The Number of Near-Coherence Classes of Ultrafilters is Either Finite or $2^{\mathfrak{c}}$

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Abstract

We prove that the number of near-coherence classes of non-principal ultrafilters on the natural numbers is either finite or $2^{\mathfrak{c}}$. Moreover, in the latter case the Stone-Čech compactification $\beta\omega$ of ω contains a closed subset C consisting of $2^{\mathfrak{c}}$ pairwise non-nearly-coherent ultrafilters. We obtain some additional information about such closed sets under certain assumptions involving the cardinal characteristics $\mathfrak{u}$ and $\mathfrak{d}$.

Applying our main result to the Stone-Čech remainder $\beta\mathbb{R}_+ - \mathbb{R}_+$ of the half-line $\mathbb{R}_+ = [0, \infty)$ we obtain that the number of composants of $\beta\mathbb{R}_+ - \mathbb{R}_+$ is either finite or $2^{\mathfrak{c}}$.

1 Introduction

In this paper, all filters are on the set ω of natural numbers, and they contain all cofinite subsets of ω. In particular, all ultrafilters are non-principal ultrafilters on ω. If $\mathcal{F}$ is a filter and $f : \omega \to \omega$ is a finite-to-one function, then $f(\mathcal{F})$ is defined to be $\{X \subseteq \omega : f^{-1}(X) \in \mathcal{F}\}$. This is a filter, and it is an ultrafilter if $\mathcal{F}$ is.

As in [5], we call two filters $\mathcal{F}_1$ and $\mathcal{F}_2$ *coherent* if $\mathcal{F}_1 \cup \mathcal{F}_2$ has the finite intersection property, i.e., if there is a filter that includes them both. We call $\mathcal{F}_1$ and $\mathcal{F}_2$ *nearly coherent* if there is a finite-to-one $f : \omega \to \omega$ such that $f(\mathcal{F}_1)$ and $f(\mathcal{F}_2)$ are coherent. Notice that a filter and an ultrafilter are coherent if and only if the former is included in the latter; in particular, two ultrafilters are coherent only if they are equal.

Near-coherence is an equivalence relation, introduced and extensively studied in [5]. It is natural to ask how many equivalence classes it has. Since the number of ultrafilters is $2^{2^{\aleph_0}}$ by a theorem of Pospíšil [17], the number of near-coherence classes of ultrafilters is obviously between 1 and $2^{2^{\aleph_0}}$, inclusive. Its exact value, however, is independent of the usual (ZFC) axioms of set theory. The known consistency results are, in chronological order, using the standard notation $\mathfrak{c}$ for the cardinality $2^{\aleph_0}$ of the continuum:

1. It is consistent relative to ZFC, and in fact it is a consequence of the continuum hypothesis (CH) or of Martin's axiom (MA), that the number of near-coherence classes is $2^{\mathfrak{c}}$.

2. It is consistent relative to ZFC that there is only one near-coherence class of ultrafilters.

The first of these consistency results follows from the fact that among selective ultrafilters (those such that every function on ω is either constant or one-to-one on a set in the ultrafilter) near coherence is the same as isomorphism (via permutations of ω). In particular, any selective ultrafilter is nearly coherent with only $\mathfrak{c}$ others. On the other hand, CH (or just MA) implies that there are $2^{\mathfrak{c}}$ selective ultrafilters, and therefore that there are $2^{\mathfrak{c}}$ near-coherence classes. The history of this result is a bit obscure. Booth [10] writes that Galvin was the first to prove the existence of selective ultrafilters under CH, but doesn't say that Galvin made the slight extension to get $2^{\mathfrak{c}}$ selective ultrafilters. Booth himself weakens the hypothesis to MA and shows that the selective ultrafilters are dense in the Stone-Čech remainder

$\omega^* = \beta\omega - \omega$. Rudin [20] proves that CH yields $2^{\mathfrak{c}}$ selective ultrafilters, but she describes the result as well known. In the second author's thesis [4], the assumption here is reduced to MA (and in fact to the hypothesis there called FRH(ω) but nowadays expressed as $\mathfrak{p} = \mathfrak{c}$ or as MA(σ-centered)), but the proof is essentially the same as under CH. In this paper we shall show that the number of near-coherence classes of ultrafilters is $2^{\mathfrak{c}}$ under the weaker hypothesis $\mathfrak{u} \geq \mathfrak{d}$ (see Section 2 for the definitions of $\mathfrak{u}$ and $\mathfrak{d}$).

The statement that there is only one near-coherence class of ultrafilters is known as the principle of near coherence of filters (NCF) and is proved consistent in [8]. For more information about it, see [5, 6].

It is shown in [7] that the statement "there are exactly two near-coherence classes of ultrafilters" follows from the statement "there are simple P_κ-points for two different cardinals κ." The consistency of the latter statement is the content of Section 6 of [8], but Dow has found an error in that section. So it is, for the time being, an open problem whether it is consistent to have exactly two near-coherence classes of ultrafilters. Shelah has proposed a new construction of a model with simple $P_{\aleph_1}$-points and $P_{\aleph_2}$-points, but the details of the construction remain to be written down carefully and checked. If correct, it will restore the previously believed result that there can be exactly two near-coherence classes of ultrafilters.

For all cardinals κ strictly between 2 and $2^{\mathfrak{c}}$, it has always been an open question whether there could be exactly κ near-coherence classes of ultrafilters. In this paper, we present the first negative result about this question, eliminating all infinite cardinals except $2^{\mathfrak{c}}$. In particular, the number of near-coherence classes of ultrafilters cannot be $\aleph_0$ or $\mathfrak{c}$.

Our principal result is:

Theorem 1 *If there are infinitely many near coherence classes of ultrafilters, then $\beta\omega$ contains a closed subset C consisting of $2^{\mathfrak{c}}$ pairwise non-nearly-coherent ultrafilters. Consequently, the number of near-coherence classes of ultrafilters is either finite or equal to $2^{\mathfrak{c}}$. This number is equal to $2^{\mathfrak{c}}$ if $\mathfrak{u} \geq \mathfrak{d}$.*

Remark 2 Mioduszewski [15, 16] has established that the number of near-coherence classes of ultrafilters is the same as the number of composants of the indecomposable continuum $\mathbb{R}_+^* = \beta\mathbb{R}_+ - \mathbb{R}_+$, the Stone-Čech remainder of a closed half-line $\mathbb{R}_+ = [0, \infty)$. (See also [6] for a discussion and a proof with less machinery.) Thus, our result implies that the number of composants of $\mathbb{R}_+^*$ is either finite or $2^{\mathfrak{c}}$. In particular, $\mathbb{R}_+^*$ cannot have exactly $\aleph_0$ or $\mathfrak{c}$

composants. Moreover, if the number of composants is infinite then there is a closed subset $\mathsf{C} \subset \mathbb{R}^*_+$ of size $2^{\mathfrak{c}}$ having at most one-point intersection with each composant. This result can be viewed as an analog of Mazurkiewicz's classical theorem [14] that a non-degenerate metrizable indecomposable continuum has an uncountable closed subset containing at most one point from each composant.

