our iteration are also trivial. We define $\mathbb{Q}_\beta$ to be the direct limit of the earlier $\mathbb{Q}_\alpha$'s, and we define i_β and t_β as the unique functions that commute with the embeddings of the iterations and the earlier i_α's and t_α's, respectively. They are a superb embedding and truncation because $\forall\exists$ statements are preserved by direct limits of structures.

It remains to define the construction at the step from an odd α to $\alpha+1$, where $\mathbb{P}_{\alpha+1}$ was obtained by a (dense subset of a) Hechler-like iteration with respect to a family $\mathcal{F}_\alpha$ of nice $\mathbb{P}_\alpha$-names of reals. In this situation, we distinguish two cases.

Suppose first that all the $\mathbb{P}_\alpha$-names in $\mathcal{F}_\alpha$ come from $\mathbb{Q}_\alpha$-names via the embedding i_α. That is, there is a set $\mathcal{F}'_\alpha$ of nice $\mathbb{Q}_\alpha$ names such that

$$\mathcal{F}_\alpha = \{(i_\alpha)_*(\dot{f}) : \dot{f} \in \mathcal{F}'_\alpha\}.$$

In this case, we define $\mathbb{Q}_{\alpha+1}$ in exact analogy to $\mathbb{P}_{\alpha+1}$ but using the family $\mathcal{F}'_\alpha$ of nice names. Thus, a condition in $\mathbb{Q}_{\alpha+1}$ has the form (q, s, A) where $q \in \mathbb{Q}_\alpha$, $s \in {}^{<\omega}\omega$, and A is a finite subset of $\mathcal{F}'_\alpha$. The ordering is defined just as in $\mathbb{P}_{\alpha+1}$.

The embedding $i_{\alpha+1} : \mathbb{Q}_{\alpha+1} \to \mathbb{P}_{\alpha+1}$ is given by

$$i_{\alpha+1}(q, s, A) = (i_\alpha(q), s, (i_\alpha)_*(A)),$$

and the truncation is given by

$$t_{\alpha+1}(p, s, A) = (t_\alpha(p), s, {(i_\alpha)_*}^{-1}(A)).$$

There remains the case that not all the names in $\mathcal{F}_\alpha$ come from $\mathbb{Q}_\alpha$. In this case, we do nothing in the $\mathbb{Q}$ iteration. That is, we define $\mathbb{Q}_{\alpha+1}$ to be $\mathbb{Q}_\alpha$ (with the identity map as the embedding). The embedding and truncation maps between $\mathbb{Q}_{\alpha+1} = \mathbb{Q}_\alpha$ and $\mathbb{P}_{\alpha+1}$ are obtained by composing the maps from stage α, between $\mathbb{Q}_\alpha$ and $\mathbb{P}_\alpha$, with the superb embedding and truncation between $\mathbb{P}_\alpha$ and $\mathbb{P}_{\alpha+1}$. (Recall that, although the complete embeddings in the $\mathbb{P}$ iteration were not all superb, those between $\mathbb{P}_\alpha$ and $\mathbb{P}_{\alpha+1}$ for odd α were superb, so the present definition makes sense.)

This completes the definition of the $\mathbb{Q}_\alpha$'s for all $\alpha \leq \lambda$, together with their superb embeddings into the $\mathbb{P}_\alpha$'s and the associated truncation functions. We leave to the reader the verification of the relevant definitions. The construction was done with a fixed ξ; only in the following lemma will we have to consider the possibility of varying ξ.

Lemma 5.8.

- *For every condition $p \in \mathbb{P}_\alpha$, for all sufficiently large $\xi < \kappa$, p is in the image of i_α.*
- *For every nice $\mathbb{P}_\alpha$-name $\dot{f}$, for all sufficiently large $\xi < \kappa$, there exists a nice $\mathbb{Q}_\alpha$-name $\dot{g}$ such that $\dot{f} = (i_\alpha)_*(\dot{g})$.*

Proof. We prove both assertions simultaneously by induction on α. Observe that the second assertion follows (for any particular α) from the first, since a nice name involves only countably many antichains, each consisting of only countably many conditions, and κ is regular and uncountable. So it suffices to prove the first assertion for an arbitrary α, assuming both assertions for all smaller α.

For $\alpha = 0$, it suffices to take ξ larger than all of the finitely many ordinals where p takes a value other than $\varnothing$.

