MAXIMAL PURE INDEPENDENT SETS

ANDREAS BLASS AND JOHN IRWIN

Dedicated to László Fuchs on his 75th birthday.

1. Introduction

Let G be a torsion-free abelian group. A subset S of G is *pure independent* if it is linearly independent and the subgroup $\langle S\rangle$ it generates is pure in G. We shall be concerned with properties of pure independent subsets of G that are maximal with respect to inclusion. These maximal pure independent sets (or, more precisely, the subgroups they generate) generalize the basic subgroups studied in [2, 3]. We shall show that some of the important properties of basic subgroups hold, with minor modifications, also for maximal pure independent sets. The main advantage of working with maximal pure independent sets is that they exist in all torsion-free abelian groups. The main disadvantage is that the "minor modifications" mentioned above are, at least in one case, definitely needed; the uniqueness up to isomorphism of basic subgroups does not always carry over to maximal pure independent sets.

To place our results in perspective, we begin with a brief review of basic subgroups and their properties. The notion of basic subgroup arose first in the theory of p-groups, where it was defined in [6] as follows.

Definition 1.1. A subgroup B of an abelian p-group G is *basic* if it is a direct sum of cyclic groups, it is pure in G, and the quotient G/B is divisible.

This concept was imported unchanged into the theory of torsion-free groups [2, 3]. Of course, in this new context, "direct sum of cyclic groups" amounts to "free," so the definition reads as follows.

Definition 1.2. A subgroup B of a torsion-free abelian group G is *basic* if it is free, it is pure in G, and the quotient G/B is divisible.

In contrast to the p-group situation, a torsion-free abelian group need not have any basic subgroups [3, Theorem 14], not even if it is separable (see definition below or in [5]). But when they exist, basic subgroups have several pleasant properties, of which we shall need two, due to Dugas and Irwin [2, 3].

1991 *Mathematics Subject Classification.* 20K20.

Convention 1.3. From now on, when we refer to groups we shall always mean torsion-free, abelian groups.

Theorem 1.4 (Dugas and Irwin)**.** *Any two basic subgroups of the same group are isomorphic.*

This is Theorem 2 of [3]. Although we shall not repeat the proof here, we mention that it explicitly computes the rank of a basic subgroup of G as the dimension of G/pG as a vector space over $\mathbb{Z}/p\mathbb{Z}$, for an arbitrary prime number p. This sheds some light on why basic subgroups need not exist — this dimension may be different for different primes. Chapter VI of [5] describes a notion of p-basic that works with one prime at a time. These p-basic groups always exist, but they are not in general pure.

Definition 1.5. A group is *torsionless* if it is isomorphic to a subgroup of $\mathbb{Z}^\kappa$ for some cardinal number κ. It is *separable* if it is isomorphic to a pure subgroup of $\mathbb{Z}^\kappa$ for some cardinal number κ.

Theorem 1.6 (Dugas and Irwin)**.** *If a separable group G has a basic subgroup of rank κ, then it is isomorphic to a pure subgroup of $\mathbb{Z}^\kappa$ for this same κ.*

This is Theorem 1 of [2].

Corollary 1.7. *A separable group G with a basic subgroup of infinite rank κ has cardinality at most 2^κ.*

Our primary purpose in this paper is to extend these two theorems and the corollary to deal with maximal pure independent sets rather than basic subgroups. In Section 3 we establish the analog of Theorem 1.4 except when finite ranks are involved. In Section 4, we show that this exception is necessary by constructing groups with maximal pure independent sets of many different finite cardinalities. Finally, in Section 5, we discuss analogs of Theorem 1.6. The analogy is not perfect, but it is good enough to get the exact analog of Corollary 1.7, and even a bit better for some values of κ.

2. Maximal Pure Independent Sets

This brief section is devoted to background material, namely the definition of maximal pure independent sets, their elementary properties, and their connection with the basic subgroups that they are intended to generalize. Recall our global assumption that "group" means "torsion-free abelian group."

Definition 2.1. A subset S of a group G is *pure independent* if it is a set of free generators of a pure subgroup of G. It is *maximal pure independent* in G if it is pure independent and is not properly included in any other pure independent subset of G.

We record for reference an immediate consequence of Zorn's Lemma.

Proposition 2.2. *Every group G has a maximal pure independent subset.*

The next proposition gives some equivalent descriptions of maximal pure independent sets.

Proposition 2.3. *Let S be a subset of a group G. The following three statements are equivalent.*

1. *S is a maximal pure independent set in G.*
2. *S freely generates a pure subgroup $\langle S\rangle$ of G which is not a proper direct summand of any other free, pure subgroup of G.*
3. *S freely generates a pure subgroup $\langle S\rangle$ of G such that the quotient $G/\langle S\rangle$ has no pure subgroup isomorphic to $\mathbb{Z}$.*

Proof Each of the three conditions requires S to be a set of free generators for a pure subgroup $\langle S\rangle$ of G. Assuming S has this property, we show that failure of any one of the three conditions implies failure of the next in cyclic order.