The proof of Theorem 1 is rather lengthy and requires some preparatory work. We shall give different proofs for the cases $\mathfrak{u} \geq \mathfrak{d}$ (Section 3) and $\mathfrak{u} < \mathfrak{d}$ (Section 4). Section 2 reviews some known results that we shall need and obtains some immediate consequences of them. Section 5 gives some additional information, under the assumption $\mathfrak{u} > \mathfrak{d}$, about the closed set whose existence Theorem 1 asserts. Finally, Section 6 presents some problems that remain open.

2 Preliminaries

This section is a review of known information that will be needed in our proofs.

2.1 Cardinals

We shall need three of the standard cardinal characteristics of the continuum, in addition to the cardinality $\mathfrak{c}$ of the continuum already used in the introduction.

The *dominating number*, $\mathfrak{d}$, is defined as the smallest cardinality of any family $\mathcal{D}$ of functions $\omega \to \omega$ such that every function $\omega \to \omega$ is eventually majorized by some member of $\mathcal{D}$. The *ultrafilter number*, $\mathfrak{u}$, is defined as the smallest size of any base for an ultrafilter. The *unsplitting number* $\mathfrak{r}$, sometimes called the *refining* or *reaping number*, is the smallest cardinality of a family $\mathcal{R}$ of infinite subsets of ω that is unsplittable in the sense that, for any $S \subseteq \omega$, there is some $R \in \mathcal{R}$ with either $R - S$ or $R \cap S$ finite, i.e., R is almost included in S or in $\omega - S$.

It is easy to see that all of these cardinals are between $\aleph_1$ and $\mathfrak{c}$, inclusive, and that $\mathfrak{r} \leq \mathfrak{u}$ (because any ultrafilter base is unsplittable). We shall need the theorem of Aubrey [1] that $\mathfrak{r} \geq \min\{\mathfrak{u}, \mathfrak{d}\}$. In other words, although each of the inequalities $\mathfrak{r} < \mathfrak{u}$ and $\mathfrak{r} < \mathfrak{d}$ is consistent (the former by [11] and the latter by [8] or [9]), their conjunction is inconsistent. We shall use Aubrey's result in the following form.

Lemma 3 *If* $\mathfrak{u} \geq \mathfrak{d}$ *then also* $\mathfrak{r} \geq \mathfrak{d}$.

If $\mathcal{F}$ is a filter, then we denote by $\chi(\mathcal{F})$ the smallest cardinality of any base for $\mathcal{F}$. Notice that this is also the smallest cardinality of any family of sets that generates $\mathcal{F}$, because closing such a family under finite intersections to produce a base will not increase the cardinality. Notice also that $\mathfrak{u} = \min\{\chi(\mathcal{U}) : \mathcal{U} \text{ an ultrafilter}\}$.

2.2 Topology in $\beta\omega$

The ultrafilters (non-principal and on ω, as always) can be identified with the points of the Stone-Čech remainder $\omega^* = \beta\omega - \omega$ of the discrete space ω. The topology of ω^* has as a basis of open sets (and also a basis of closed sets) the sets of the form $[A] = \{\mathcal{U} : A \in \mathcal{U}\}$ for infinite $A \subseteq \omega$. These sets $[A]$ are exactly the nonempty clopen subsets of ω^*. The nonempty closed subsets of ω^* are exactly those of the form $[\mathcal{F}] = \{\mathcal{U} : \mathcal{F} \subseteq \mathcal{U}\}$ for filters $\mathcal{F}$. The smallest number $\chi(\mathcal{F})$ of generators for $\mathcal{F}$ is also the smallest number of open sets whose intersection is $[\mathcal{F}]$.

By a *discrete sequence* (in ω^*), we mean a sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of distinct ultrafilters whose range is a discrete set. Equivalently, for each $n \in \omega$, there is a set that belongs to $\mathcal{U}_n$ but not to $\mathcal{U}_m$ for any $m \neq n$.

Because ω^* is a compact Hausdorff space, any sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ has a limit along any ultrafilter $\mathcal{V}$. We denote the limit by $\mathcal{V}\text{-}\lim_n \mathcal{U}_n$; this ultrafilter consists of those $A \subseteq \omega$ for which $\{n \in \omega : A \in \mathcal{U}_n\} \in \mathcal{V}$.

The following result of Rudin [19, Lemma 2] will be important in our proofs.

Lemma 4 *If* $\langle \mathcal{U}_n : n \in \omega \rangle$ *is a discrete sequence of ultrafilters, then distinct ultrafilters* $\mathcal{V}$ *yield distinct limits* $\mathcal{V}\text{-}\lim_n \mathcal{U}_n$.

Since any infinite set in ω^* (or indeed in any regular space) has an infinite discrete subset, it follows that every infinite closed subset of ω^* has the same cardinality as the whole space ω^*. By a theorem of Pospíšil [17], that cardinality is $2^{\mathfrak{c}}$. We summarize for future reference:

Lemma 5 *If a filter* $\mathcal{F}$ *is included in infinitely many ultrafilters, then it is included in* $2^{\mathfrak{c}}$ *ultrafilters, and these constitute a closed subset of* ω^*.

In connection with this result, it will be useful to also have the following information about the contrary case.

Lemma 6 *If a filter $\mathcal{F}$ is included in only finitely many ultrafilters, then each of those ultrafilters $\mathcal{U}$ has $\chi(\mathcal{U}) \leq \chi(\mathcal{F})$.*

Proof As $[\mathcal{F}]$ is finite, each of its points $\mathcal{U}$ is isolated in $[\mathcal{F}]$. This means that there is a set $A \subseteq \omega$ such that $\mathcal{U}$ is the only ultrafilter extending $\mathcal{F}$ and containing A. Then $\mathcal{U}$ is generated by A together with any system of generators for $\mathcal{F}$. □

If f is any finite-to-one map $\omega \to \omega$, then the induced map $\mathcal{U} \mapsto f(\mathcal{U})$ of ultrafilters is a continuous function from ω^* to itself. In particular,

$$f(\mathcal{V}\text{-}\lim_n \mathcal{U}_n) = \mathcal{V}\text{-}\lim_n f(\mathcal{U}_n).$$

2.3 P-points

An ultrafilter $\mathcal{U}$ is called a *P-point* if, whenever $\{A_n : n \in \omega\}$ is a countable family of elements of $\mathcal{U}$, then there is some $B \in \mathcal{U}$ that is almost included in each of them, i.e., $B - A_n$ is finite for every n. Such a B is called a *pseudointersection* of the A_n's.

A P-point $\mathcal{U}$ cannot be in the closure of a countable set of other ultrafilters in ω^*. Indeed, we could find, for each of those other ultrafilters, a set in $\mathcal{U}$ but not in the other ultrafilter. A psudointersection of those sets in $\mathcal{U}$ gives a neighborhood of $\mathcal{U}$ containing none of the countably many other ultrafilters.

If $\mathcal{U}$ is a P-point, then so is $f(\mathcal{U})$ for any finite-to-one f.

Shelah has shown [23] that the existence of P-points is not provable in ZFC, though it is a consequence of CH ([21]). We shall need the following lemma and corollary, due to Ketonen [12] though a version of the lemma was already in [18].

