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# A MODEL-THEORETIC VIEW OF SOME SPECIAL ULTRAFILTERS

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Combinatorial definitions of special sorts of ultrafilters can often be formulated rather neatly in terms of the associated ultrapowers, and some of the combinatorial lemmas used in the study of such ultrafilters have simpler and more natural formulations as assertions about ultrapowers. The purpose of this paper is to survey a number of such formulations and to indicate how they can be used to prove combinatorial results.

In Section 1, we review the results of Puritz [15] which characterize the most popular special ultrafilters ($\delta$-stable, selective, etc.) in terms of ultrapowers. We also review the results of [5] and give a fairly typical example of a model-theoretic presentation of a combinatorial argument. In Section 2, we use the concept of conservative extensions to generalize a theorem of Puritz [16]; this generalization is used to give simplified proofs of a result in [5] and a related unpublished result of Galvin. In Section 3, we treat Daguenet's rangé and affable ultrafilters [9] and Baumgartner and Taylor's arrow ultrafilters [1] in terms of amalgamation of ultrapowers. Finally, in Section 4, we consider situations where an amalgamation of two models is nearly unique; these are related to square-bracket partition relations.

## 1. Skies, Constellations, and Amalgamations

All ultrafilters in this paper are assumed to be on the set $\omega$ of natural numbers. Using a pairing function, we identify $\omega \times \omega$ with $\omega$, and we write the projections from $\omega \times \omega$ to $\omega$ as $p_1, p_2 \colon \omega \to \omega$. All models in this paper are assumed to be elementary extensions of the structure N that consists of $\omega$ with all relations and functions. Thus, all submodels are elementary submodels. We often use the same notation for a model and its universe. Also, we use the same notation for a function on $\omega$, the symbol denoting it in the language of N, and the interpretation of this symbol in any model A; such interpretations are called the standard functions of A. A similar convention applies to predicates.

Recall (from [15], for example) that a model A is generated by a single element a if and only if it is isomorphic to an ultrapower U-prod N of N. Here

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$U$ can be taken to be the type of $a$,

$$\text{Type }(a) = \{X \subseteq \omega \mid A \models X(a)\},$$

and the isomorphism sends $f(a)$ to the equivalence class $[f]$ of $f$ in the ultrapower. The type of a pair of elements is defined analogously; note that $p_1$ and $p_2$ map Type $(a,b)$ to Type $(a)$ and Type $(b)$ respectively. Similar remarks apply to types of more than two elements.

Recall also from [15] that two nonstandard elements of a model $A$ are said to be in the same constellation (resp. sky) if each is equal to (resp. less than or equal to) a standard function of the other. We write $\mathrm{Con}(a)$ and $\mathrm{Sk}(a)$ for the constellation and sky of $a$. Skies are convex in the natural ordering of $A$, so the set $\mathrm{Sk}(A)$ of skies inherits a linear ordering from $A$. Constellations are partially ordered by the convention that $\mathrm{Con}(f(a)) \leq \mathrm{Con}(a)$ for all standard unary functions $f$. In an ultrapower of $N$, the generators constitute the highest constellation, and the sky containing them is the highest sky.

In the following theorem, we list the model-theoretic characterizations, as well as the usual combinatorial definitions, of the best-known special sorts of ultrafilters. We omit the proof, since the first three parts were proved and the fourth announced in [15]; the fifth part is quite similar to the fourth.

*Theorem 1* (Puritz). *Let* $U$ *be an ultrafilter on* $\omega$.