If condition 1 fails, i.e., if S isn't maximal, then there is a proper superset $T \supsetneq S$ that is also a set of free generators for a pure subgroup $\langle T\rangle$ of G. Then $\langle S\rangle$ is a proper direct summand of $\langle T\rangle$, the other summand being $\langle T-S\rangle$. So condition 2 also fails.

Suppose next that condition 2 fails; say $\langle S\rangle$ is a proper direct summand of a pure, free subgroup F of G. Then $F/\langle S\rangle$ is a non-trivial, free group and is pure in $G/\langle S\rangle$. Thus condition 3 also fails.

Finally, suppose condition 3 fails. Consider any pure subgroup P of $G/\langle S\rangle$ that is isomorphic to $\mathbb{Z}$, and let $x \in G$ be an element whose coset $[x] \in G/\langle S\rangle$ corresponds to $1 \in \mathbb{Z}$ under the isomorphism. Then $S \cup \{x\}$ is an independent set generating a pure subgroup of G, namely the preimage of P under the projection $G \to G/\langle S\rangle$. In other words, $S \cup \{x\}$ is a pure independent set and, since $x \notin S$, condition 1 also fails. □

Remark 2.4. In connection with condition 2 in Proposition 2.3, we emphasize that $\langle S\rangle$ need not be a maximal, pure, free subgroup of G. It may well be a proper subgroup — just not a summand — of another pure, free subgroup of G.

Corollary 2.5. *If B is a basic subgroup of G, then every set S of free generators of B is a maximal pure independent subset of G.*

Proof Condition 3 of the theorem is trivial to verify, since a divisible group $G/B = G/\langle S\rangle$ cannot have a pure subgroup isomorphic to $\mathbb{Z}$. □

Remark 2.6. It is well known that there are torsion-free groups with no basic subgroups. Thus, Corollary 2.5 and Proposition 2.2 show that the notion of (subgroup generated by a) maximal pure independent set is strictly more general that that of basic subgroup. In fact, even when basic subgroups exist, maximal pure independent sets are more general. For example, let F be a free group of countably infinite rank, and let $h : F \to \mathbb{Q}$ be a homomorphism whose range is a proper, non-free subgroup of $\mathbb{Q}$, such as the group of dyadic rational numbers. Then the kernel of h is not basic in F (for the quotient, the image of h, isn't divisible), but any free generating set for this kernel is a maximal pure independent set in F (for the quotient contains no pure copy of $\mathbb{Z}$). Of course F has basic subgroups, for example all of F or the kernel of a homomorphism from F onto $\mathbb{Q}$.

We close this section with an observation that will exclude some exceptional cases when we deal with torsionless groups.

Proposition 2.7. *Suppose G is a torsionless group with a finite maximal pure independent set S. Then S generates the whole group G, and therefore G is a free group of finite rank.*

Proof Suppose, toward a contradiction, that $\langle S\rangle$ is a proper subgroup of G. Fix some element $x \in G - \langle S\rangle$, and consider the purification H of $\langle S, x\rangle$ in G. This H is a torsionless group of rank $n+1$, where n is the number of elements of S, so H is free. But then, since $\langle S\rangle$ is a pure subgroup of rank n, the quotient $H/\langle S\rangle$ is isomorphic to $\mathbb{Z}$. This quotient is pure in $G/\langle S\rangle$, so, by Proposition 2.3, S cannot be a maximal pure independent set in G. □

3. Invariance of Cardinality

Our main goal in this section is to establish an analog of Theorem 1.4 for maximal pure independent sets. A perfect analog would say that any two such sets in the same group have the same cardinality. This is not generally true — counterexamples are given in Section 4 — but it is true if the maximal pure independent sets are both infinite. That will be a corollary of the following more precise result.

Theorem 3.1. *Let G be a group with a pure, free subgroup F of rank κ. Suppose that G also has a subgroup M of cardinality μ such that G/M has no pure subgroup isomorphic to $\mathbb{Z}$. Then $\kappa \leq \mu$.*

Proof We first eliminate the trivial case where μ is finite. Since G is torsion-free, this case would make M the trivial group, so G would have no pure subgroup isomorphic to $\mathbb{Z}$. Then F would also have to be trivial. So we would have $\kappa = 0$ and $\mu = 1$, which verifies the conclusion of the theorem.

From now on, we assume that μ is infinite. Thus $\mu \cdot \aleph_0 = \mu$.

Suppose, toward a contradiction, that $\mu < \kappa$. Define H to be the purification in G of the subgroup $M+F$, and observe that H/M, being a pure subgroup of G/M, has no pure subgroup isomorphic to $\mathbb{Z}$.

For each element $m \in M$ and each non-negative integer n, if there is some $f \in F$ such that $m+f$ is divisible by n, then choose one such f and call it $f_{m,n}$. We adopt the natural convention that "divisible by 0" means equal to 0, so $f_{m,0}$ is $-m$ if $m \in F$ and undefined otherwise. Since there are only μ values for m and only $\aleph_0$ values for n, there are at most μ elements of the form $f_{m,n}$. These elements lie in the free group F, so they lie in a direct summand F_1 of F of rank at most μ. Write F_2 for a complementary summand of F. Notice that F_2 is nontrivial, because F has rank κ and F_1 has strictly smaller rank, at most μ.