For limit α, we have that p comes from an earlier $\mathbb{P}_\beta$. The ξ's that worked for p in $\mathbb{P}_\beta$ continue to work at level α.

If the lemma is true at an even α, then to see that it remains true at $\alpha + 1$, consider an arbitrary $p \in \mathbb{P}_{\alpha+1} = {\mathbb{P}_\alpha}^\mu / D$, and represent it by a μ-sequence of elements of $\mathbb{P}_\alpha$, say $p = [\langle p_i \rangle_{i<\mu}]_D$. Apply the induction hypothesis to the p_i's and use the fact that κ is regular and $> \mu$ to conclude that, for any sufficiently large ξ, there are $q_i \in \mathbb{Q}_\alpha$ with $i_\alpha(q_i) = p_i$ for all $i < \mu$. Then $[\langle q_i \rangle_{i<\mu}]_D$ is a member of $\mathbb{Q}_{\alpha+1}$ mapped by $i_{\alpha+1}$ to p.

Finally, suppose the lemma is true at an odd stage α and consider an arbitrary condition $(p, s, A) \in \mathbb{P}_{\alpha+1}$. Once ξ is large enough, we know, by induction hypothesis, that p is in the image of i_α. Furthermore, each of the names in $\mathcal{F}_\alpha$ comes from $\mathbb{Q}_\alpha$ for sufficiently large ξ. Since the number of these names is $< \mu < \kappa$, all sufficiently large ξ will have all of $\mathcal{F}_\alpha$ coming from $\mathbb{Q}_\alpha$. For these ξ, $((i_\alpha)^{-1}(p), s, {(i_\alpha)_*}^{-1}(A))$ is a condition in $\mathbb{Q}_{\alpha+1}$ mapped to (p, s, A) by $i_{\alpha+1}$. □

Recall that our plan is to prove that $\mathfrak{nm} < \lambda$ in the $\mathbb{P}_\lambda$ forcing extension by showing that the κ Cohen reals added by $\mathbb{P}_0$ witness that $\mathfrak{nm} \leq \kappa$. So we must take an arbitrary nice $\mathbb{P}_\lambda$-name $\dot{x}$ for a real and find a condition (extending an arbitrarily given condition) in $\mathbb{P}_\lambda$ forcing that a certain one of the original Cohen reals agrees with $\dot{x}$ at infinitely many points in ω.

Let an arbitrary nice $\mathbb{P}_\lambda$-name $\dot{x}$ for a real be given, and apply Lemma 5.8 to obtain a $\xi < \kappa$ such that $\dot{x}$ comes from $\mathbb{Q}_\lambda$ via i_λ, say $\dot{x} = (i_\lambda)_*(\dot{y})$, where $\dot{y}$ is a nice $\mathbb{Q}_\lambda$-name. We fix $\dot{x}, \dot{y}$, and ξ for the rest of the proof. The remainder of the proof will proceed as follows. We shall define, for each $p \in \mathbb{P}_\lambda$, a finite sequence $s_p \in {}^{<\omega}\omega$, called p's *opinion* about the ξ^{th} Cohen real $\dot{c}$. We shall also define, for any condition $p \in \mathbb{P}_\lambda$ and any $s \in {}^{<\omega}\omega$ that extends s_p, a condition called $p + s$ with the following properties.

(1) $p + s \leq p$.
(2) $p + s$ forces that s is an initial segment of $e_0^\lambda(\dot{c})$, the ξ^{th} Cohen real.
(3) The truncations of $p + s$ and p agree, i.e., $t_\lambda(p + s) = t_\lambda(p)$ in $\mathbb{Q}_\lambda$.

Here $\dot{c}$ denotes the canonical nice $\mathbb{P}_0$-name for the ξ^{th} Cohen real, so $e_0^\lambda(\dot{c})$ serves as a standard $\mathbb{P}_\lambda$-name for the same real. Once these definitions and properties are in place, we can complete the proof as follows.