Lemma 7 *Suppose $\mathcal{F}$ is a filter with $\chi(\mathcal{F}) < \mathfrak{d}$, and suppose we are given a decreasing ω-sequence of sets $A_0 \supseteq A_1 \supseteq \dots$ such that each A_n intersects every set in $\mathcal{F}$. Then the A_n's have a pseudointersection B that also intersects every set in $\mathcal{F}$.*

In the special case where $\mathcal{F}$ is an ultrafilter, so that "intersects every set in $\mathcal{F}$" is synonymous with "is in $\mathcal{F}$," the lemma reduces to the following.

Corollary 8 *Every ultrafilter $\mathcal{U}$ with $\chi(\mathcal{U}) < \mathfrak{d}$ is a P-point.*

It is well known that, if $\mathcal{U}$ is a P-point and f is a one-to-one function from ω into ω^* (or into any compact Hausdorff space), then there is a set $A \in \mathcal{U}$ whose image $f(A)$ is discrete. We shall need the following slightly more general result, whose proof, though essentially the same as for the result just quoted, we give for the sake of completeness.

Lemma 9 *Suppose X is a compact Hausdorff space, $f : \omega \to X$ is a one-to-one map, and $\mathcal{U}$ is an ultrafilter. Then there is a decreasing ω-sequence $A_0 \supseteq A_1 \supseteq \dots$ of sets in $\mathcal{U}$ such that, if $B \subseteq A_0$ is a pseudointersection of the A_n's, then $f(B)$ is discrete and has at most one limit point in the image of f.*

Proof Since X is a compact Hausdorff space, the sequence f has a limit x with respect to $\mathcal{U}$. For each $k \in \omega$ choose a closed neighborhood N_k of x that does not contain $f(k)$ unless $f(k) = x$. (Note that the exceptional situation $f(k) = x$ arises for at most one k.) Define

$$A_n = \bigcap_{k \leq n} f^{-1}(N_k - \{x\}).$$

Clearly, these A_n's form a decreasing sequence of sets in $\mathcal{U}$. Let B be any pseudointersection of them. To finish the proof, we show that no $f(k)$ is a limit point of $f(B)$ unless $f(k) = x$. Indeed, if $f(k) \neq x$ then $f(k) \notin N_k$. As N_k is closed, $X - N_k$ is a neighborhood of $f(k)$. As B is a pseudointersection of the A_n's, there are only finitely many elements $m \in B$ for which $f(m) \in X - N_k$, so $f(k)$ cannot be a limit point of $f(B)$. □

2.4 Discrete and ω^*-discrete ultrafilters

Baumgartner generalized the notion of P-point to the notion of discrete ultrafilter. We shall need the further generalization to the notion of Y-discrete ultrafilter, defined as follows.

Definition 10 An ultrafilter $\mathcal{U}$ is a *discrete ultrafilter* if for any one-to-one map $f : \omega \to \mathbb{R}$ there is a set $A \in \mathcal{U}$ such that $f(A)$ is discrete. More generally, for any topological space Y, an ultrafilter $\mathcal{U}$ is *Y-discrete* if, whenever $f : \omega \to Y$ is one-to-one, the image of some $U \in \mathcal{U}$ is discrete.

Thus, “discrete” is the same as $\mathbb{R}$-discrete. We shall be interested primarily in ω^*-discreteness.

It follows immediately from Lemma 9 that all P-points are discrete ultrafilters. We shall need the fact that the converse fails, unless both classes are empty; see [3, Corollary 2.9].

Lemma 11 *If there exists a P-point then there also exists a discrete ultrafilter $\mathcal{U}$ that is not a P-point.*

The following lemma will allow us to transfer information about discreteness to the more general notion of Y-discreteness, in particular for $Y = \omega^*$.

Proposition 12 *Every discrete ultrafilter is ω^*-discrete and in fact Y-discrete for every functionally Hausdorff space Y.*

A space is called functionally Hausdorff if its points can be separated by real-valued continuous functions.

Proof Since we shall need this proposition only for the ω^* case, we prove this case in detail and then indicate briefly the proof for arbitrary functionally Hausdorff spaces.

Let $f : \omega \to \omega^*$ be a one-to-one map. There is a continuous function g from ω^* to the product $\{0,1\}^\omega$ of discrete two-point spaces, such that the restriction of g to $f(\omega)$ is one-to-one. To construct such a g, it suffices to choose countably many sets $A_k \subseteq \omega$ such that, for each pair $m \neq n$ in ω, there is at least one A_k that belongs to $f(m)$ but not to $f(n)$. Then define the k^{th} component of $g(\mathcal{V})$, for any ultrafilter $\mathcal{V}$, to be 1 or 0 according to whether A_k is or is not in $\mathcal{V}$. Since $\{0,1\}^\omega$ is homeomorphic to the Cantor set $\subseteq \mathbb{R}$, any discrete ultrafilter contains an X such that $g(f(X))$ is discrete. But any space with a one-to-one continuous map to a discrete space is itself discrete, so $f(X)$ is discrete.

The argument with Y in place of ω^* is similar. The function g will now map Y into $\mathbb{R}^\omega$, and its components are provided by the assumption that Y is functionally Hausdorff. Unlike the Cantor set above, this $\mathbb{R}^\omega$ cannot be regarded as a subspace of $\mathbb{R}$, but all we really need to embed into $\mathbb{R}$ is the range of $g \circ f$. As a countable, metrizable space, this can be embedded in $\mathbb{R}$, and then the proof can be completed as above. □

2.5 Testing near coherence

The definition of near-coherence says, for ultrafilters $\mathcal{U}$ and $\mathcal{V}$, that $f(\mathcal{U}) = f(\mathcal{V})$ for some finite-to-one $f : \omega \to \omega$. We shall need to know that, under certain circumstances, the search for such an f need not extend to all finite-to-one functions.

Definition 13 Let $\mathcal{F}$ be a filter. A family $\mathcal{T}$ of finite-to-one functions $\omega \to \omega$ is called a *test family over* $\mathcal{F}$ if, whenever two ultrafilters $\mathcal{U}$ and $\mathcal{V}$ in $[\mathcal{F}]$ are nearly coherent, then there is an $f \in \mathcal{T}$ with $f(\mathcal{U}) = f(\mathcal{V})$. When $\mathcal{F}$ is the filter of cofinite sets, then we abbreviate "test family over the cofinite filter" to "test family."

We shall need the following result, which is Lemma 10 of [5].

Lemma 14 *There is a test family of cardinality* $\mathfrak{d}$.

In Section 4, we shall obtain even smaller test families over certain filters under the hypothesis $\mathfrak{u} < \mathfrak{d}$. The use of test families in our proofs will be by way of the following main lemma.