Let H_1 be the purification in G of $M+F_1$. We intend to show that H is the direct sum $H_1 \oplus F_2$. If we can show this, then the proof will be complete. Indeed, since $M \subseteq H_1$, we shall have

$$\frac{H}{M} = \frac{H_1}{M} \oplus F_2,$$

which contradicts, since F_2 is a nontrivial free group, the fact that H/M contains no pure copy of $\mathbb{Z}$.

So it remains only to verify that $H = H_1 \oplus F_2$. Consider any $h \in H$. By definition of H, we have $nh = m+f$ for some positive integer n, some $m \in M$, and some $f \in F$. So $f_{m,n}$ exists, and n divides both $m+f$ and $m+f_{m,n}$ and therefore also their difference $f - f_{m,n}$. As F is pure in G, we have $f - f_{m,n} = nf'$ for some $f' \in F$. Write $f' = f'_1 + f'_2$ with $f'_1 \in F_1$ and $f'_2 \in F_2$. Then

(1)
$$\begin{aligned} n(h-f'_2) = nh - nf'_2 = m+f-nf'_2 &= m + f_{m,n} + f - f_{m,n} - nf'_2 \\ = m + f_{m,n} + nf' - nf'_2 &= m + f_{m,n} + nf'_1 \in M + F_1. \end{aligned}$$

By definition of H_1, it follows that $h - f'_2 \in H_1$, and therefore $h \in H_1 + F_2$.

This establishes that $H = H_1 + F_2$. It remains to prove that this sum is direct. Suppose $x \in H_1 \cap F_2$. By definition of H_1, we can write $nx = m+f$ with n a positive integer, $m \in M$, and $f \in F_1$. Since $x \in F_2 \subseteq F$, we have $m = nx - f \in F$. Therefore, $f_{m,0}$ exists and equals $-m$, which gives us that $m \in F_1$. Now nx is in F_1 (because it equals $m + f$ and both m and f are in F_1) and also in F_2 (because $x \in F_2$). Since $F = F_1 \oplus F_2$ is a direct sum, this implies $nx = 0$ and thus $x = 0$. □

Corollary 3.2. *If a group has maximal pure independent sets of cardinalities κ and λ, then $\kappa \cdot \aleph_0 = \lambda \cdot \aleph_0$.*

Proof It is easy to check (as at the beginning of the proof of the theorem), that if one of κ and λ is zero then so is the other. We assume from now on that they are not zero.

Apply the theorem with F being the subgroup generated by a maximal pure independent set of cardinality κ and M being the subgroup generated by a maximal pure independent set of cardinality λ. The cardinality μ of M is thus $\lambda \cdot \aleph_0$. By the theorem, $\kappa \leq \lambda \cdot \aleph_0$, and it follows, since ${\aleph_0}^2 = \aleph_0$, that $\kappa \cdot \aleph_0 \leq \lambda \cdot \aleph_0$.

Since the hypotheses of the theorem are symmetrical between κ and λ, the reverse inequality also holds. □

Corollary 3.3. *If a group has maximal pure independent sets of infinite cardinalities κ and λ, then $\kappa = \lambda$.*

Proof For infinite cardinals κ, we have $\kappa \cdot \aleph_0 = \kappa$. □

Corollary 3.4. *In a torsionless group, all maximal pure independent sets have the same cardinality.*

Proof If the group has infinite rank, then, by Proposition 2.7, its maximal pure independent sets are infinite, so the preceding corollary applies. If the group has finite rank, then, again by Proposition 2.7, the cardinality of each maximal pure independent set equals the rank of the group. □

Thus, we have the analog of Theorem 1.4 except that we might have, in one and the same (non-torsionless) group G, two maximal pure independent sets of different (non-zero) finite cardinalities, or one of finite cardinality and one of cardinality $\aleph_0$. We shall see in Section 4 that this exception really occurs.

4. Non-Invariance of Cardinality

In this section, we shall construct examples showing that a group may have maximal pure independent sets of different finite cardinalities or of a finite cardinality and $\aleph_0$. In fact, the following theorem says that "anything is possible", i.e., we can arbitrarily prescribe the cardinalities of maximal pure independent sets, subject only to Corollary 3.2 and the trivial observation that, if the empty set is maximal pure independent in G, then no other set is.

Theorem 4.1. *Let R be a nonempty subset of $\{1, 2, \ldots, \aleph_0\}$. There exists a group G with maximal pure independent sets of all cardinalities in R and of no other cardinalities.*

This theorem will be an easy consequence of the following more explicit description of the pure, free subgroups of the G's that we construct.