Let an arbitrary condition $p \in \mathbb{P}_\lambda$ and an arbitrary $N \in \omega$ be given; we must find an extension of p that forces $\dot{x}$ and $e_0^\lambda(\dot{c})$ to agree at some $n > N$. Fix an n greater than both N and the length of p's opinion s_p about $\dot{c}$. Take the truncation $t_\lambda(p) \in \mathbb{Q}_\lambda$, and extend it to a condition q that decides a specific value $m \in \omega$ for $\dot{y}(n)$. Then extend s_p to some $s \in {}^{<\omega}\omega$ that has n in its domain with $s(n) = m$. By property (3) of $p+s$ and by our choice of q, we have $q \leq t_\lambda(p) = t_\lambda(p+s)$. Because t_λ is the truncation function associated to i_λ, it follows that $i_\lambda(q)$ is compatible with $p + s$. Let r be a common extension. Then, thanks to properties (1) and (2) of $p + s$, we find that r extends p and forces $e_0^\lambda(\dot{c})(n) = m$. Furthermore, since q forces $\dot{y}(n) = m$, since $\dot{x} = i_\lambda(\dot{y})$, and since r extends $i_\lambda(q)$, we also have that r forces $\dot{x}(n) = m$. Thus, r forces an agreement between $\dot{x}$ and $e_0^\lambda(\dot{c})$ beyond N, as required.

It remains to define opinions s_p and the operation $p + s$ and to verify the properties (1)–(3) used in the preceding argument. Although these refer to the final forcings $\mathbb{P}_\lambda$ and $\mathbb{Q}_\lambda$, we shall give the definitions and verify the properties for all $\alpha \leq \lambda$, by induction on α. To make the induction work, we shall also arrange that the definitions commute with the embeddings $e_\alpha^\beta : \mathbb{P}_\alpha \to \mathbb{P}_\beta$.

For $\alpha = 0$, a condition $p \in \mathbb{P}_0$ is a function from κ to ${}^{<\omega}\omega$, and we define its opinion s_p to be $p(\xi)$. (Recall that ξ is a fixed ordinal $< \kappa$.) If s is any extension of s_p, then $p + s$ is defined by changing the value of p at ξ to s, i.e.,

$$(p + s)(\xi) = s \quad \text{and} \quad (p + s)(\eta) = p(\eta) \text{ for } \eta \neq \xi.$$

The verification of the three required properties is trivial.

Suppose next that β is a limit ordinal and we already have the required definitions and properties for all smaller ordinals α. Then any $p \in \mathbb{P}_\beta$ is of the form $e_\alpha^\beta(p')$ for some $\alpha < \beta$ and some $p' \in \mathbb{P}_\alpha$. We define s_p to be $s_{p'}$ and we define $p + s$ to be $e_\alpha^\beta(p' + s)$. The assumption that our definitions for smaller ordinals commute with the embeddings implies that this definition at the limit stage β is unambiguous and also commutes with the embeddings. Properties (1) through (3) are again immediate consequences of the definitions and elementary properties of forcing. For example, in (2), we have by induction hypothesis that

$$p' + s \Vdash_{\mathbb{P}_\alpha} s \subseteq e_0^\alpha(\dot{c}),$$

and it follows that

$$e_\alpha^\beta(p' + s) \Vdash_{\mathbb{P}_\beta} s \subseteq e_\alpha^\beta(e_0^\alpha(\dot{c})),$$

which is exactly the statement

$$p + s \Vdash_{\mathbb{P}_\beta} s \subseteq e_0^\beta(\dot{c})$$

required by (2) at stage β.

Suppose next that $\beta = \alpha + 1$ with α even, so we are at an ultrapower stage of the construction. Given any $p \in \mathbb{P}_\beta = {\mathbb{P}_\alpha}^\mu/D$, express it as the equivalence class modulo D of a μ-sequence, $p = [\langle p_i \rangle_{i<\mu}]_D$. Because the opinions s_{p_i} lie in the countable set ${}^{<\omega}\omega$ and because D is countably complete, a single sequence is s_{p_i} for D-almost all i. Define s_p to be that sequence. So the set $Z = \{i < \mu : s_{p_i} = s_p\}$ is in D. Now if $s \in {}^{<\omega}\omega$ is an extension of s_p, then $p_i + s$ is defined for all $i \in Z$. Define $p + s$ to be the equivalence class $[\langle p'_i \rangle_{i<\mu}]$ where $p'_i = p_i + s$ for $i \in Z$ and p'_i is arbitrary for $i \notin Z$. (The arbitrary choices have no effect on the equivalence class $p + s$, since $Z \in D$.) As before, the required properties (1) through (3) are immediate consequences of the definitions and the induction hypotheses. So is the commutativity with $e_\alpha^{\alpha+1}$, and commutativity with embeddings from earlier $\mathbb{P}_\gamma$'s follows via the induction hypothesis.