(a) $U$ *is selective* (*also called minimal, Ramsey, absolute*) $\Leftrightarrow$ (def)
*Every function on* $\omega$ *is one-to-one or constant on some set in* $U$ $\Leftrightarrow$
$U$-prod $N$ *has only one constellation.*

(b) $U$ *is* $\delta$-*stable* (*also called* P-*point*) $\Leftrightarrow$ (def)
*Every function on* $\omega$ *is finite-to-one or constant on some set in* $U$ $\Leftrightarrow$
$U$-prod $N$ *has only one sky.*

(c) $U$ *is rare* (*also called* Q-*point*) $\Leftrightarrow$ (def)
*Every finite-to-one function on* $\omega$ *is one-to-one on some set in* $U$ $\Leftrightarrow$
*The highest sky of* $U$-prod $N$ *is a single constellation.*

(d) $U$ *is rapid* $\Leftrightarrow$(def)
*For each* $f : \omega \to \omega$ *there is a set in* $U$ *whose* $n^{th}$ *element is* $\geq f(n)$ *for all* $n$ $\Leftrightarrow$
*The top constellation of* $U$-prod $N$ *is coinitial in its sky.*

(e) $U$ *is semi-selective* [12] $\Leftrightarrow$(def)
*For any sequence of sets* $X_n \in U$ *there are elements* $x_n \in X_n$ *with* $\{x_n \mid n \in \omega\} \in U$ $\Leftrightarrow$
*The top constellation is coinitial in the nonstandard part of* $U$-prod $N$. □

Notice that these characterizations make it obvious that rapidity is invariant under permutations of $\omega$ and that semi-selective ultrafilters are the same as rapid P-points. However, both these facts also have easy combinatorial proofs.

We next recall some facts about amalgamation of models; a general reference for this material is [5]. If two models $A$ and $B$ are embedded in a third, say by maps $\alpha:A\to C$, $\beta:B\to C$ (which are automatically elementary embeddings), then the way their images intersect in $C$ can be described by specifying the submodels $A' = \alpha^{-1}(\beta(B))$ and $B' = \beta^{-1}(\alpha(A))$ (of $A$ and $B$) and the isomorphism $\theta = \beta^{-1}\alpha:A'\xrightarrow{\sim}B'$. We say that $\alpha$ and $\beta$ embed $A$ and $B$ in $C$ with intersection $\theta$. If $A = B$, then the submodel

$$A'' = \{a \in A \mid \theta(a) = a\} = \{a \in A \mid \alpha(a) = \beta(a)\}$$

is called the straight intersection of the two copies of $A$ in $C$. If $A'' = A'$, we say that $\alpha$ and $\beta$ embed $A$ into $C$ with purely straight intersection.

An amalgamation of $A$ and $B$, that is, an ordered pair of embeddings $\alpha:A\to C$, $\beta:B\to C$ whose images generate $C$, is considered equivalent to another amalgamation $\alpha':A\to C'$, $\beta':B\to C'$ if there is an isomorphism $\gamma:C\xrightarrow{\sim}C'$ with $\gamma\alpha = \alpha'$ and $\gamma\beta = \beta'$. Equivalent amalgamations have the same intersection, but the converse fails in general. If $A$ and $B$ are generated by $a$ and $b$ respectively, then an amalgamation is determined, up to equivalence, by the type of $(\alpha(a), \beta(b))$, which can be any ultrafilter extending $\mathrm{Type}(a) \times \mathrm{Type}(b)$.

The following theorem summarizes the information about amalgamations that we shall need later. Note that an amalgamation induces order-preserving embeddings of $Sk(A)$ and $Sk(B)$ into $Sk(C)$; we use $\alpha$ and $\beta$ to denote these embeddings as well as the embeddings of the models.