Theorem 4.2. *Let R be a nonempty subset of $\{1, 2, \ldots, \aleph_0\}$. There exists a group G with pure independent sets S_r, for $r \in R$, such that:*

1. *S_r has cardinality r.*
2. *The different S_r's are linearly independent.*
3. *Every pure, free subgroup of G is included in one of the groups $\langle S_r \rangle$.*

Proof We begin by observing that requirement 3 in the theorem will be satisfied if every pure copy of $\mathbb{Z}$ in G is included in some $\langle S_r \rangle$. Indeed, if this weakened form of 3 holds and if F is any pure, free subgroup of G, then each element of F is in some $\langle S_r \rangle$ (for it lies in a pure copy of $\mathbb{Z}$). And we cannot have two non-zero elements of F lying in different $\langle S_r \rangle$'s, for then their sum could not be in any $\langle S_r \rangle$, because of requirement 2. Thus, all of F lies in a single $\langle S_r \rangle$.

It is convenient to isolate part of the construction of G in the following technical lemma.

Lemma 4.3. *Suppose we are given, in the rational vector space $\mathbb{Q}^n$,*

- *finitely many rational subspaces H_i,*
- *a finitely generated additive subgroup L of $\mathbb{Q}^n$, and*
- *a vector $v \in L$ lying in none of the H_i.*

Then there is an integer $p > 1$ such that the subgroup $\langle L, v/p \rangle$ generated by L and v/p contains no elements in any H_i except those already in L, i.e.,

$$\langle L, v/p \rangle \cap H_i \subseteq L \quad \text{for each } i.$$

Proof We may assume without loss of generality that each H_i is a hyperplane not containing v, because each of the given H_i is included in such a hyperplane.

Temporarily fix i. Since $\mathbb{Q}^n = H_i \oplus \mathbb{Q}v$, let $\pi_i : \mathbb{Q}^n \to \mathbb{Q}v$ be the projection with kernel H_i. Then $\pi_i(L)$ is a finitely generated subgroup of $\mathbb{Q}v$.

Now un-fix i. The subgroup of $\mathbb{Q}v$ generated by all the $\pi_i(L)$ together is again finitely generated, so it has the form $\mathbb{Z}\alpha v$ for some rational number α. Since $v \in L$, this subgroup contains v, so $\alpha = 1/k$ for some $k \in \mathbb{Z} - \{0\}$.

Let p be a prime number not dividing k. We shall show that this p satisfies the conclusion of the lemma. That is, $\langle L, v/p\rangle \cap H_i \subseteq L$ for each i.

So consider any vector in $\langle L, v/p\rangle$, say $l + (mv/p)$ where $l \in L$ and $m \in \mathbb{Z}$, and assume that $l + \frac{mv}{p} \in H_i$ for a certain i. We must show $l + (mv/p) \in L$. Recall that π_i has kernel H_i, so we have $\pi_i(l + (mv/p)) = 0$. That is, since π_i fixes v,

$$\frac{mv}{p} = \pi_i\left(\frac{mv}{p}\right) = -\pi_i(l) \in \pi_i(L) \subseteq \mathbb{Z}\frac{1}{k}v.$$

So $\dfrac{m}{p} = \dfrac{j}{k}$ for some $j \in \mathbb{Z}$. Then $km = pj$ and, since p was chosen as a prime not dividing k, we infer that p divides m. Thus, m/p is an integer and $l + \frac{mv}{p} \in \langle L, v\rangle = L$ because $v \in L$. □

Returning to the proof of Theorem 4.2, we construct the desired G as an additive subgroup of the rational vector space

$$\bigoplus_{r\in R} \bigoplus_{0\leq i<r} \mathbb{Q}e_{r,i},$$

i.e., the rational vector space having as basis a doubly indexed family of vectors $e_{r,i}$ where the first index r ranges over R and, for each fixed r, the second index i takes r values. For each $r \in R$, let $S_r = \{e_{r,i} : 0 \leq i < r\}$. We begin the construction of G by putting into G all members of

$$G_0 = \bigoplus_{r\in R} \bigoplus_{0\leq i<r} \mathbb{Z}e_{r,i},$$

i.e., the group freely generated by the union of all the S_r's.

The rest of the construction of G is an induction, putting into G at each stage one new element along with whatever that element and previous members of G generate. We write G_k for the group obtained after k steps. So $G_{k+1} = \langle G_k, x_k\rangle$ for some x_k, and the final result is

$$G = \bigcup_{k=0}^{\infty} G_k.$$

Notice that each of our S_r's is an independent set and they are independent of each other. In view of the remarks at the beginning of the proof (and the obvious fact that S_r has cardinality r), the proof of the theorem will be complete if we can arrange the construction so that each $\langle S_r\rangle$ is pure in G and so that, for every $v \in G$ that is not in any $\langle S_r\rangle$, the cyclic subgroup $\langle v\rangle$ is not pure in G.

To make each $\langle S_r \rangle$ pure in G, it suffices to make it pure in each G_k. It is obviously pure in G_0 (being a direct summand of G_0); we shall arrange each step, from G_k to G_{k+1}, so as to preserve this purity. That is, this step should not add to G any new elements of the rational vector subspace

$$H_r = \bigoplus_{0 \le i < r} \mathbb{Q} e_{r,i}.$$

By the *support* of a vector $v \in \bigoplus_{r \in R} \bigoplus_{0 \le i < r} \mathbb{Q} e_{r,i}$, we mean the set of $r \in R$ such that, for some i, the component of v in $\mathbb{Q} e_{r,i}$ is not zero. Thus, to maintain purity of $\langle S_r \rangle$, we must avoid putting into G any new elements whose support is $\{r\}$.