Lemma 15 *Assume*

- $\mathcal{F}$ *and* $\mathcal{G}$ *are filters,*
- $\mathcal{U}_n$ *are ultrafilters extending* $\mathcal{G}$*, for* $n \in \omega$*,*
- $\mathcal{T}$ *is a test family over* $\mathcal{G}$*, and*
- *for each* $f \in \mathcal{T}$*, there is* $A \in \mathcal{F}$ *for which the sequence* $\langle f(\mathcal{U}_n) : n \in A \rangle$ *is discrete.*

Then the ultrafilters $\mathcal{V}\text{-}\lim_n \mathcal{U}_n$ *for* $\mathcal{V} \in [\mathcal{F}]$ *are pairwise distinct and not nearly coherent. These ultrafilters* $\mathcal{V}\text{-}\lim_n \mathcal{U}_n$ *form a closed set in* ω^**, whose cardinality is that of* $[\mathcal{F}]$.

Proof For any $f \in \mathcal{T}$, let A be as in the fourth assumption. Then the ultrafilters

$$f(\mathcal{V}\text{-}\lim_n \mathcal{U}_n) = \mathcal{V}\text{-}\lim_n f(\mathcal{U}_n)$$

for varying $\mathcal{V} \in [\mathcal{F}]$ are distinct by Lemma 4, because $\langle f(\mathcal{U}_n) : n \in A\rangle$ is discrete and $A \in \mathcal{F} \subseteq \mathcal{V}$. Since $\mathcal{T}$ is a test family over $\mathcal{G}$, and all the ultrafilters $\mathcal{V}\text{-}\lim_n \mathcal{U}_n$ (for $\mathcal{V} \in [\mathcal{F}]$) extend $\mathcal{G}$, we have that these ultrafilters are pairwise distinct and not nearly coherent. Furthermore, these ultrafilters are exactly the images of the ultrafilters $\mathcal{V} \in [\mathcal{F}]$ under the continuous extension $\beta\omega \to \beta\omega$ of the map $\omega \to \beta\omega$ sending each n to $\mathcal{U}_n$. So they constitute the image of the compact set $[\mathcal{F}]$ under a continuous map, and such an image is, of course, closed. Finally, the lemma's assertion about cardinality follows from the previous assertion of pairwise distinctness. □

3 Proof When $\mathfrak{u} \geq \mathfrak{d}$

This section is devoted to the proof of Theorem 1 under the assumption that $\mathfrak{u} \geq \mathfrak{d}$. Notice that in this situation the theorem's last sentence means that the hypothesis in the first sentence, that there are infinitely many near-coherence classes of ultrafilters, must be proved rather than being assumed. The following proposition summarizes the part of the theorem to be proved in this section.

Proposition 16 *Assume $\mathfrak{u} \geq \mathfrak{d}$. There is an infinite closed set $\mathsf{C} \subseteq \omega^*$ no two of whose elements are nearly coherent. Thus, the number of near-coherence classes of ultrafilters is $2^{\mathfrak{c}}$.*

Proof By Lemma 14, fix a test family $\mathcal{T}$ of cardinality $\mathfrak{d}$. We intend to construct ultrafilters $\mathcal{U}_n$ for $n \in \omega$ so that the assumptions of Lemma 15 are satisfied by the $\mathcal{U}_n$'s, $\mathcal{T}$, and the filter of cofinite sets (in the roles of both the $\mathcal{F}$ and the $\mathcal{G}$ of the lemma). In fact, we shall arrange that the whole sequence $\langle f(\mathcal{U}_n) : n \in \omega\rangle$ is discrete for every $f \in \mathcal{T}$. Then Lemma 15 will complete the proof of the proposition.

The ultrafilters $\mathcal{U}_n$ will be constructed simultaneously in a transfinite induction of length $\mathfrak{d}$, in which the stages are indexed by pairs $(f, k) \in \mathcal{T} \times \omega$. Since there are only $\mathfrak{d}$ of these pairs, such an indexing is possible. At each stage, we shall have constructed filters $\mathcal{G}_n$, each of which will be included in the corresponding $\mathcal{U}_n$. Although $\mathcal{G}_n$ will change — in fact grow — from one stage to the next, it will not be necessary to clutter the notation with an explicit mention of the stage. From each stage to the next, at most one new generator will be added to each $\mathcal{G}_n$, so that, since the induction has length

$\mathfrak{d}$, we shall have $\chi(\mathcal{G}_n) < \mathfrak{d}$ at each stage. Indeed, since only countably many generators are added at any one stage (at most one new generator for each $\mathcal{G}_n$), there will be, at each stage, a set of size $< \mathfrak{d}$ that contains generators for all the $\mathcal{G}_n$. (In fact, this also follows directly from $\chi(\mathcal{G}_n) < \mathfrak{d}$, since $\mathfrak{d}$ cannot have cofinality ω.)

At the initial stage of the induction, we let each $\mathcal{G}_n$ be the filter of cofinite sets. At limit stages, we take, for each n, the union of the filters $\mathcal{G}_n$ from all the previous stages. It remains to describe the successor stages of the induction, so consider the step from the stage indexed by (f, k) to the next stage, and consider the filters $\mathcal{G}_n$ produced by all the previous stages. As noted above, there is a set $\mathcal{X}$ of size $< \mathfrak{d}$ containing generators for all the $\mathcal{G}_n$. Since we are assuming $\mathfrak{u} \geq \mathfrak{d}$, Lemma 3 implies that there is a set $S \subseteq \omega$ such that both $f(X) \cap S$ and $f(X) - S$ are infinite for all $X \in \mathcal{X}$. This means that we can adjoin $f^{-1}(S)$ as a new generator to $\mathcal{G}_k$ and adjoin $\omega - f^{-1}(S)$ as a new generator to all $\mathcal{G}_n$ for $n \neq k$ and still have filters, i.e., the new generators do not destroy the finite intersection property. Adjoining these generators gives the filters $\mathcal{G}_n$ for the next stage. Now, with these updated filters, we have that $f(\mathcal{G}_k)$ contains a set, namely S, whose complement is in $f(\mathcal{G}_n)$ for all $n \neq k$. Thus, this stage of the construction ensures that, for the ultimately constructed ultrafilters $\mathcal{U}_n \supseteq \mathcal{G}_n$, the point $f(\mathcal{U}_k)$ is isolated in $\{f(\mathcal{U}_n) : n \in \omega\}$.

After all $\mathfrak{d}$ of the stages are completed, take, for each n, the union of the filters $\mathcal{G}_n$ constructed during the induction, and extend this filter (arbitrarily) to an ultrafilter $\mathcal{U}_n$. Since there was a stage for every pair (f, k), we have ensured that, for each $f \in \mathcal{T}$, every element of $\{f(\mathcal{U}_n) : n \in \omega\}$ is isolated, i.e., this set is discrete. Thus, Lemma 15 applies, and the proof of the proposition is complete. □

The following is a corollary not of the proposition but of its proof; we shall obtain an improvement later, in Proposition 24.