Theorem 2. *Let $A$ and $B$ be elementary extensions of $N$.*

(a) (Theorem 1 of [3] or Theorem 3.10 of [11]) *If $\alpha:A\to C$ and $\beta:B\to C$ are embeddings and if $a \in A$, $b \in B$ with $Sk(\alpha(a)) = Sk(\beta(b))$, then there exist $a' \in Sk(a)$, $b' \in Sk(b)$ with $\alpha(a') = \beta(b')$.*

(b) (Theorem 3 of [5], see also Prop. 2.7 of [11] and Props. 6 and 10 of [8]) *Let $A'$ and $B'$ be elementary submodels of $A$ and $B$, and let $\theta:A'\xrightarrow{\sim}B'$. Let $Sk^*(A) = \{Sk(a) \mid a \in A, (\forall x \in A')x < a\}$ and similarly for $B$. Let $\leq^*$ be a linear ordering of the disjoint union of $Sk^*(A)$ and $Sk^*(B)$ that agrees with the natural orderings of $Sk(A)$ and $Sk(B)$. Then there is an amalgamation, $\alpha:A\to C$, $\beta:B\to C$, with intersection $\theta$, such that $\leq^*$ agrees with the ordering induced via $\alpha$ and $\beta$ by the natural ordering of $Sk(C)$.*

(c) (Theorem 3.4 of [16]) *If, in part (b), $A' = B' = N$ and if $\leq^*$ has all of $Sk(A)$ before all of $Sk(B)$, then the amalgamation in (b) is unique up to equivalence.* □

As an example of how such results can be applied, let us consider the situation where the number of inequivalent amalgamations of two nonstandard models A and B is as small as possible. In Theorem 2b, we can take A' and B' to be N , and we can have $\leq^*$ put the skies of A either entirely before or entirely after those of B . Thus, there are at least two inequivalent amalgamations. To avoid having more than two, we must require that (1) no nonstandard submodels of A and B are isomorphic (lest there be more choices for $\theta$) and (2) A and B have only one sky each (lest there be more choices for $\leq^*$). But these two conditions are sufficient, for when two such models are amalgamated, the intersection can only be N by (1), and the unique sky of A must precede or follow the unique sky of B by Theorem 2a, so there are only two amalgamations by Theorem 2c. Applying this result to ultrapowers, U-prod N and V-prod N , we find the following result of Daguenet [8]. The filter $U \times V$ (generated by sets $X \times Y$ with $X \in U$ , $Y \in V$) has only two extensions to ultrafilters if and only if (1) no nonprincipal ultrafilter is below both U and V in the Rudin-Keisler ordering and (2) U and V are $\delta$-stable. It should be pointed out that this proof of Daguenet's theorem is essentially the same as the proof in [8]; in particular, Theorem 2b is a generalized form of the model-theoretic translation of Propositions 6 and 10 of [8]. However, the model-theoretic form seems easier to apply in a variety of situations.

## 2. Conservative Extensions

Let A be an elementary submodel of B . Following Phillips [14], we call B a conservative extension of A if the intersection of A with any parametrically definable subset of B is parametrically definable in A . (In the terminology of Gaifman [11], this means that every element of B realizes a parametrically definable type over A .) It is well known that every conservative extension is an end extension (see [14], [11], or [4]) but the converse is not provable [4]. The following theorem shows why conservative extensions are relevant to amalgamation questions. Since any model is a conservative extension of N , it generalizes Theorem 2c.

Theorem 3. *Let* B *be an end extension of* A . *Then* B *is a conservative extension of* A *if and only if, for every extension* C *of* A , *there is only one amalgamation (up to equivalence) of* B *and* C *with intersection* $\mathrm{id}: A \xrightarrow{\sim} A$ *and with all elements of* B - A *embedded above all elements of* C .

Note that the existence of such an amalgamation of B and C is guaranteed by Theorem 2b; it is the uniqueness that is important here. We prove only the "only if" part of the theorem, since we shall not need the converse. The full proof will appear in [6].