List, in a sequence (v_k), all the vectors in $\bigoplus_{r \in R} \bigoplus_{0 \le i < r} \mathbb{Q} e_{r,i}$ whose support contains at least two elements. This is possible because this vector space is countable. For technical reasons, choose the sequence so that each such vector occurs infinitely often in it.

At stage k of our inductive construction of G, when we have G_k and want to enlarge it to $G_{k+1} = \langle G_k, x_k \rangle$, we shall choose x_k so as to satisfy the following two requirements.

Purity: $G_{k+1} \cap H_r \subseteq G_k$ for all $r \in R$.
Division: If $v_k \in G_k$ then $v_k/p \in G_{k+1}$ for some integer $p > 1$.

As mentioned earlier (and as its name suggests), the purity requirement ensures that each $\langle S_r \rangle$ is pure in G.

The division requirement ensures that no $v \in G - \bigcup_{r \in R} \langle S_r \rangle$ generates a pure copy of $\mathbb{Z}$ in G. Indeed, any such v must have at least two elements in its support, because of the purity requirement, and v is in some G_m. But then $v = v_k$ for infinitely many k, including some $k \ge m$. Then the division requirement puts some v/p into G_{k+1} and therefore into G. So $\mathbb{Z} v$ is not pure in G.

So all that remains is to carry out the step from G_k to G_{k+1} in such a way as to satisfy the two requirements. If $v_k \notin G_k$, then we simply set $G_{k+1} = G_k$, and the requirements are trivially satisfied.

From now on, assume $v_k \in G_k$. We intend to choose an integer $p > 1$ and set $G_{k+1} = \langle G_k, v_k/p \rangle$. This will certainly satisfy the division requirement; we show that p can be chosen so as to also satisfy the purity requirement.

Let R_0 be a finite subset of R so large that

$$V = \bigoplus_{r \in R_0} \bigoplus_{0 \le i < r} \mathbb{Q} e_{r,i}$$

contains v_k and all the generators x_j $(j < k)$ from earlier stages of the inductive construction. Thus,

$$G_k = (G_k \cap V) \oplus \bigoplus_{r \in R - R_0} \bigoplus_{0 \le i < r} \mathbb{Z} e_{r,i}.$$

No matter what p we choose, v_k/p will be in V, so G_{k+1} will have the form

$$(G_{k+1} \cap V) \oplus \bigoplus_{r \in R - R_0} \bigoplus_{0 \le i < r} \mathbb{Z} e_{r,i}$$

and will therefore satisfy the purity requirement for all $r \in R - R_0$. To satisfy this requirement also for $r \in R_0$, we apply Lemma 4.3. For notational simplicity, suppose first that $\aleph_0 \notin R_0$.

Start with the finite-dimensional rational vector space V (which can be identified with a $\mathbb{Q}^n$ as in the lemma), the finitely many subspaces H_r for $r \in R_0$, the subgroup $L = G_k \cap V$, and the vector $v = v_k$. Note that L is finitely generated, the generators being the $e_{r,i}$ for $r \in R_0$ and $i < r$ along with the x_j for $j < k$. So the hypotheses of the lemma are satisfied, and we obtain an integer $p > 1$ such that, for each $r \in R_0$,

$$\langle G_k \cap V, v_k/p \rangle \cap H_r \subseteq G_k \cap V, \quad \text{i.e.,} \quad (G_{k+1} \cap V) \cap H_r \subseteq G_k.$$

But for $r \in R_0$, we have $H_r \subseteq V$, so the previous formula simplifies to $G_{k+1} \cap H_r \subseteq G_k$.

If $\aleph_0 \in R_0$ the argument is similar, but V should have $\mathbb{Q} e_{\aleph_0,i}$ as a summand for only finitely many i, enough so that v_k and all earlier x_j are in V. This completes the proof of the purity requirement and thus of the theorem. □

An immediate consequence of the theorem, by taking $R = \mathbb{N}$, is the following answer to a question posed (in private communication) by J. Reid.

Corollary 4.4. *There is a group G with pure free subgroups of all finite ranks but no pure free subgroup of infinite rank.*

Finally, we infer Theorem 4.1 from Theorem 4.2.

Proof of Theorem 4.1 Given R, let G and the sets S_r be as in Theorem 4.2. This G has maximal pure independent sets of all cardinalities in R, namely the S_r, which are maximal by condition 3.

Now consider any maximal pure independent set S in G, say of cardinality s; we must show that $s \in R$. By condition 3 of Theorem 4.2, S is included in $\langle S_r \rangle$ for some $r \in R$; so clearly $s \le r$. If s is infinite, then so is r and we have $s = r = \aleph_0$, so the proof is complete. If s is finite then, as a finite-rank, pure subgroup of the free group $\langle S_r \rangle$, $\langle S \rangle$ must be a direct summand of $\langle S_r \rangle$. But by Proposition 2.3, it cannot

be a proper direct summand. So $\langle S \rangle = \langle S_r \rangle$ and therefore $s = r \in R$.
□

5. Embeddings and Cardinalities

In this section, we connect the size of a maximal pure independent set in a group G with the number κ of factors needed in a product $\mathbb{Z}^\kappa$ in order that G be isomorphic to a subgroup (or to a pure subgroup) of this product. We also consider related bounds on the cardinality of G.

