

Nearly Adequate Sets

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ABSTRACT. When a cardinal characteristic of the continuum is defined as the minimum cardinality of any family of reals with a certain property, we call families with this property “adequate” for that characteristic. In many cases, there are weakenings of “adequate” that still imply that the family’s cardinality is at least the characteristic in question. We analyze a few such weakenings. Our main results are partition theorems relating these weakenings to each other or to the original notions of adequacy.

1. Introduction

Many cardinal characteristics of the continuum are defined as the smallest cardinality of any adequate family of reals, where the meaning of “adequate” varies from one characteristic to another. Here are some examples that will play a role in this paper.

DEFINITION 1.1. A family \mathcal{D} of functions $\omega \rightarrow \omega$ is *dominating* if every function $\omega \rightarrow \omega$ is eventually majorized by one from \mathcal{D} . The *dominating number* \mathfrak{d} is the smallest cardinality of any dominating family.

DEFINITION 1.2. A family \mathcal{S} of subsets of ω is *splitting* if every infinite subset X of ω is split by some $S \in \mathcal{S}$, i.e., $X \cap S$ and $X - S$ are both infinite. The *splitting number* \mathfrak{s} is the smallest cardinality of any splitting family.

DEFINITION 1.3. A family \mathcal{B} of subsets of ω is an *ultrafilter base* if its upward closure $\{X \subseteq \omega : (\exists B \in \mathcal{B}) B \subseteq X\}$ is a non-principal ultrafilter on ω . The *ultrafilter number* \mathfrak{u} is the smallest cardinality of any ultrafilter base.

CONVENTION 1.4. All ultrafilters considered in this paper are non-principal. Although we sometimes say “non-principal” for emphasis, we usually omit it.

For more information on cardinal characteristics, see [10, 21, 7].

Pawlikowski and Reclaw [17] introduced a general method of weakening notions of adequacy without changing the corresponding cardinals. They call a set X of reals *small* (with respect to a specified notion of adequacy) if there is no Borel map f such that $f[X]$ is adequate. For our purposes, it is the negation of smallness that

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is important; a set X is not small if and only if it can be mapped onto an adequate set by a Borel function. This property of X is clearly a weakening of adequacy. Yet the corresponding cardinal characteristic is unchanged; the minimum cardinality of a non-small set is the same as the minimum cardinality of an adequate set. (Pawlikowski and Reclaw also consider a variant of smallness, using continuous functions in place of Borel functions. This variant is more sensitive to the details of how objects are coded as reals. For more on the notions of smallness associated to the cardinal characteristics in Cichoń's diagram, see [2, 17].)

We shall present, in Section 2, one result concerning a weakening of this sort, but most of this paper is concerned with a different style of weakening of notions of adequacy. It is not generally applicable to arbitrary notions of adequacy (as non-smallness is), but when it is applicable it seems to lead to quite interesting results. Here are two definitions exhibiting this style of weakening; the first will be used in Section 3 and the second will play a central role in all the later sections.

DEFINITION 1.5. Let \mathcal{U} be an ultrafilter on ω and let $k \in \omega$. A subfamily \mathcal{B} of \mathcal{U} is a k -base for \mathcal{U} if

$$\{B_1 \cap B_2 \cap \dots \cap B_k : \text{All } B_i \in \mathcal{B}\}$$

is a base for \mathcal{U} .

Thus a 1-base is the same as a base. If we did not fix k but only asked that all intersections of finitely many sets from \mathcal{B} form a base, then \mathcal{B} would be what is sometimes called a subbase for \mathcal{U} . Thus, we have the following chain of implications

$$\begin{aligned} \text{base} &\implies \text{2-base} \implies \dots \implies \text{\(k\)-base} \implies \\ &\implies \text{\(k+1\)-base} \implies \dots \implies \text{subbase.} \end{aligned}$$

Notice also that the minimum possible cardinality for k -bases of ultrafilters (for any fixed k) or for subbases of ultrafilters is \mathfrak{u} , because the number of k -fold intersections (or of finite intersections) of sets from an infinite family \mathcal{B} is the same as the cardinality of \mathcal{B} itself.

DEFINITION 1.6. Let $\mathcal{D} \subseteq {}^\omega\omega$ and let $k \in \omega$. Then \mathcal{D} is k -dominating if, for every $g \in {}^\omega\omega$ there are $f_1, \dots, f_k \in \mathcal{D}$ such that, for all but finitely many $n \in \omega$,

$$g(n) \leq \max\{f_1(n), \dots, f_k(n)\}.$$

In other words, the set of functions obtainable as the pointwise maximum of k functions from \mathcal{D} is a dominating family. We call \mathcal{D} *finitely dominating* if every $g \in {}^\omega\omega$ is eventually majorized by the pointwise maximum of some finitely many functions from \mathcal{D} .

Thus, 1-dominating is the same as dominating, and we have the chain of implications

$$\begin{aligned} \text{dominating} &\implies \text{2-dominating} \implies \dots \implies \text{\(k\)-dominating} \implies \\ &\implies \text{\(k+1\)-dominating} \implies \dots \implies \text{finitely dominating} \implies \text{unbounded.} \end{aligned}$$

Again, the minimum size of a k -dominating family (for any fixed k) or of a finitely dominating family is \mathfrak{d} .