Finally, suppose $\beta = \alpha + 1$ with α odd. A condition in $\mathbb{P}_\beta$ has the form (p, r, A) with $p \in \mathbb{P}_\alpha$, $r \in {}^{<\omega}\omega$, and A a finite subset of $\mathcal{F}_\alpha$. Define $s_{(p,r,A)} = s_p$. For an extension s of s_p, define $(p, r, A)+s = (p+s, r, A)$. Once again, properties (1) through (3) and commutativity with the embeddings follow directly from the definitions and the induction hypotheses.

This concludes the definitions of s_p and $p+s$ for all $\mathbb{P}_\alpha$, in particular for $\mathbb{P}_\lambda$, and so it concludes the proof of Theorem 5.1. □

We close by indicating the aspect of this proof that causes trouble if we try to use it to get $\mathfrak{b}(\omega_1)$ or even $\mathfrak{nm}(\omega_1)$ to be smaller than the various almost-disjointness numbers for ω_1.

For each odd α, let $\dot{h}_\alpha$ be the standard nice $\mathbb{P}_{\alpha+1}$-name for the Hechler-like real adjoined by $\mathbb{P}_{\alpha+1}$ to dominate all the (reals denoted by) members of $\mathcal{F}_\alpha$. Because of our choice of the families $\mathcal{F}_\alpha$ as covering all families of fewer than μ names of the final model, there will be a set Z of odd ordinals $< \lambda$, such that the order type of Z is μ and, whenever $\alpha < \beta$ are in Z, then

$$\mathbb{P}_{\beta+1} \Vdash e_{\alpha+1}^{\beta+1}(\dot{h}_\alpha) <^* \dot{h}_\beta.$$

Consider any even γ in the range $\sup Z < \gamma < \lambda$ (which is nonempty as $|Z| = \mu < \lambda$ and λ is regular). Then for all $\alpha < \beta$ in Z,

$$\mathbb{P}_\gamma \Vdash e_{\alpha+1}^\gamma(\dot{h}_\alpha) <^* e_{\beta+1}^\gamma(\dot{h}_\beta).$$

Of course this persists to the next stage, i.e., we can replace γ with $\gamma+1$ throughout the preceding formula. But, in addition, the ultrapower operation that takes $\mathbb{P}_\gamma$ to $\mathbb{P}_{\gamma+1}$ produces what might be called $\dot{h}_x$ for a "new element x of Z". More precisely, if we let $z : \mu \to Z$ be the enumeration of Z in increasing order and if we write the nice names $e_{z(i)+1}^\gamma(\dot{h}_{z(i)})$ (for $i < \mu$) as $\langle A_n^i, h_n^i \rangle_{n\in\omega}$, then this $\dot{h}_x$ means the nice $\mathbb{P}_{\gamma+1}$-name $\langle A_n, h_n \rangle_{n\in\omega}$ where A_n consists of those equivalence classes $[\langle p_i : i < \mu \rangle]_D$ for which $p_i \in A_n^i$ for D-almost all i, and where $h_n([\langle p_i : i < \mu \rangle]_D) = h_n^i(p_i)$ for D-almost all n. By Łoś's theorem and the fact that D is μ-complete and non-principal, it follows that

$$\mathbb{P}_{\gamma+1} \Vdash e_{\alpha+1}^{\gamma+1}(\dot{h}_\alpha) <^* \dot{h}_x$$

for all $\alpha \in Z$.

Repeating the argument with a larger, even ordinal δ in place of γ, we get a nice $\mathbb{P}_{\delta+1}$-name, say $\dot{h}_y$, with

$$\mathbb{P}_{\delta+1} \Vdash e^{\delta+1}_{\alpha+1}(\dot{h}_\alpha) <^* \dot{h}_y$$

for all $\alpha \in Z$ and with the additional property, also obtained by means of Łoś's theorem, that

$$\mathbb{P}_{\delta+1} \Vdash \dot{h}_y <^* e^{\delta+1}_{\gamma+1}(\dot{h}_x).$$

Repeating the same argument infinitely often, with larger and larger even ordinals in place of γ and δ, we obtain, in the final model of our iteration, an infinite sequence

$$e^{\lambda}_{\gamma+1}(\dot{h}_x) >^* e^{\lambda}_{\delta+1}(\dot{h}_y) >^* \dots$$

forced to be decreasing in the mod-finite ordering $>^*$.