We begin by stating for reference a well-known lemma describing embeddings of the sort we seek.

Lemma 5.1. *A group G is embeddable as a subgroup of $\mathbb{Z}^\kappa$ if and only if there is a family $\mathcal{F}$ of at most κ homomorphisms $G \to \mathbb{Z}$ such that, for any non-zero $x \in G$, there is at least one $f \in \mathcal{F}$ such that $f(x) \neq 0$.*

A reduced group G is embeddable as a pure subgroup of $\mathbb{Z}^\kappa$ if and only if there is a family $\mathcal{F}$ of at most κ homomorphisms $G \to \mathbb{Z}$ such that, for any non-zero $x \in G$, the only integers that divide $f(x)$ for every $f \in \mathcal{F}$ are those that divide x. Furthermore, if no non-zero element of G is divisible by infinitely many primes, then $\mathcal{F}$ will satisfy this condition provided, for each non-zero $x \in G$, the only primes that divide $f(x)$ for every $f \in \mathcal{F}$ are those that divide x.

Proof For both directions of both parts, just take the homomorphisms $f \in \mathcal{F}$ to be the coordinate projections $\mathbb{Z}^\kappa \to \mathbb{Z}$ restricted to a copy of G. □

The hypotheses "reduced" in the second sentence of the lemma and "no non-zero element of G is divisible by infinitely many primes" in the third serve only to ensure that $\mathcal{F}$ satisfies the condition in the first sentence. Notice that these hypotheses are satisfied by all torsionless groups; these are the only groups to which we shall apply the lemma.

Corollary 5.2. *If G is embeddable as a pure subgroup of $\mathbb{Z}^\kappa$, then there is a family $\mathcal{F}$ of at most κ homomorphisms $G \to \mathbb{Z}$ such that, for any $x \in G$, there is $f_x \in \mathcal{F}$ with $f_x(x)$ a divisor of x in G.*

Proof Beginning with a family $\mathcal{F}$ as given by the second part of the lemma, close it under the group operations in $\mathrm{Hom}(G, \mathbb{Z})$. The resulting $\mathcal{F}'$ will satisfy the requirement of the corollary. Indeed, for any non-zero $x \in G$ (the case of $x = 0$ being trivial), $V = \{f(x) : f \in \mathcal{F}\}$ is a set of integers with no common divisor except for divisors of x. The greatest common divisor d of V is thus a divisor of x, and it is a linear combination, with integer coefficients, of members of V. It is therefore of the form $f(x)$ for some $f \in \mathcal{F}'$, and this f can serve as f_x. □

Corollary 5.3. *If G is a separable group of cardinality κ, then G is embeddable as a pure subgroup of $\mathbb{Z}^\kappa$.*

Proof As G is separable, there is a family $\mathcal{F}$ of homomorphisms as in the preceding corollary, except that it may have cardinality greater than κ. But if we choose, for each $x \in G$, one $f_x \in \mathcal{F}$ as in that corollary, then the resulting subfamily of $\mathcal{F}$ still satifies the conditions of the second part of the lemma. □

After these preliminaries, we are ready to present our results relating maximal pure independent sets to embeddings in powers of $\mathbb{Z}$.

Theorem 5.4. *Suppose G is a separable group with a maximal pure independent set of cardinality κ. Then G can be embedded as a subgroup of $\mathbb{Z}^\kappa$. In particular, the cardinality of G is at most 2^κ.*

Proof We assume that κ is infinite, for otherwise the result is trivial by Proposition 2.7.

Fix a maximal pure independent set S of size κ. Since κ is infinite, the subgroup $\langle S\rangle$ of G generated by S also has cardinality κ. For each $m \in \langle S\rangle$, apply Corollary 5.2 (with some possibly larger cardinal in place of κ) to choose a homomorphism $f_m : G \to \mathbb{Z}$ such that $f_m(m)$ divides m in G. We shall show that the κ (or fewer) homomorphisms so chosen satisfy the conditions in the first part of Lemma 5.1 and thus establish the desired embeddability of G in $\mathbb{Z}^\kappa$.

So consider any non-zero element $x \in G$. We must find $m \in \langle S\rangle$ with $f_m(x) \neq 0$. Since G is separable and $x \neq 0$, the purification in G of the cyclic group generated by x is also cyclic; let y be a generator of it. So $x = ny$ for some non-zero integer n, and y is not divisible in G by any primes.