Corollary 17 *Assume $\mathfrak{u} \geq \mathfrak{d}$. Let $\mathcal{F}$ be a filter with $\chi(\mathcal{F}) < \mathfrak{r}$. Then there is an infinite closed set C of pairwise not nearly coherent ultrafilters with $\mathsf{C} \subseteq [\mathcal{F}]$. Thus, if a closed set in ω^* is an intersection of fewer than $\mathfrak{r}$ open sets, then it includes a set C as in the proposition.*

Proof Repeat the proof of the proposition, using $\mathcal{F}$ in place of the cofinite filter as the initial $\mathcal{G}_n$ for all n. The induction still works, because we still have, at each stage, a family $\mathcal{X}$ of size $< \mathfrak{r}$ containing generators for all the

$\mathcal{G}_n$. At the end, all the $\mathcal{U}_n$ will, by construction, be in $[\mathcal{F}]$. As $[\mathcal{F}]$ is closed, it will also contain all the limits of the $\mathcal{U}_n$, which, according to the proof of Lemma 15, are the elements of the C that we finally obtain. $\square$

4 Proof When $\mathfrak{u} < \mathfrak{d}$

To prove our main theorem in the case $\mathfrak{u} < \mathfrak{d}$, we shall need three preliminary propositions, which may be of some interest in their own right. Two of them involve the dominating number of a filter, defined as the cofinality of the reduced power of ω. More explicitly:

Definition 18 For any filter $\mathcal{F}$, its *dominating number* $\mathfrak{d}(\mathcal{F})$ is defined as the smallest cardinality of any family $\mathcal{D}$ of functions $\omega \to \omega$ such that every function $\omega \to \omega$ is majorized by some function from $\mathcal{D}$ on some set in $\mathcal{F}$.

Notice that, when $\mathcal{F}$ is the filter of cofinite sets, then $\mathfrak{d}(\mathcal{F})$ is simply $\mathfrak{d}$, and that, for all filters, $\mathfrak{d}(\mathcal{F}) \leq \mathfrak{d}$.

The first of our three preliminary propositions can be deduced from Theorem 5.5.3 and Corollary 10.3.2 of [2], but for convenience we give a self-contained proof here.

Proposition 19 *If a filter $\mathcal{F}$ and an ultrafilter $\mathcal{U}$ are not nearly coherent, then $\mathfrak{d}(\mathcal{F}) \leq \chi(\mathcal{U})$.*

Proof Let $\mathcal{F}$ and $\mathcal{U}$ be as in the hypothesis of the proposition, and fix a base $\mathcal{B}$ for $\mathcal{U}$ with cardinality $|\mathcal{B}| = \chi(\mathcal{U})$. For each $X \in \mathcal{B}$ and each $n \in \omega$, define $\text{next}(X, n)$ to be the first element of X that is $> n$.

Suppose, toward a contradiction, that $\chi(\mathcal{U}) < \mathfrak{d}(\mathcal{F})$. In particular, the functions $\text{next}(X, -)$ for $X \in \mathcal{B}$ do not form a family $\mathcal{D}$ as in the definition of $\mathfrak{d}(\mathcal{F})$. So fix $h : \omega \to \omega$ that is not majorized on any set in $\mathcal{F}$ by any of the functions $\text{next}(X, -)$ for $X \in \mathcal{B}$. Partition ω into intervals $I_0 = [i_0, i_1)$, $I_1 = [i_1, i_2)$, etc., where $0 = i_0 < i_1 < i_2 < \dots$ and where each i_{k+1} is chosen to be larger than $h(n)$ for all $n \leq i_k$. Let $g : \omega \to \omega$ be the function that sends all elements of any interval I_k to k. Then $g(\mathcal{F})$, not being coherent with $g(\mathcal{U})$, cannot be the cofinite filter. Thus, we can find a set $A \in \mathcal{F}$ that is the union of some of the intervals I_k but that omits infinitely many of the I_k.

From the sequence of intervals $\langle I_k \rangle$, extract the subsequence $\langle J_k = [l_k, r_k) : k \in \omega \rangle$ consisting of those intervals that are outside A. Define $f : \omega \to \omega$ to map all elements of $[r_{k-1}, r_k)$ to k (where r_{-1} means 0). Less formally, f takes the value k on J_k, the k^{th} of the intervals omitted by A, and also on the block of I-intervals in A that immediately precede this J_k.

By assumption, $f(\mathcal{F})$ and $f(\mathcal{U})$ are not coherent. So we can find $F \in \mathcal{F}$ and $X \in \mathcal{U}$ with $f(F) \cap f(X) = \varnothing$. Shrinking these sets if necessary, we can assume that $F \subseteq A$ and that $X \in \mathcal{B}$. By our choice of h, there is some $n \in F$ with $\text{next}(X, n) < h(n)$. Letting k be the (unique) index with $n \in I_k$, we have that $I_k \subseteq A$ because $n \in F \subseteq A$. Thus, f takes the same value on I_k and I_{k+1}. Also, since $n \leq i_{k+1}$, we have $h(n) < i_{k+2}$, and so

$$i_k \leq n < \text{next}(X, n) < h(n) < i_{k+2}.$$

Thus, both the element n of F and the element $\text{next}(X, n)$ of X lie in the set $I_k \cup I_{k+1}$ on which f is constant. This contradicts the fact that $f(F)$ and $f(X)$ are disjoint, and so the proof is complete. □

Proposition 20 *Suppose $\mathcal{V}$ is an ultrafilter with $\chi(\mathcal{V}) < \mathfrak{d}$, and suppose $\mathcal{U}_n$, for $n \in \omega$, are ultrafilters that are not nearly coherent with $\mathcal{V}$. Then the filter $\bigcap_{n\in\omega} \mathcal{U}_n$ is also not nearly coherent with $\mathcal{V}$.*

Proof Suppose the contrary, and let f be a finite-to-one function such that $f(\bigcap_{n\in\omega} \mathcal{U}_n)$ and $f(\mathcal{V})$ are coherent. Since the latter is an ultrafilter, we have $f(\bigcap_{n\in\omega} \mathcal{U}_n) \subseteq f(\mathcal{V})$. That is, $f(\mathcal{V})$ belongs to $[f(\bigcap_{n\in\omega} \mathcal{U}_n)]$, which is the closure of the set $\{f(\mathcal{U}_n) : n \in \omega\}$ in ω^*. But by Corollary 8, we know that $f(\mathcal{V})$ is a P-point, so it cannot be in the closure of a countable set of other ultrafilters. Therefore $f(\mathcal{V}) = f(\mathcal{U}_n)$ for some n, contrary to the assumption that no $\mathcal{U}_n$ is nearly coherent with $\mathcal{V}$. □

Proposition 21 *For any filter $\mathcal{F}$, there is a test family of cardinality $\mathfrak{d}(\mathcal{F})$ over $\mathcal{F}$.*

Proof We first handle the case that, for some finite-to-one $f : \omega \to \omega$, $f(\mathcal{F})$ is the cofinite filter. In this case, we have $\mathfrak{d}(\mathcal{F}) = \mathfrak{d}$. Indeed, if $\mathcal{D}$ is as in the definition of $\mathfrak{d}(\mathcal{F})$, then it is straightforward to check that the family of functions

$$x \mapsto \max\{g(y) : f(y) = x\}$$

for $g \in \mathcal{D}$ is a dominating family. Since there always exists a test family of size $\mathfrak{d}$, and since a test family serves as a test family over any filter, the proposition is proved in this case.