Proof: Since the theorem is trivial if $B = A$, we assume that $B$ is a proper conservative extension of $A$ and that $\beta: B \to D$, $\gamma: C \to D$ is an amalgamation of the sort described in the theorem. For each formula $\phi$, each $b \in B$, and each $c \in C$, we shall show that the truth value of $\phi(\beta(b), \gamma(c))$ in $D$ depends only on $\phi$, $b$, $c$, $B$, and $C$, not on $\beta$, $\gamma$, or $D$. Once this is shown, the desired uniqueness follows easily, for if $\beta': B \to D'$, $\gamma': C \to D'$ is another such amalgamation, then the isomorphism, defined by sending $f(\beta(b), \gamma(c)) \in D$ to $f(\beta'(b), \gamma'(c)) \in D'$ for all standard $f$, all $b \in B$, and all $c \in C$, shows that the amalgamations are equivalent.

Let $\phi$, $b$, $c$ be given. As $B$ is a conservative extension of $A$, we have a formula $\psi$ and an element $a \in A$ such that the equivalence $\phi(b, x) \longleftrightarrow \psi(a, x)$ holds in $B$ for all $x \in A$. Let $q$ be the first $x \in B$ for which this equivalence fails (or any element of $B - A$ if it never fails), so $q \in B - A$. Since

$$B \models (\forall x < q)(\phi(b, x) \longleftrightarrow \psi(a, x)),$$

$$D \models (\forall x < \beta(q))(\phi(\beta(b), x) \longleftrightarrow \psi(\beta(a), x)).$$

Remembering that $\gamma(c) < \beta(q)$ and that $\beta$ agrees with $\gamma$ on $A$, we see that $D \models \phi(\beta(b), \gamma(c))$ iff $D \models \psi(\gamma(a), \gamma(c))$ iff $C \models \psi(a, c)$. Thus, the truth value in $D$ of $\phi(\beta(b), \gamma(c))$ can be determined by first working in $B$ to find $\psi$ and $a$ and then evaluating $\psi(a, c)$ in $C$; one does not need $\beta$, $\gamma$, or $D$. □

To make use of this result, we need to know that models have proper conservative extensions. For ultrapowers of $N$, this is easy, because if $B_n$ $(n \in \omega)$ are nonstandard models, then their ultraproduct U-prod $B_n$ is a conservative extension of U-prod $N$ (see [13] or [4]). In the general case, one can write the model as a direct limit of ultrapowers of $N$ and obtain a proper conservative extension as the corresponding direct limit of ultrapowers of some nonstandard model. If we assume the continuum hypothesis (CH), then we have, for any u power $A =$ U-prod $N$, a proper conservative extension $B$ which is minimal $A$ in the strict sense that all proper submodels of $B$ are included in $A$. achieve this, one need only take $B =$ U-prod($V_n$-prod $N$) where the $V_n$ are pairwise non-isomorphic selective ultrafilters; the strict minimality is Theorem 16.1 of [2]. (Of course, CH could be replaced by Martin's axiom or any other assumption that provides the $V_n$'s.)

To illustrate how Theorem 3 can be applied, we consider the problem of finding a model $B$ with two skies, such that the number of inequivalent amalgamations of two copies of $B$ is as small as possible. It was shown in [5] that this number is at least nine (by exhibiting nine pairs $\theta$, $\leq^*$ as in Theorem 2b), and an example was constructed, using CH, where the lower bound of nine is attained. The proof that the example works was a rather messy combinatorial argument; we shall give a much simpler proof (covering a slightly larger class of examples) based on Theorem 3 plus the following well-known fact.

Theorem 4. An ultrapower of $N$ has no nontrivial automorphisms.

