

NEARLY COUNTABLE CARDINALS

ANDREAS BLASS

To Professor C. Ryll-Nardzewski on the occasion of his seventieth birthday

One of set theory's first contributions to the rest of mathematics, and still one of the most important, is the distinction between different infinite cardinalities, especially between countable infinity and the cardinality $\mathfrak{c} = 2^{\aleph_0}$ of the continuum. This distinction made possible the theory of Lebesgue measure (where countable additivity is an essential ingredient but continuum additivity is impossible) and the Baire category theorem (where again "countable" clearly cannot be replaced with \mathfrak{c}). Another familiar example is connected with the Bolzano-Weierstraß theorem that every bounded sequence $(x_n)_{n \in \mathbb{N}}$ of real numbers has a convergent subsequence $(x_n)_{n \in A}$. An easy, classical result asserts that for any countably many bounded sequences there is a single A on which they all converge; again this is clearly false in general for \mathfrak{c} sequences.

In these and similar situations, it is reasonable to ask where the transition from countable-like to continuum-like behavior occurs. Of course, if one believes the continuum hypothesis, under which countable infinity \aleph_0 and the continuum \mathfrak{c} are consecutive cardinals, then there is nothing more to be said. But if, as is known to be consistent with the usual axioms of set theory (ZFC), there are cardinals strictly between \aleph_0 and \mathfrak{c} , then it makes sense to ask whether these cardinals behave like \aleph_0 or like \mathfrak{c} with respect to additivity of Lebesgue measure, the Baire category theorem, the Bolzano-Weierstraß theorem, etc. Questions of this sort are studied in the theory of *cardinal characteristics of the continuum*. (I shall comment later on what this theory can tell us when the continuum hypothesis holds.)

Although the title, "Nearly countable cardinals," refers to cardinals that act like \aleph_0 in one way or another, the usual way of indicating the boundary between \aleph_0 -like and \mathfrak{c} -like behavior is by the first cardinal of the latter sort. Here are some typical definitions.

The additivity of measure, $\mathbf{add}(L)$, is the smallest cardinality of a family of sets of Lebesgue measure 0 whose union does not have Lebesgue measure 0. (Note that "does not have Lebesgue measure 0" is not the same as "has positive Lebesgue

Partially supported by NSF grant DMS-9505118.

This paper is based on the survey lecture that I gave at the Wierzba conference in honor of Professor Ryll-Nardzewski's seventieth birthday. I thank the organizers for inviting me to participate in this very pleasant conference.

measure” because the union may fail to be measurable. Note also that, though the definition refers only to measure 0 sets, a more general notion of additivity would give nothing different, for in any family of pairwise disjoint, Lebesgue measurable sets, only countably many have non-zero measure.) The “L” stands for “Lebesgue”; “B” for “Baire” is used when “measure zero” is replaced with “meager” (i.e., “first category”).

The covering number for category, $\mathbf{cov}(B)$, is the minimum number of meager sets needed to cover \mathbb{R} (and $\mathbf{cov}(L)$ is of course the measure analog).

The uniformity number for category, $\mathbf{non}(B)$, is the minimum cardinality of any non-meager set of reals.

The cardinal similarly connected with the Bolzano-Weierstraß theorem is the splitting number \mathfrak{s} , defined as the minimum number of subsets S of \mathbb{N} needed to split every infinite $A \subseteq \mathbb{N}$; here “ S splits A ” means that both $A \cap S$ and $A - S$ are infinite. To see the connection with the Bolzano-Weierstraß theorem, think of the characteristic function of S as a bounded sequence of zeros and ones to find that \mathfrak{s} is the minimum number of such two-valued sequences $(x_n)_{n \in \mathbb{N}}$ such that no single A has all the sequences $(x_n)_{n \in A}$ convergent. It is not difficult to extend this to arbitrary bounded sequences, by considering the binary expansions of the numbers in the sequences.

It may, however, be worth remarking that the transition from two-valued sequences to general bounded sequences is not always so easy as this. Consider the minimum number of infinite sets $A \subseteq \mathbb{N}$ such that every bounded sequence becomes convergent when restricted to one of these A ’s; it is an open problem whether this cardinal is provably unchanged if one replaces “bounded” by “two-valued.”

The dominating number \mathfrak{d} is defined as the minimum number of functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that every $g : \mathbb{N} \rightarrow \mathbb{N}$ is eventually majorized by one of these f ’s (i.e., $g(n) \leq f(n)$ for all but finitely many $n \in \mathbb{N}$, usually abbreviated $g \leq^* f$).