Recall from Proposition 2.3 that $G/\langle S\rangle$ has no pure subgroup isomorphic to $\mathbb{Z}$. This applies in particular to the subgroup generated by the coset $[y]$ of y. Thus, in $G/\langle S\rangle$, this coset is divisible by some prime p, say $[y] = p[z]$. Back in G, this means that $y = pz + m$ for some $m \in \langle S\rangle$. Notice that p doesn't divide m, because it doesn't divide y. Therefore $f_m(m)$ is not divisible by p either. But then $f_m(y) = pf_m(z) + f_m(m)$ isn't divisible by p either. In particular, $f_m(y) \neq 0$. Therefore $f_m(x) = nf_m(y) \neq 0$. □

Remark 5.5. Unlike Theorem 1.6, the preceding theorem does not guarantee that the image of the embedding can be taken to be pure in $\mathbb{Z}^\kappa$. We do not know whether the theorem can be improved to provide this guarantee. Of course the cardinality bound in the theorem implies, via Corollary 5.3, that G is embeddable as a pure subgroup of $\mathbb{Z}^{2^\kappa}$. We shall obtain a smaller exponent, for some values of κ, in Corollary 5.9 below.

Remark 5.6. Leaving the conclusion of the theorem as it is, without purity, one might hope to reduce the hypothesis from "separable" to "torsionless." But this too is an open problem.

The following result is the analog, for maximal independent sets in torsion-free groups, of a result for basic subgroups in p-groups given in [5, Theorem 34.3]. An analog for basic subgroups of torsion-free groups is mentioned as background in [1, Proposition 1].

Theorem 5.7. *Suppose G is a torsionless group with a maximal pure independent set of infinite cardinality κ. Then G has cardinality at most $\kappa^{\aleph_0}$.*

Proof Fix a maximal pure independent set S in G of cardinality κ. We shall define a one-to-one function from G into the set Φ of all functions whose domains are infinite subsets of the positive integers and whose values are in $\langle S\rangle$, the subgroup of G generated by S. Since $\langle S\rangle$, like S, has cardinality κ, Φ has cardinality $\kappa^{\aleph_0}$, so the existence of j will suffice to establish the theorem.

To define $j(x)$ for an arbitrary $x \in G$, proceed as follows. The domain of $j(x)$ is the set of those positive integers that divide $[x]$, the coset of x in $G/\langle S\rangle$. Since $G/\langle S\rangle$ has no pure subgroup isomorphic to $\mathbb{Z}$ (by Proposition 2.3), each of its elements is divisible by infinitely many integers, and so the domain of $j(x)$ is infinite. To complete the definition of $j(x)$, consider any n in its domain. Then since n divides $[x]$, we have $x = ny+h$ for some $y \in G$ and $h \in \langle S\rangle$. For given x and n, there will be many choices for y and h; pick one such pair arbitrarily. Then set $j(x)(n) = h$.

This completes the definition of our function $j : G \to \Phi$. It remains to check that j is one-to-one. So suppose $j(x) = j(x')$. Thus, for infinitely many positive integers n (namely all n in the domain of $j(x)$) we can write $x = ny + h$ and $x' = ny' + h$, with the same h. Then $x - x'$ is divisible by these infinitely many positive integers n. But G is torsionless, so $x-x'$ must be 0, i.e., $x = x'$, and the proof is complete. □

Remark 5.8. In this theorem, the hypothesis that G is torsionless can be weakened. All we needed in the proof is that the only element divisible by infinitely many integers is 0. This is equivalent to requiring all subgroups of rank 1 to be isomorphic to $\mathbb{Z}$.

Corollary 5.9. *If a separable group G has a maximal pure independent set of infinite cardinality κ, then G can be embedded as a pure subgroup in $\mathbb{Z}^{\kappa^{\aleph_0}}$.*

Proof Combine the theorem with Corollary 5.3. □

Remark 5.10. To clarify the relationship between the cardinality estimates in Theorems 5.4 and 5.7, we mention some of the relevant facts of cardinal arithmetic. For all infinite cardinals, $\kappa^{\aleph_0} \leq 2^\kappa$, so Theorem 5.7 is never worse than the estimate in Theorem 5.4. Whether it is actually better depends on κ. There are cardinals κ for which $\kappa^{\aleph_0} = 2^\kappa$; an obvious example is $\kappa = \aleph_0$. But there are also cardinals κ for which $\kappa^{\aleph_0} < 2^\kappa$. Examples include the cardinality $\mathfrak{c} = 2^{\aleph_0}$ of the continuum as well as cardinals finitely many steps beyond it, i.e., the next cardinal $\mathfrak{c}^+$, and $\mathfrak{c}^{++}$, and so forth for a simple (not transfinite) sequence. In fact, for these particular examples, $\kappa^{\aleph_0} = \kappa$, so we recover in Theorem 5.7 the cardinality estimate of Theorem 1.6 for these values of κ.

The following theorem provides improvements of some of the preceding work if we consider groups satisfying stronger hypotheses than separability. We refer to [4, Sections IV.2 and VII.4] for the definitions of the hypotheses "Whitehead group" and "coseparable" used here.