Assume from now on that no finite-to-one function sends $\mathcal{F}$ to the cofinite filter. Fix a family $\mathcal{H}$ of functions $\omega \to \omega$ such that $|\mathcal{H}| = \mathfrak{d}(\mathcal{F})$ and such that every function $\omega \to \omega$ is majorized, on some set in $\mathcal{F}$, by some $h \in \mathcal{H}$.

For each $h \in \mathcal{H}$, proceed as in the proof of Proposition 19 to construct a partition of ω into intervals I_k, a function g, a set $A \in \mathcal{F}$, a subsequence of intervals J_k, bigger intervals $[r_{k-1}, r_k)$, and the function f that is constant on just these bigger intervals. The key property of f that we shall need is that, whenever $a \leq b \leq h(a)$ and $a \in A$, then $f(a) = f(b)$. To verify this property, first use the definition of the intervals I_k to see that a and b are either in the same one of these intervals or in consecutive ones. Then use the assumption that $a \in A$ to see that, even if they are in consecutive intervals, the first of these intervals (the one containing a) is not among the J_k's, and so a and b lie in the same $[r_{k-1}, r_k)$. Thus $f(a) = f(b)$.

We have associated to each $h \in \mathcal{H}$ a function f, and we shall show that these f's constitute a test family over $\mathcal{F}$. Since the number of these f's is at most $|\mathcal{H}| = \mathfrak{d}(\mathcal{F})$, this will complete the proof of the proposition.

Suppose, therefore, that $\mathcal{U}$ and $\mathcal{V}$ are nearly coherent ultrafilters in $[\mathcal{F}]$. Fix a finite-to-one function $j : \omega \to \omega$ such that $j(\mathcal{U}) = j(\mathcal{V})$. By the defining property of $\mathcal{H}$, fix some $F \in \mathcal{F}$ and some $h \in \mathcal{H}$ such that, for all $n \in F$, $h(n) > \max(j^{-1}(j([0, n])))$. If we view j as partitioning ω into finite pieces $j^{-1}(n)$, then the requirement on $h(n)$, for $n \in F$, is that it be larger than the right endpoints of all the (finitely many) pieces whose left endpoints are $\leq n$. Let f be the function constructed from h above. We shall complete the proof by showing that $f(\mathcal{U}) = f(\mathcal{V})$.

We may assume, by shrinking F if necessary, that $F \subseteq A$. To prove that $f(\mathcal{U}) = f(\mathcal{V})$ it suffices, since these are ultrafilters, to show that $f(U) \cap f(V) \neq \varnothing$ for all $U \in \mathcal{U}$ and $V \in \mathcal{V}$. So consider any such U and V. Shrinking them if necessary, we may assume, since $F \in \mathcal{F} \subseteq \mathcal{U}, \mathcal{V}$, that $U, V \subseteq F \subseteq A$. As $j(\mathcal{U}) = j(\mathcal{V})$, the sets $j(U)$ and $j(V)$ must intersect. Fix some $a \in U$ and $b \in V$ such that $j(a) = j(b)$. Without loss of generality, suppose $a \leq b$. It follows, by our choice of h, that $b \leq h(a)$. Thanks to the key property of f verified above, we conclude that $f(a) = f(b)$, and the proof is complete. $\square$

After these preparatory results, we are ready to prove the case $\mathfrak{u} < \mathfrak{d}$ of the main theorem; we state the result, with some additional information, as a separate proposition.

Proposition 22 *Assume that $\mathfrak{u} < \mathfrak{d}$ and that $\{\mathcal{U}_n : n \in \omega\}$ is a set of distinct ultrafilters that are pairwise not nearly coherent. Then the closure of $\{\mathcal{U}_n : n \in \omega\}$ in ω^* has an infinite closed subset (hence of size $2^{\mathfrak{c}}$) whose members are pairwise not nearly coherent.*

In contrast to Proposition 16, here we must assume, rather than prove, that infinitely many non-nearly-coherent ultrafilters are given. As is shown in [8], it is consistent to have only a single near-coherence class of ultrafilters (with $\mathfrak{u} < \mathfrak{d}$).

Proof of Proposition Assume the hypotheses. We intend to find filters $\mathcal{F}$ and $\mathcal{G}$ and a family $\mathcal{T}$ of functions satisfying the hypotheses of Lemma 15. Furthermore, we shall ensure that $[\mathcal{F}]$ is infinite. Since the closed set of non-nearly-coherent ultrafilters produced by the lemma consists of ultrafilters of the form $\mathcal{V}\text{-}\lim_n \mathcal{U}_n$, which are in the closure of $\{\mathcal{U}_n : n \in \omega\}$, and since this closed set has the same cardinality as $[\mathcal{F}]$, this will suffice to complete the proof of the proposition.

Let $\mathcal{W}$ be an ultrafilter with $\chi(\mathcal{W}) = \mathfrak{u} < \mathfrak{d}$. Recall that, by Corollary 8, $\mathcal{W}$ is a P-point.

Since the $\mathcal{U}_n$ are pairwise not nearly coherent, at most one of them is nearly coherent with $\mathcal{W}$. Discarding one of the $\mathcal{U}_n$'s, we assume from now on that none of them are nearly coherent with $\mathcal{W}$. Define the filter $\mathcal{G}$ to be the intersection of the $\mathcal{U}_n$'s, and note that, by Proposition 20, $\mathcal{G}$ is not nearly coherent with $\mathcal{W}$. By Proposition 19, it follows that $\mathfrak{d}(\mathcal{G}) \leq \chi(\mathcal{W}) = \mathfrak{u}$. By Proposition 21, fix a test family $\mathcal{T}$ over $\mathcal{G}$ with $|\mathcal{T}| = \mathfrak{u}$. We have defined $\mathcal{G}$ and $\mathcal{T}$ in such a way that the hypotheses of Lemma 15 are satisfied insofar as they do not mention $\mathcal{F}$. It remains to define a filter $\mathcal{F}$ such that, for each $f \in \mathcal{T}$, there is $A \in \mathcal{F}$ for which the sequence $\langle f(\mathcal{U}_n) : n \in A\rangle$ is discrete.

Since $\mathcal{W}$ is a P-point, Lemma 11 gives us a discrete ultrafilter $\mathcal{V}$ that is not a P-point. By Proposition 12, $\mathcal{V}$ is ω^*-discrete, and by Corollary 8 it has character $\chi(\mathcal{V}) \geq \mathfrak{d} > \mathfrak{u}$. We shall use $\mathcal{V}$ to help us build the required $\mathcal{F}$, in a transfinite induction of length $\mathfrak{u}$, constructing an increasing sequence of filters. The stages of the induction will be indexed by the functions $f \in \mathcal{T}$; this can be arranged since $|\mathcal{T}| = \mathfrak{u}$. We begin with the filter of cofinite sets, and at limit stages we form the union of the previously built filters. It remains to describe the successor steps. At each of these steps, we shall add at most one new generator to our filter, and this generator will be a set in $\mathcal{V}$. This ensures that, at each stage during the construction, the current filter is generated by fewer than $\mathfrak{u}$ sets and that the filter obtained at the end of the construction is generated by at most $\mathfrak{u}$ sets and is included in $\mathcal{V}$. Furthermore, by always using sets from the ultrafilter $\mathcal{V}$, we avoid any concern about whether the added sets preserve the finite intersection property and so generate a proper filter; they will automatically do so since $\mathcal{V}$ has the finite intersection property.