Proof: See Theorems 2.5 and 11.5 of [2], or 4.2 of [11], or [10].□

Assuming CH, let $A$ be a minimal proper extension of $N$ (automatically strictly minimal and conservative) and let $B$ be a strictly minimal proper conservative extension of $A$. Strict minimality ensures that $A$ and $B$ are ultrapowers of $N$ and that $B$ has only two constellations, $H = B - A$ and $L = A - N$. $H$ and $L$ are also the skies of $B$, as $B$ is an end extension of $A$. We shall classify the amalgamations of two copies of $B$ according to the intersection. As $N$, $A$, $B$ are pairwise non-isomorphic and are the only submodels of $B$, Theorem 4 tells us that the intersection is purely straight. Obviously, $B$ itself is the only amalgamation with intersection $B$. With intersection $A$, there are only two amalgamations, one with $\alpha(H)$ preceding $\beta(H)$ and one vice versa, by Theorem 3. Finally, if the intersection is $N$, there are six possible orderings of the skies: $\beta(L)\alpha(L)\alpha(H)\beta(H)$, $\alpha(L)\beta(L)\alpha(H)\beta(H)$, $\alpha(L)\alpha(H)\beta(L)\beta(H)$, and similar ones with $\alpha$ and $\beta$ interchanged. For each ordering, we find by applying Theorem 3 three times that there is only one amalgamation. Consider, for example, the first ordering in our list. One shows successively that the models $M_0 = \beta(A)$, $M_1$ generated by $\beta(A)$ and $\alpha(A)$, $M_2$ generated by $\beta(A)$ and $\alpha(B)$, and $M_3$ generated by $\beta(B)$ and $\alpha(B)$ are unique. Each except $M_0$ is obtained by amalgamating the previous one with $A$ or $B$, the intersection being $N$ or $A$, and the new elements of $A$ or $B$ coming after the elements of the previous $M_i$. As $A$ and $B$ are conservative over $N$ and $A$, Theorem 3 gives the required uniqueness, and the proof is complete.

The same sort of argument shows that, if we have a sequence $N \subsetneq A_1 \subsetneq \cdots \subsetneq A_n$ in which each $A_i$ is a strictly minimal conservative extension of its predecessor, then the number of amalgamations of two (or, more generally, of $r$) copies of $A_n$ is the smallest number permitted by Theorem 2b for a model with $n$ skies. For example, two copies of a model with three skies can be amalgamated in at least 29 ways (just list the possible intersections $\theta$ and orderings $\leq^*$, and apply Theorem 2b), and the bound of 29 is attainable (CH) by any $A_3$ as above.

These results are closely related to square-bracket partition relations for ultrafilters. The relation $U \to [U]^n_{r+1}$ means that, whenever the n-element subsets of $\omega$ are partitioned into $r+1$ classes, there is a set in $U$ whose n-element subsets lie in only $r$ of the classes. If we identify the set of n-element subsets of $\omega$ with the set $\omega_+^n$ of increasing n-tuples, we see that the partition relation says that there do not exist $r+1$ disjoint sets all of which meet all sets in the filter $F$ generated by $\omega_+^n$ and sets $X^n$ with $X \in U$. In other words, the Boolean algebra of subsets of $\omega^n$ modulo $F$ has at most $r$ pairwise disjoint elements, which means that it has at most $r$ ultrafilters, which means that $F$ has at most $r$ extensions to ultrafilters on $\omega^n$. But such extensions are precisely the types of increasing n-tuples each term of which has type $U$. Thus, we have the following slight generalization (from 2 to arbitrary n) of an observation in [5] and [9].

<u>Theorem 5</u>. $U \to [U]^n_{r+1}$ <u>if and only if there are at most</u> $r$ <u>ways to amalgamate</u> $n$ <u>copies of</u> U-prod N <u>with a specified ordering for the</u> $n$ <u>images of</u> [id] .□

Thus, for example, the counting arguments above show that, if $U \to [U]^2_4$ then $U$ is $\delta$-stable, if $U \to [U]^2_{14}$ then U-prod N has at most two skies, and (CH) the 4 and 14 are best possible. The same technique, of applying Theorems 2b and 3 and counting the possible intersections $\theta$ and orderings $\leq^*$, clearly gives a host of similar results. A rather impressive example is the following unpublished result of Galvin, originally proved by a direct combinatorial argument; I thank Alan Taylor for bringing it to my attention. If $[U] \to [U]^{24}_{468,744,135,800,126,572,558,268,335,357,952}$, then $U$ is $\delta$-stable, and the subscript is the best possible.