The bounding number \mathfrak{b} is defined as the minimum number of functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that no single g eventually majorizes them all. That is, for each $g : \mathbb{N} \rightarrow \mathbb{N}$, one of the chosen f ’s satisfies $g(n) \leq f(n)$ for infinitely many $n \in \mathbb{N}$.

For each of these cardinals, it is easy to see that it is at least \aleph_1 and at most \mathfrak{c} . Each is the first cardinal behaving like \mathfrak{c} rather than like \aleph_0 in one or another sense. (One uses the first of the larger cardinals rather than the last of the smaller ones, because the former always exists, thanks to the cardinals’ being well-ordered, while the latter may fail to exist.) A great many such cardinals have been defined; see [14] for a recent survey and references to the earlier literature.

Very little can be said about any one of these cardinals κ . In each case, it is consistent with ZFC that any combination of equality and strict inequality holds in $\aleph_1 \leq \kappa \leq \mathfrak{c}$. The situation changes, however, when we consider two (or more) of these cardinals together, for there are often provable (non-strict) inequalities (or subtler connections) between them. (Note that there cannot be provable strict inequalities, for the continuum hypothesis, which is consistent with ZFC, makes all these cardinal characteristics equal.) Some of these inequalities are trivial; for example, $\mathbf{add} \leq \mathbf{cov}$, $\mathbf{add} \leq \mathbf{non}$ (for both measure and category), and $\mathfrak{b} \leq \mathfrak{d}$. Others are not difficult but require a clever idea. Examples of this sort are

Rothberger's theorems [11] that $\mathbf{cov}(B) \leq \mathbf{non}(L)$ and $\mathbf{cov}(L) \leq \mathbf{non}(B)$ and the theorem of Miller [9] and Truss [13] that $\mathbf{add}(B) = \min\{\mathbf{cov}(B), \mathfrak{b}\}$. This category also includes the (folklore) result that $\mathfrak{s} \leq \mathfrak{d}$, whose proof I'll give below. Finally, as an example of a more difficult result, let me cite the theorem, found independently by Bartoszyński [1] and Raisonier and Stern [10] that $\mathbf{add}(L) \leq \mathbf{add}(B)$. (The reverse inequality is not provable, so this is an asymmetry between measure and category.)

For future reference, let me outline the proof that $\mathfrak{s} \leq \mathfrak{d}$. It involves constructing, from any $f : \mathbb{N} \rightarrow \mathbb{N}$, a subset S_f of \mathbb{N} such that, if we begin with a dominating family \mathcal{D} of f 's, then the resulting S_f 's form a splitting family. This construction is as follows. Given f , partition \mathbb{N} into finite intervals $[a, b]$ such that $f(a) < b$; then let S_f be the union of every second of these intervals. To show that, when \mathcal{D} is a dominating family, then $\{S_f \mid f \in \mathcal{D}\}$ is a splitting family, we use a second construction, sending any infinite $A \subseteq \mathbb{N}$ to the function $\nu_A : \mathbb{N} \rightarrow \mathbb{N}$ that sends any $k \in \mathbb{N}$ to the next element of A after k . Then it is easy to check that if $\nu_A \leq^* f$ then S_f splits A (because A meets all but finitely many of the intervals used in defining S_f). This implication immediately shows that the first construction sends dominating families to splitting families, so $\mathfrak{s} \leq \mathfrak{d}$.

Many (though not all) cardinal characteristics of the continuum, like those defined above, are of the form: the smallest number of ... such that every ... is related to one of them by the relation ... Many (though not all) proofs of inequalities between these cardinals can be presented as "two constructions and an implication," like the constructions $f \mapsto S_f$ and $A \mapsto \nu_A$ and the implication "if $\nu_A \leq^* f$ then S_f splits A " in the proof above. Vojtáš [15] introduced the name "generalized Galois-Tukey connection" for such a pair of functions subject to an implication and studied both the general concept and numerous realizations of it in analysis.

The existence of a generalized Galois-Tukey connection implies an inequality between cardinal characteristics (as in the proof of $\mathfrak{s} \leq \mathfrak{d}$ above), but it seems to contain more information. At one time, I hoped that the generalized Galois-Tukey connections capture the essential content of the proofs of inequalities, in a form that remains non-trivial even when, for example, the continuum hypothesis holds and the inequalities themselves are therefore trivial. Yiparaki [16] showed, however, that the continuum hypothesis also provides generalized Galois-Tukey connections for any two cardinal characteristics of the sort considered here. These connections are, however, highly non-constructive, in contrast to the ones produced in proofs like that above. It turns out that the generalized Galois-Tukey connections produced in any of the usual proofs are pairs of *Borel* functions, and that, even if the continuum hypothesis holds, there cannot be Borel generalized Galois-Tukey connections for those inequalities that hold "only because of the continuum hypothesis," i.e., that are false in other models of set theory (forcing extensions). It therefore seems reasonable to regard the existence of Borel generalized Galois-Tukey connections as the essential content of proofs of cardinal characteristic inequalities. This essential content remains meaningful even in situations, like universes satisfying the continuum hypothesis, where the inequalities themselves become trivial.