Consider the step from the stage indexed by f to the next stage, and let $\mathcal{F}'$ be the filter constructed so far. Consider the ultrafilters $f(\mathcal{U}_n)$. They are distinct, because the $\mathcal{U}_n$ are pairwise not nearly coherent. Since $\mathcal{V}$ is ω^*-discrete, there is a set $A \in \mathcal{V}$ such that $\{f(\mathcal{U}_n) : n \in A\}$ is discrete. Choose one such A and adjoin it to $\mathcal{F}'$ as a new generator. This ensures that, at the end of our induction, $\mathcal{F}$, which is the union of all the filters $\mathcal{F}'$ built during the inductive stages, will satisfy the hypothesis of Lemma 15.

According to that lemma, we obtain, in the closure of $\{\mathcal{U}_n : n \in \omega\}$, a closed set of pairwise not nearly coherent ultrafilters, and this closed set has the same cardinality as $[\mathcal{F}]$. It remains to verify that this cardinality is infinite.

Suppose, toward a contradiction, that $[\mathcal{F}]$ were finite. By Lemma 6, we infer that $\mathcal{V}$, being an ultrafilter extending $\mathcal{F}$, has $\chi(\mathcal{V}) \leq \chi(\mathcal{F}) \leq \mathfrak{u}$, whereas we saw, immediately after choosing $\mathcal{V}$, that its character is $> \mathfrak{u}$. □

Propositions 16 and 22 together complete the proof of Theorem 1.

5 Closed Sets When $\mathfrak{u} > \mathfrak{d}$

Comparing Propositions 16 and 22, we see that the latter has not only an additional hypothesis, namely that infinitely many non-nearly-coherent ultrafilters must be given, but also an additional conclusion, namely that $2^{\mathfrak{c}}$ non-nearly-coherent ultrafilters (in fact an infinite closed set of them) can be found in the closure of the given ultrafilters. It is natural to ask whether this additional conclusion can also be obtained, of course under the same additional hypothesis, when $\mathfrak{u} \geq \mathfrak{d}$. We do not know the answer to this question, but we have an affirmative answer under the stronger assumption that $\mathfrak{u} > \mathfrak{d}$. The result is exactly like Proposition 22 except that the inequality between $\mathfrak{u}$ and $\mathfrak{d}$ is reversed.

Proposition 23 *Assume that* $\mathfrak{u} > \mathfrak{d}$ *and that* $\{\mathcal{U}_n : n \in \omega\}$ *is a set of distinct ultrafilters that are pairwise not nearly coherent. Then the closure of* $\{\mathcal{U}_n : n \in \omega\}$ *in* ω^* *has an infinite closed subset (hence of size* $2^{\mathfrak{c}}$*) whose members are pairwise not nearly coherent.*

Proof Assume the hypotheses of the proposition. As in previous proofs, we shall produce $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{T}$ to satisfy the hypotheses of Lemma 15. Furthermore, we shall ensure that $[\mathcal{F}]$ is infinite. Then Lemma 15 will complete the proof of the proposition. We take $\mathcal{G}$ to be the filter of cofinite sets, and, invoking Lemma 14, we take $\mathcal{T}$ to be a test family of cardinality $\mathfrak{d}$. It remains to construct a filter $\mathcal{F}$ such that $[\mathcal{F}]$ is infinite and, for each $f \in \mathcal{T}$, there is $A \in \mathcal{F}$ for which the sequence $\langle f(\mathcal{U}_n) : n \in A\rangle$ is discrete. As in previous proofs, the construction is a transfinite induction, starting with the cofinite filter, taking unions at limit steps, and adding at most one new generator to our filter at each successor step. There will be $\mathfrak{d}$ steps in the induction, indexed by the elements of $\mathcal{T}$. So at every stage, the part of $\mathcal{F}$ already constructed will be a filter $\mathcal{F}'$ with $\chi(\mathcal{F}') < \mathfrak{d}$. The final result will therefore be a filter $\mathcal{F}$ with $\chi(\mathcal{F}) \leq \mathfrak{d} < \mathfrak{u}$. In view of Lemma 6, it follows that there are infinitely many ultrafilters in $[\mathcal{F}]$.

It remains to carry out the successor step, from the stage indexed by $f \in \mathcal{T}$ to the next stage, adding, to the filter $\mathcal{F}'$ produced by the previous stages, some set A such that $\langle f(\mathcal{U}_n) : n \in A\rangle$ is discrete. To obtain an appropriate A, first apply Lemma 9 to the compact Hausdorff space ω^*, the function $\omega \to \omega^*$ that sends n to $f(\mathcal{U}_n)$, and an arbitrary ultrafilter $\mathcal{V} \supseteq \mathcal{F}'$. The result is a decreasing sequence of sets $A_0 \supseteq A_1 \supseteq \dots$ in $\mathcal{V}$ such that, if $B \subseteq A_0$ is a pseudointersection of the A_n's, then $\langle f(\mathcal{U}_n) : n \in B\rangle$ is discrete. So we need only obtain such a pseudointersection B that can be added to $\mathcal{F}'$, i.e., that meets every set in $\mathcal{F}'$. Noting that each A_n, being in $\mathcal{V}$, meets every set in $\mathcal{F}'$, and remembering that $\chi(\mathcal{F}') < \mathfrak{d}$, we obtain the required B from Lemma 7. □

Using the preceding result, we can improve Corollary 17 as follows, replacing the bound $\mathfrak{r}$ for $\chi(\mathcal{F})$ by the possibly larger cardinal $\mathfrak{u}$.

Proposition 24 *Assume* $\mathfrak{u} \geq \mathfrak{d}$. *Let* $\mathcal{F}$ *be a filter with* $\chi(\mathcal{F}) < \mathfrak{u}$. *Then there is an infinite closed set* C *of pairwise not nearly coherent ultrafilters with* $\mathsf{C} \subseteq [\mathcal{F}]$. *Thus, such a closed set, of size* $2^{\mathfrak{c}}$, *can be found inside any non-empty subset of* ω^* *that is the intersection of fewer than* $\mathfrak{u}$ *open sets.*

Proof Assume that $\mathfrak{u} \geq \mathfrak{d}$ and that $\mathcal{F}$ is as in the prpoposition. To obtain the required C, we may assume $\mathfrak{r} < \mathfrak{u}$, for otherwise we get C immediately from Corollary 17. By Lemma 3 and our assumption that $\mathfrak{u} \geq \mathfrak{d}$, we have $\mathfrak{d} \leq \mathfrak{r} < \mathfrak{u}$. Thus, Proposition 23 applies, and we need only show, given any filter $\mathcal{F}$ with $\chi(\mathcal{F}) < \mathfrak{u}$, that there are infinitely many, pairwise not nearly coherent ultrafilters extending $\mathcal{F}$. It therefore suffices, given any finite number n (possibly zero) of ultrafilters $\mathcal{U}_1, \dots, \mathcal{U}_n$, to find another ultrafilter $\mathcal{U} \supseteq \mathcal{F}$ that is not nearly coherent with any of them.