## 3. <u>Rangé and Arrow Ultrafilters</u>

In this section we discuss two recently introduced sorts of special ultrafilters, the rangé ultrafilters of Daguenet [9] and the arrow ultrafilters of Baumgartner and Taylor [1]. An ultrafilter $U$ is n-rangé (where $n \geq 2$) if, for every proper submodel $A$ of U-prod N, there is only one (up to equivalence) amalgamation of $n$ copies of U-prod N such that (1) the $n$ images of the generator [id] are in increasing order and (2) the straight intersection of any two of the $n$ copies is $A$. (Daguenet gave a combinatorial definition of rangé; the present definition is essentially Theorem 5.3 of [9].) Note that, although nothing is said explicitly about the non-straight part of the intersection, it follows from the definition that there isn't any, i.e. that the intersection is purely straight. For, given an amalgamation with intersection larger than the straight part $A$, we could apply Theorem 2b to produce an inequivalent amalgamation with purely straight intersection $A$, thereby violating the uniqueness requirement. Thus, if $U$ is n-rangé, then no two distinct submodels of $U$-prod $N$ are isomorphic (lest Theorem 2b provide a non-straight amalgamation), so if $f(U) = g(U)$ then $f = g$ on a set in $U$. (This is called property $C$ in [9].) One could weaken the definition of n-rangé by specifying the entire intersection in clause (2). When $n \geq 3$, this weakening is only apparent, for the intersection has to be straight, but for $n = 2$ it may be a real weakening.

Some of the properties of rangé ultrafilters obtained combinatorially in [9] are quite easily seen using the model-theoretic approach. For example, to show that (n+1)-rangé implies n-rangé, one takes two supposedly inequivalent amalgamations of $n$ copies of $U$-prod $N$ satisfying (1) and (2), one amalgamates them with intersection (the image of) $A$ by Theorem 2b, one notes that all properly ordered (n+1)-tuples formed from the $2n$ images of $[id]$ have the same type by (n+1)-range, and one infers (using $n + 1 < 2n$) that all properly ordered n-tuples of these $2n$ images (in particular the two coming from the two original amalgamations) have the same type, contrary to the supposed inequivalence of these amalgamations. It is even easier to see that all 2-rangé ultrafilters are $\delta$-stable, for if $U$-prod $N$ has two or more skies, then Theorem 2b provides several amalgamations of two copies of $U$-prod $N$ with purely straight intersection $N$ but with different orderings of the skies. Similarly, if $U$ is n-range and $V = f(U)$, then $V$ is n-rangé, because $V$-prod $N$ is (up to an isomorphism induced by $f$) a submodel of $U$-prod $N$, so inequivalent amalgamations of $n$ copies of the former could be extended, without changing the intersections, to amalgamations of $n$ copies of the latter, by Theorem 2b.

If one weakens the definition of n-rangé by adding to clause (1) that the $n$ images of $[id]$ are order-indiscernible, then one has the model-theoretic definition of n-affable ultrafilters [9]. It is then quite easy to verify that $U$ is n-rangé if and only if it is m-affable for all $m \leq n$.

Baumgartner and Taylor [1] consider partition relations of the form $U \to (U, k)^2$, meaning that, whenever the two-element subsets of $\omega$ are partitioned into two classes, either some set in $U$ has all its pairs in the first class or some k-element set has all its pairs in the second class. These k-arrow ultrafilters have the following model-theoretic characterization.

Theorem 6. $U \to (U, k)^2$ if and only if, for any given amalgamation of two copies of $U$-prod $N$, there is an amalgamation of $k$ copies such that every two are amalgamated in the given way.