There are inequalities that involve three cardinal characteristics. An example, $\mathbf{add}(B) = \min\{\mathbf{cov}(B), \mathfrak{b}\}$, was mentioned earlier. Of course the \leq direction here amounts to two ordinary inequalities, the trivial $\mathbf{add}(B) \leq \mathbf{cov}(B)$ and Miller's nontrivial $\mathbf{add}(B) \leq \mathfrak{b}$. But the \geq direction really involves all three cardinals. The structure of its proof involves several functions, arranged in what Vojtáš calls the max-min diagram; they can be viewed as forming a generalized Galois-Tukey connection between the sets involved in the characteristic $\mathbf{add}(B)$ and a combination, called sequential composition, of the sets involved in the other two characteristics; see [4] for details. The concept of sequential composition used here is not commutative and captures (1) the order in which hypotheses are used in the proofs, (2) the interdependencies of the maps in the max-min diagram, and (3) the order of iterated forcing extensions needed to change these cardinal characteristics. The theory shows that, in some sense, (1), (2), and (3) are different perspectives of the same phenomenon.

Finally, to counteract the impression that cardinal characteristics are related only to analysis and topology, let me describe a rather recently discovered connection between cardinal characteristics and algebra. The algebra involved concerns the group $\Pi = \mathbb{Z}^{\aleph_0}$ of infinite sequences of integers, the group operation being componentwise addition. The “unit vectors” e_n , consisting of a 1 in position n and 0's elsewhere, generate a countable subgroup Σ of Π . Despite the fact that Σ is much smaller than Π , Specker [12] proved that every homomorphism $h : \Pi \rightarrow \mathbb{Z}$ is completely determined by where it sends the e_n 's; furthermore, any such h must send all but finitely many e_n 's to 0. Specker also proved the same results with Π replaced by certain subgroups G . Noticing that all those G 's had, like Π , the cardinality of the continuum, Eda [7] asked whether Π has subgroups G of smaller cardinality than \mathfrak{c} such that every homomorphism $G \rightarrow \mathbb{Z}$ annihilates all but finitely many e_n 's. (The corresponding question for homomorphisms being determined by where they send the e_n 's has a trivial affirmative answer, as one can take $G = \Sigma$.) Eda proved that the answer to his question is independent of the ZFC axioms. To state his result more precisely, let \mathfrak{se} be the smallest cardinality of any subgroup G of Π such that every homomorphism $G \rightarrow \mathbb{Z}$ annihilates all but finitely many e_n 's. Eda's proof of the consistency and independence of $\mathfrak{se} = \mathfrak{c}$ actually showed that $\mathfrak{p} \leq \mathfrak{se} \leq \mathfrak{d}$ (where \mathfrak{p} is a characteristic that I haven't defined because it plays only a minor role here). In [3], I showed that $\mathbf{add}(L) \leq \mathfrak{se} \leq \mathfrak{b}$. The \mathfrak{b} part of this result improves the \mathfrak{d} part of Eda's since $\mathfrak{b} \leq \mathfrak{d}$; the $\mathbf{add}(L)$ part is incomparable with Eda's \mathfrak{p} part since either of these two characteristics can consistently be larger than the other.

The occurrence of $\mathbf{add}(L)$ in this result raises a question: What could Lebesgue measure have to do with these group homomorphisms? The honest answer is “nothing.” But Bartoszyński [2] has given the following combinatorial description of $\mathbf{add}(L)$ that is relevant to the homomorphisms. Define a *slalom* to be a function S assigning to each $n \in \mathbb{N}$ a subset $S(n)$ of \mathbb{N} of cardinality n . We say that S *predicts* a function $f : \mathbb{N} \rightarrow \mathbb{N}$ if $f(n) \in S(n)$ for all but finitely many $n \in \mathbb{N}$. Then $\mathbf{add}(L)$ is minimum size of a family of functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that no single slalom predicts them all.