This is no problem in our proof, but it is a serious problem in an attempt to apply the proof to, say, ω_1 in place of ω. There cannot be an infinite sequence in ${}^{\omega_1}\omega_1$ strictly decreasing mod countable. So our forcing, by producing such a sequence, must collapse ω_1, and with ω_1 the proof would also collapse.

References

[1] S. A. Adeleke, "Embeddings of infinite permutation groups in sharp, highly transitive, and homogeneous groups," *Proc. of Edinburgh Math. Soc. (2)*, 31 (1988) 169–178.
[2] T. Bartoszyński, "Combinatorial aspects of measure and category," *Fund. Math.* 127 (1987) 225–239.
[3] T. Bartoszyński and H. Judah, *Set Theory: On the Structure of the Real Line*, A K Peters, (1995).
[4] A. Blass, "Combinatorial cardinal characteristics of the continuum," in *Handbook of Set Theory* (M. Foreman, A. Kanamori, and M. Magidor, eds.), to appear.
[5] J. Brendle, "Mad families and iteration theory," in *Logic and Algebra* (Y. Zhang, ed.), Contemp. Math., 302 (2002) 1–31.
[6] J. Brendle, "Mob families and mad families," *Arch. Math. Logic*, 37 (1998) 183–197.
[7] J.Brendle, "The almost disjointness number may have countable cofinality," *Trans. Amer. Math. Soc.*, 355 (2003) 2633–2649.
[8] J. Brendle, O. Spinas and Y. Zhang, "Uniformity of the meager ideal and maximal cofinitary groups," *J. Algebra*, 232 (2000) 209–225.
[9] P. J. Cameron, "Cofinitary permutation groups," *Bull. London Math. Soc.*, 28 (1996) 113–140.
[10] E. van Douwen, "The integers and topology," in *Handbook of Set-Theoretic Topology* (K. Kunen and J. Vaughan, eds.), North-Holland, (1984) 111–167.
[11] F. Galvin: "Almost disjoint permutation groups," *Proc. Amer. Math. Soc.* 124 (1996) 1723–1725.
[12] T. Hyttinen, "Cardinal invariants of eventually different functions," *Bull. London Math. Soc.*, to appear.
[13] T. Jech, *Set Theory*, Academic Press (1978).
[14] K. Kunen, *Set Theory–An Introduction to Independence Proofs*, North Holland, (1980).
[15] A. Miller, "Some properties of measure and category," *Trans. Amer. Math. Soc.* 266 (1981) 93–114.
[16] J. Roitman and L. Soukup, "Luzin and anti-Luzin almost disjoint families," *Fund. Math.* 158 (1998) 51–67.
[17] S. Shelah, "On cardinal invariants of the continuum," in *Axiomatic Set Theory* (J. Baumgartner, D. Martin, and S. Shelah, eds.), Contemp. Math. 31 (1984) 183-207.
[18] S. Shelah, *Cardinal Arithmetic*, Oxford University Press (1994).
[19] S. Shelah, "Two cardinal invariants of the continuum ($\mathfrak{d} < \mathfrak{a}$) and FS linearly ordered iterated forcing" (also known under the title "Are $\mathfrak{a}$ and $\mathfrak{d}$ your cup of tea?") *Acta Math.* 192 (2004) 187–223.

[20] W. Sierpiński, "Sur une décomposition d'ensembles," *Monatshefte Math. Phys.* 35 (1928) 239–242.
[21] R. C. Solomon, "Families of sets and functions," *Czechoslovak Math. J.*, 27 (1977) 556–559.
[22] M. Talagrand, "Compacts de fonctions mesurables et filtres non mesurables," *Studia Math.* 67 (1980) 13–43.
[23] A. Tarski, "Sur la décomposition des ensembles en sous-ensembles presque disjoints," *Fund. Math.* 12 (1928) 188–205.
[24] J. K. Truss, "Embeddings of infinite permutation groups," in *Proceedings of Groups–St Andrews 1985* (E. F. Robertson and C. M. Cambell eds.), London Math. Soc. Lecture Note Ser. 121 (1986) 335-351.
[25] Y. Zhang, "On a class of mad families," *J. Symbolic Logic*, 64 (1999) 1259–1276.
[26] Y. Zhang, "Permutation groups and covering properties," *J. London Math. Soc. (2)*, 63 (2001) 1–15.
[27] Y. Zhang, "Maximal cofinitary groups," *Arch. Math. Logic*, 39 (2000) 41–52.

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