Proof: The model-theoretic condition is trivial if the two images of $[id]$ in the given amalgamation are the same, so we may assume that the first image precedes the second. In terms of the 2-type and the k-type of the images of [id] in the given and the required amalgamation, this condition reads: If $V$ is an ultrafilter containing $\omega_+^2$ and extending $U \times U$, then there is an ultrafilter $W$ containing $\omega_+^k$ and satisfying $p_{ij}(W) = V$ for all $i < j$, where $p_{ij}: \omega^k \to \omega^2$ is the projection to the $i^{th}$ and $j^{th}$ factors. Since any family with the finite intersection property can be extended to an ultrafilter, the condition is equivalent to: If $P \subseteq \omega_+^2$ meets every set of the form $X^2$ with $X \in U$, then there is an $s \in \omega_+^k$ such that $p_{ij}(s) \in P$ for all $i < j$. But this is precisely $U \to (U, k)^2$. □

As a corollary, we get the following result, which was stated without proof in [9].

Corollary. If $U$ is 2-rangé, then $U \to (U, k)^2$ for all $k$.

Proof: Given an amalgamation of two copies of U-prod N, let $A$ be the straight intersection. By Theorem 2b, we can amalgamate $k$ copies of U-prod N so that every two of the copies have straight intersection $A$. But then 2-rangé says that every two of the copies are amalgamated in the given way. □

It is possible to use Theorems 2b and 6 to give simple model-theoretic proofs of several of the combinatorial properties of arrow ultrafilters; for example, if $U \to (U, 3)^2$ then U-prod N has at most two skies (a result of Kanamori cited in [1]) and has no two isomorphic submodels (Prop. 4.6 of [9]). We leave the proofs to the reader.

Baumgartner and Taylor showed in [1] that even the weakest analogous partition relation with higher exponent, $U \to (U, 4)^3$ (which means that triples rather than pairs are patitioned), implies that $U$ is selective (which implies that $U \to (U)^n$ for all $n$, by a well-known result of Kunen in [7]). This phenomenon can be understood model-theoretically as follows. $U \to (U, 4)^3$ means that, given any amalgamation of three copies of U-prod N with the images of [id] in increasing order, we can amalgamate four copies so that every three of them are amalgamated in the given way. In this amalgamation of four copies, the four images of [id] clearly form a set of order-indiscernibles. A fortiori, so do any three of them, so the three images of [id] in the given amalgamation must have been order-indiscernible. If $U$ were not selective, there would be a proper nonstandard submodel $A$ in U-prod N, and Theorem 2b would provide an amalgamation of three copies of U-prod N in which the first and second copies of $A$ are identified but not the third. Then the three images of [id] are clearly not indiscernible, so nonselective ultrafilters cannot satisfy $U \to (U, 4)^3$. This discussion suggests that a more appropriate generalization of $U \to (U, k)^2$ to higher exponents $r$ might be: Given any amalgamation of $r$ copies of U-prod N in which the $r$ images of [id] are indiscernible, we can amalgamate $k$ copies so that any $r$ of them are amalgamated in the given way. The reasonableness of this generalization is attested by the fact that all r-rangé ultrafilters have this property.

## 4. Nearly Unique Amalgamations

In this section we prove a theorem relating the number of inequivalent amalgamations of two models to a structural property of these amalgamations. Let $\alpha: A \to C$, $\beta: B \to C$ be an amalgamation of $A$ and $B$. We say that it has new low skies if there is a sky of $C$ which precedes the $\alpha$-images of all skies of $A$ and the $\beta$-images of all skies of $B$, that is, if $(\alpha(A) \cup \beta(B)) - N$ is not coinitial in $C - N$.

Theorem 7. Two ultrapowers of N have infinitely many inequivalent amalgamations if and only if at least one of their amalgamations has a new low sky.