Let such $\mathcal{U}_i$ be given, and, by Lemma 14, let $\mathcal{T}$ be a test family of size $\mathfrak{d}$. We shall construct the required $\mathcal{U}$ in an induction of length $\mathfrak{d}$, indexed by the functions $f \in \mathcal{T}$. We begin the induction with the given filter $\mathcal{F}$, and at limit stages we take unions. At any successor step, we shall add at most one new generator to the filter under construction. Since $\chi(\mathcal{F}) < \mathfrak{u}$ and since there are only $\mathfrak{d} < \mathfrak{u}$ steps in the induction, we shall have at each stage of the induction a filter generated by fewer than $\mathfrak{u}$ sets.

At the step from the stage indexed by some $f \in \mathcal{T}$ to the next stage, the generator to be added to the current filter, say $\mathcal{G}$, is obtained as follows. Since $\mathcal{G}$ is generated by fewer than $\mathfrak{u}$ sets, so is $f(\mathcal{G})$. By Lemma 6, there are infinitely many ultrafilters extending $f(\mathcal{G})$. Let $\mathcal{V}$ be an ultrafilter extending $f(\mathcal{G})$ and distinct from the finitely many ultrafilters $f(\mathcal{U}_i)$. Let B be a set that is in $\mathcal{V}$ but in none of the $f(\mathcal{U}_i)$. Then adjoin $f^{-1}(B)$ to $\mathcal{G}$ as a new generator.

At the end of the induction, we have produced a filter $\mathcal{G} \supseteq \mathcal{F}$ such that, for each $f \in \mathcal{T}$, none of the $f(\mathcal{U}_i)$ extend $f(\mathcal{G})$, because the latter filter contains the set B obtained during the induction step for f, while all of the former contain $\omega - B$. Thus, we can get the required $\mathcal{U}$ by extending $\mathcal{G}$ arbitrarily to an ultrafilter.

Finally, to prove the last assertion in the proposition, we first observe that it follows from what is already proved if the given set is closed. To establish the general case, let X be a non-empty intersection of fewer than $\mathfrak{u}$ open subsets G_i of ω^*, and let $\mathcal{U}$ be an arbitrary element of X. For each i, choose a basic open set $[A_i]$ such that $\mathcal{U} \in [A_i] \subseteq \mathsf{G}_i$. Since all the A_i are in $\mathcal{U}$, they have the finite intersection property and so generate a filter $\mathcal{F}$. By the part of the proposition already proved, there is an infinite closed set C of pairwise not nearly coherent ultrafilters with

$$\mathsf{C} \subseteq [\mathcal{F}] = \bigcap_i [A_i] \subseteq \bigcap_i \mathsf{G}_i = \mathsf{X}.$$

□

6 Some Open Problems

As indicated before Proposition 23, there is a gap between it and the very similar Proposition 22, namely the case $\mathfrak{u} = \mathfrak{d}$.

Question 25 Given an infinite set of ultrafilters that are pairwise not nearly coherent, must its closure include an infinite closed subset no two of whose members are nearly coherent?

By Propositions 22 and 23, the answer is affirmative as long as $\mathfrak{u} \neq \mathfrak{d}$. Inspecting the proof of Proposition 22, we can see that the answer to Question 25 is also affirmative under the assumption that there is an ω^*-discrete ultrafilter $\mathcal{V}$ with character $\chi(\mathcal{V}) > \min\{\mathfrak{u}, \mathfrak{d}\}$.

In the light of this observation it is interesting to mention that discrete ultrafilters need not exist in ZFC: according to [22] there is a model of ZFC without nowhere dense ultrafilters, i.e., a model where every ultrafilter has an image, under a map $\omega \to \mathbb{R}$, containing no nowhere dense subset of $\mathbb{R}$, and therefore certainly no discrete set.

Question 26 Is the existence of a $\beta\omega$-discrete ultrafilter provable in ZFC?

We may pose this question is a more general form.

Question 27 Describe Tychonov spaces Y for which Y-discrete ultrafilters exist in ZFC. Given such a space Y describe the structure of Y-discrete ultrafilters.

It can be shown that the class of spaces Y described in Question 27 includes all scattered spaces (when Y is scattered, each OK-point, as defined in [13], will be Y-discrete) and excludes all spaces containing a copy of the space $\mathbb{Q}$ of rationals (because $\mathbb{Q}$-discrete ultrafilters are discrete, and thus need not exist in ZFC).

There are natural questions concerning the relation of ω^*-discrete ultrafilters with discrete or nowhere dense ultrafilters; the following seems particularly basic.

Question 28 Is each ω^*-discrete ultrafilter discrete? Is it nowhere dense?

Recall that a *weak P-point* is an ultrafilter that is not in the closure of any countable set of other ultrafilters. Kunen [13] showed (in ZFC) that weak P-points exist. One can show that an ω^*-discrete ultrafilter has the additional "at most one limit point not in the range" property (as in Lemma 9) if and only if it is a weak P-point. Baumgartner's proof of Lemma 11 produces, from any such ultrafilter, another ω^*-discrete ultrafilter that is not a weak P-point. Given this, and given the connection between discrete ultrafilters and P-points, it is natural to ask about further connections.

Question 29 Is there a weak P-point that is not ω^*-discrete? Can an ω^*-discrete ultrafilter be a weak P-point without being a P-point?

The next question is motivated by the circumstance that we proved that the number of near-coherence clases is infinite if $\mathfrak{u} \geq \mathfrak{d}$ (Proposition 16) but had to assume it when $\mathfrak{u} < \mathfrak{d}$ (Proposition 22).

Question 30 Is $\mathfrak{u} < \mathfrak{d}$ provably equivalent to the assertion that there are only finitely many near-coherence classes of ultrafilters?

Of course, Proposition 16 gives the implication in one direction, so the question reduces to asking whether a model of $\mathfrak{u} < \mathfrak{d}$ can have infinitely many near-coherence classes of ultrafilters. The difficulty of answering this question arises from the paucity of known models of $\mathfrak{u} < \mathfrak{d}$. Apart from models of NCF (i.e., models with only one near-coherence class) and the model recently proposed by Shelah to replace the erroneous one in [8, Section 6] (with two near-coherence classes), there are the models constructed in [9]. Michael Canjar (unpublished) has shown that the latter models do not satisfy NCF, but we do not know whether they have infinitely many near-coherence classes.

Finally, we repeat the central open question motivating this work.

Question 31 What cardinals can consistently be the number of near-coherence classes of ultrafilters?

The known results — those quoted in the introduction and those established in this paper — say that the cardinals in question include 1 and $2^{\mathfrak{c}}$ but no other infinite cardinals. Once Shelah's replacement for the model of [8, Section 6] is checked, the cardinal 2 can be added to the list. So the remaining question will concern finite cardinals ≥ 3.

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