Theorem 5.11. *Suppose G is a Whitehead group (or only a coseparable group) of cardinality at most 2^κ, where κ is an infinite cardinal. Then G can be embedded as a pure subgroup in $\mathbb{Z}^\kappa$.*

Proof Given G as in the hypothesis, we intend to produce a family $\mathcal{F}$ of at most κ homomorphisms $G \to \mathbb{Z}$ satisfying the conditions of the second part of Lemma 5.1. Thus, for every $g \in G$ and every prime number p not dividing g, there should be an $f \in \mathcal{F}$ with p not dividing $f(g)$. Equivalently, the induced homomorphisms $\bar{f} : G/pG \to \mathbb{Z}/p\mathbb{Z}$, for $f \in \mathcal{F}$, should not all simultaneously annihilate any non-zero element of G/pG.

Thus, to complete the proof, it suffices to find, for each prime p, a collection of at most κ homomorphisms $G/pG \to \mathbb{Z}/p\mathbb{Z}$ that do not all annihilate any non-zero element of G/pG and to lift these to homomorphisms $G \to Z$. Then, by taking all these lifted homomorphisms, for all primes p (for a total of at most $\kappa \cdot \aleph_0 = \kappa$ homomorphisms), we obtain the desired $\mathcal{F}$.

Consider, therefore, any fixed prime p. Since G has cardinality at most 2^κ, the dimension of G/pG as a vector space over $\mathbb{Z}/p\mathbb{Z}$ is at most 2^κ. On the other hand, $(\mathbb{Z}/p\mathbb{Z})^\kappa$ has dimension exactly 2^κ. So there is an embedding of vector spaces, $G/pG \to (\mathbb{Z}/p\mathbb{Z})^\kappa$. The coordinates of this embedding are κ homomorphisms $G/pG \to \mathbb{Z}/p\mathbb{Z}$ that do not all annihilate any non-zero element of G/pG.

All that remains is to lift these homomorphisms $G/pG \to \mathbb{Z}/p\mathbb{Z}$ to homomorphisms $G \to \mathbb{Z}$. Of course we can compose our homomorphisms with the projection $G \to G/pG$; we must then lift the resulting homomorphisms $G \to \mathbb{Z}/p\mathbb{Z}$ to homomorphisms $G \to \mathbb{Z}$. In fact, we claim that *every* homomorphism $G \to \mathbb{Z}/p\mathbb{Z}$ can be so lifted.

To see this, consider the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \frac{\mathbb{Z}}{p\mathbb{Z}} \to 0$$

where the map $\mathbb{Z} \to \mathbb{Z}$ is multiplication by p. Applying $\mathrm{Hom}(G, -)$, we get a long exact sequence, part of which is

$$\mathrm{Hom}(G, \mathbb{Z}) \to \mathrm{Hom}(G, \frac{\mathbb{Z}}{p\mathbb{Z}}) \to \mathrm{Ext}(G, \mathbb{Z}) \to \mathrm{Ext}(G, \mathbb{Z}),$$

where the last map is multiplication by p. Our claim about lifting arbitrary homomorphisms amounts to saying that the first map in this (segment of the) exact sequence is surjective, or equivalently (by exactness) that the second map is 0, or equivalently (by exactness) that the third map is one-to-one. For Whitehead groups, this last reformulation is obviously true, since the Ext terms are zero. For coseparable G, we need to show that multiplication by the prime p is one-to-one on $\mathrm{Ext}(G, \mathbb{Z})$. But one of the equivalent characterizations of coseparability in [4, Section IV.2] is that $\mathrm{Ext}(G, \mathbb{Z})$ is torsion-free. So the proof is complete. □

Corollary 5.12. *Let G be a Whitehead (or coseparable) group with a maximal pure independent set of infinite cardinality κ. Then G can be embedded as a pure subgroup of $\mathbb{Z}^\kappa$.*

Proof Combine Theorems 5.4 and 5.11. □

Thus, for Whitehead groups, a maximal pure independent set gives a pure embedding in as small a product of $\mathbb{Z}$'s as a basic subgroup would give via Theorem 1.6.

References

[1] A. Blass, "On the divisible parts of quotient groups," in *Abelian Group Theory and Related Topics*, edited by R. Göbel, P. Hill, and W. Liebert, Contemp. Math. 171, Amer. Math. Soc. (1994) 37–50.

[2] M. Dugas and J. Irwin, "On pure subgroups of Cartesian products of integers," *Resultate Math.* 15 (1989), 35–52.

[3] M. Dugas and J. Irwin, "On basic subgroups of $\prod \mathbf{Z}$," *Comm. Algebra* 19 (1991), 2907–2921.

[4] P. Eklof and A. Mekler, *Almost Free Modules: Set-Theoretic Methods*, North-Holland, 1990
[5] L. Fuchs, *Infinite Abelian Groups, vol. I*, Academic Press, 1970.
[6] L. Ya. Kulikov, "On the theory of abelian groups of arbitrary cardinality" [Russian], *Mat. Sbornik* 16 (1945) 129–162.

Mathematics Department, University of Michigan, Ann Arbor, MI 48109–1109, U.S.A.
E-mail address: ablass@umich.edu

9001 Starmount, Las Vegas, NV 89134, U.S.A.
E-mail address: drjmirwin@lvcm.com