Proof: Suppose that U-prod N and V-prod N have infinitely many inequivalent amalgamations, so infinitely many ultrafilters $W$ extend $U \times V$. It is easy to find an infinite sequence of disjoint sets $R_n \subseteq \omega^2$ such that each $R_n$ is in at least one $W_n$ extending $U \times V$. By increasing one of the $R_n$'s, we assume that these sets partition $\omega^2$, and we define $f: \omega^2 \to \omega : z \longmapsto$ the unique $n$ with $z \in R_n$. Then, for all nonstandard elements $[g] \in$ U-prod N, $[h] \in$ V-prod N and all $k \leq n$,

(*) $$W_n\text{-prod } N \models k \leq [f] \leq g([p_1]),\ h([p_2]) ,$$

because, in $W_n$-prod N, $[f] = n$, while $g([p_1]) = [gp_1]$ is infinite, being the image of $[g]$ under the embedding U-prod N $\to W_n$-prod N induced by $p_1$, and similarly $h([p_2])$ is infinite.

Fix an arbitrary nonprincipal ultrafilter $Z$ on $\omega$. The embeddings of U-prod N into the models $W_n$-prod N induce an embedding $\alpha$ of U-prod N into the ultraproduct $C = Z\text{-prod}(W_n\text{-prod } N)$. Similarly, we have $\beta$: V-prod N $\to C$. By (*) and Łoś's theorem, we have, for all $k \in \omega$ and $g$, $h$ as above,

$$C \models k \leq f(\alpha[\mathrm{id}], \beta(\mathrm{id}]) \leq g(\alpha[\mathrm{id}]),\ h(\beta[\mathrm{id}]) .$$

As $g(\alpha[\mathrm{id}]) = \alpha(g[\mathrm{id}]) = \alpha([g])$ and $h(\beta[\mathrm{id}]) = \beta([h])$, we see that in $C$ (or in the submodel generated by $\alpha$(U-prod N) and $\beta$(V-prod N)) $f(\alpha[\mathrm{id}], \beta[\mathrm{id}])$ is in a new low sky.

Conversely, suppose U-prod N and V-prod N have only finitely many amalgamations. Each of them is equivalent to a W-prod N, where W extends $U \times V$, and the embeddings of U-prod N and V-prod N are $[g] \mapsto [gp_1]$ and $[h] \mapsto [hp_2]$. Since there are only finitely many such W's by assumption, each one contains a set that is in none of the others. Consider any particular $W \supseteq U \times V$, and let S be such a set; thus $(U \times V) \cup \{S\}$ generates W. To show that W-prod N has no new low skies, we consider any infinite $[f] \in$ W-prod N and show that it is above either $[gp_1]$ for some infinite $[g] \in$ U-prod N or $[hp_2]$ for some infinite $[h] \in$ V-prod N. As $[f]$ is inifinite in W-prod N, W contains $\{z | f(z) > n\}$ for each $n \in \omega$. Thus, we can find $X_n \in U$, $Y_n \in V$ such that $f(z) > n$ for all $z \in (X_n \times Y_n) \cap S$. Without loss of generality, $X_n \supseteq X_{n+1}$ and $\bigcap_n X_n = \emptyset$, and similarly for the $Y_n$'s. Define, for $x \in \omega$,

$$g(x) = \text{the smallest } n \text{ such that } x \notin X_n \text{, and}$$

$$h(x) = \text{the smallest } n \text{ such that } x \notin Y_n \text{.}$$

Note that, as $X_n \in U$ and $g(x) \neq n$ on $X_n$, $[g]$ is inifinite in U-prod N, and similarly $[h]$ is inifinite in V-prod N. Furthermore, if $gp_1(z)$ and $hp_2(z)$ are larger than n and if $z \in S$, then $z \in (X_n \times Y_n) \cap S$, so $f(z) > n$. Thus, for all $z \in S$, $f(z) \geq \min(gp_1(z), hp_2(z))$. As $S \in W$, Łoś's theorem says that, in W-prod N, $[f] \geq \min([gp_1], [hp_2])$ as required. □


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