A different notion of prediction connects more directly to homomorphisms from subgroups of Π to \mathbb{Z} . In this notion, the predictor P predicts for each n a single value (rather than n values as predicted by a slalom), but it is required to make predictions only for infinitely many n (rather than all n) and when predicting a value at n it is allowed to use earlier values of the function being predicted. More formally, such a predictor consists of an infinite subset D of \mathbb{N} and, for each $n \in D$, a function $P_n : \mathbb{N}^n \rightarrow \mathbb{N}$; it predicts f if, for all but finitely many $n \in D$, $f(n) = P_n(f \upharpoonright \{0, 1, \dots, n-1\})$. The *evasion number* ϵ is defined to be the minimum number of f 's such that no single predictor predicts them all; the *linear evasion number* ϵ_l is the same except that one considers predictors where each P_n is a linear function $\mathbb{N}^n \rightarrow \mathbb{Q}$. Connections between these characteristics and $\mathbf{add}(L)$ were used to prove that the latter is a lower bound for \mathfrak{se} . Subsequently, Shelah [6] clarified the connections and produced an exact match (rather than just inequalities) between the algebraic and combinatorial characteristics. He showed that $\mathfrak{se} = \epsilon_l = \min\{\mathfrak{e}, \mathfrak{b}\}$. Meanwhile, Brendle [5], Eisworth (unpublished), and Laflamme [8] have found other algebraic and combinatorial applications of ϵ and related cardinals. Here, as in many earlier situations, problems from quite different parts of mathematics show essential similarities when one reduces them to their combinatorial essence. Remarkably often, when problems are independent of ZFC, that essence can be captured by a cardinal characteristic and connections between problems can be captured by equations or inequalities between the corresponding characteristics.

REFERENCES

1. T. Bartoszyński, *Additivity of measure implies additivity of category*, Trans. Amer. Math. Soc. **281** (1984), 209–213.
2. T. Bartoszyński, *Combinatorial aspects of measure and category*, Fund. Math. **127** (1987), 225–239.
3. A. Blass, *Cardinal characteristics and the product of countably many infinite cyclic groups*, J. Algebra **169** (1994), 512–540.
4. A. Blass, *Reductions between cardinal characteristics of the continuum*, BEST Proceedings (T. Bartoszyński and M. Scheepers, ed.), Contemporary Math. series, Amer. Math. Soc. (to appear).
5. J. Brendle, *Evasion and prediction — the Specker phenomenon and Gross spaces*, Forum Math. **7** (1995), 513–541.
6. J. Brendle and S. Shelah, *Evasion and prediction, II*, J. London Math. Soc. (2) **53** (1996), 19–27.
7. K. Eda, *A note on subgroups of $Z^{\mathbb{N}}$* , Abelian Group Theory, Proceedings, Honolulu 1982/83 (R. Göbel, L. Lady, and A. Mader, eds.), Lecture Notes in Mathematics 1006, Springer-Verlag, 1983, pp. 371–374.
8. C. Laflamme, *Combinatorial aspects of F_{σ} filters with an application to \mathcal{N} -sets*, Proc. Amer. Math. Soc. (to appear).
9. A. W. Miller, *Some properties of measure and category*, Trans. Amer. Math. Soc. **266** (1981), 93–114.
10. J. Raisonier and J. Stern, *The strength of measurability hypotheses*, Israel J. Math. **50** (1985), 337–349.
11. F. Rothberger, *Eine Äquivalenz zwischen der Kontinuumshypothese und der Existenz der Lusinschen und Sierpińskischen Mengen*, Fund. Math. **30** (1938), 215–217.
12. E. Specker, *Additive Gruppen von Folgen ganzer Zahlen*, Portugaliae Math. **9** (1950), 131–140.

13. J. Truss, *Sets having calibre \aleph_1* , Logic Colloquium 76 (R.O. Gandy and J.M.E. Hyland, eds.), Studies in Logic and the Foundations of Mathematics, vol. 87, North-Holland, 1977, pp. 595–612.
14. J. Vaughan, *Small uncountable cardinals and topology*, Open Problems in Topology (J. van Mill and G. Reed, eds.), North-Holland, 1990, pp. 195–218.
15. P. Vojtáš, *Generalized Galois-Tukey connections between explicit relations on classical objects of real analysis*, Set Theory of the Reals (H. Judah, ed.), Israel Mathematical Conference Proceedings 6, American Mathematical Society, 1993, pp. 619–643.
16. O. Yiparaki, *On Some Tree Partitions*, Ph.D. thesis, University of Michigan, 1994.

MATHEMATICS DEPT., UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109, U.S.A.
E-mail address: `ablass@umich.edu`