The following simple lemma says that the terminology “finitely dominating” is redundant; we retain the terminology because it is occasionally convenient.

LEMMA 1.7. *A family $\mathcal{D} \subseteq {}^\omega\omega$ is finitely dominating if and only if it is k -dominating for some $k \in \omega$.*

Proof The “if” half is obvious; we prove “only if” by contradiction. Suppose therefore that \mathcal{D} is not k -dominating for any k . Choose, for each k , a function $g_k \in {}^\omega\omega$ that is not eventually majorized by the maximum of any k functions from \mathcal{D} . Let g be a function that eventually majorizes all the g_k , for example $g(n) = \max\{g_k(n) : k \leq n\}$. Then g is not eventually majorized by the maximum of any k functions from \mathcal{D} for any k . So g witnesses that \mathcal{D} is not finitely dominating. \square

We conclude this section with some notational conventions and the definitions of three more cardinal characteristics (in addition to the \mathfrak{d} , \mathfrak{s} , and \mathfrak{u} defined above), which will play a role later in this paper.

DEFINITION 1.8. We write $[\omega]^\omega$ for the family of infinite subsets of ω and we write $\omega \nearrow \omega$ for the subset of ${}^\omega\omega$ consisting of the non-decreasing functions from ω to ω . If f eventually majorizes g , i.e., if $f(n) \geq g(n)$ for all but finitely many n , we write $f \geq^* g$ and $g \leq^* f$. If a set X is almost included in another set Y , i.e., if $X - Y$ is finite, we write $X \subseteq^* Y$ and $Y \supseteq^* X$. We use the quantifiers \exists^∞ and \forall^∞ to mean “for infinitely many” and “for all but finitely many,” respectively.

DEFINITION 1.9. A set $\mathcal{B} \subseteq {}^\omega\omega$ is *unbounded* if no single function in ${}^\omega\omega$ eventually majorizes all members of \mathcal{B} . The *bounding number* \mathfrak{b} is the minimum cardinality of an unbounded set.

DEFINITION 1.10. A family $\mathcal{D} \subseteq [\omega]^\omega$ is *dense* if

- whenever $X \in [\omega]^\omega$ and $X \subseteq^* Y \in \mathcal{D}$ then $X \in \mathcal{D}$, and
- for every $Y \in [\omega]^\omega$ there is $X \in \mathcal{D}$ with $X \subseteq Y$.

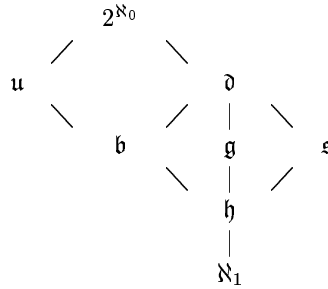
The *distributivity number* \mathfrak{h} is the smallest number of dense families with empty intersection.

DEFINITION 1.11. A family $\mathcal{G} \subseteq [\omega]^\omega$ is *groupwise dense* if

- whenever $X \in [\omega]^\omega$ and $X \subseteq^* Y \in \mathcal{G}$ then $X \in \mathcal{G}$, and
- for every partition of ω into finite intervals, there is an $X \in \mathcal{G}$ that is the union of some (infinitely many) of these intervals.

The *groupwise density number* \mathfrak{g} is the smallest number of groupwise dense families with empty intersection.

The provable (in ZFC) inequalities between these cardinals are given by the following Hasse diagram. A line joining two cardinals means that the lower one in the diagram is provably less than or equal to the upper.



2. Weak Splitting

In the introduction to [1], Bartoszyński pointed out that the small sets with respect to splitting constitute a σ -ideal. Equivalently, if a splitting family is partitioned into countably many pieces, then some piece can be mapped onto a splitting family by a Borel function. We give a slight improvement of this result.

THEOREM 2.1. *If a splitting family is partitioned into fewer than \mathfrak{h} pieces, then one of the pieces can be mapped onto a splitting family by a Borel function.*

Proof Let a splitting family \mathcal{S} be partitioned as $\mathcal{S} = \bigcup_{\alpha < \kappa} \mathcal{S}_\alpha$, where the number of pieces is $\kappa < \mathfrak{h}$. For each $\alpha < \kappa$, define

$$\mathcal{D}_\alpha = \{X \in [\omega]^\omega : \text{No set in } \mathcal{S}_\alpha \text{ splits } X\}.$$

Clearly, $\bigcap_{\alpha < \kappa} \mathcal{D}_\alpha = \emptyset$ because \mathcal{S} is splitting. By definition of \mathfrak{h} , it follows that for at least one α the family \mathcal{D}_α is not dense. Fix such an α and observe that \mathcal{D}_α trivially satisfies the first clause in the definition of dense. So there must be some $Y \in [\omega]^\omega$ such that no subset X of Y is in \mathcal{D}_α . That is, every infinite subset of Y is split by some set in \mathcal{S}_α . If we let $e : \omega \rightarrow Y$ be any bijection, then $\mathcal{S}^* = \{e^{-1}[S] : S \in \mathcal{S}_\alpha\}$ is a splitting family; indeed, if \mathcal{S}^* failed to split some infinite $X \subseteq \omega$ then \mathcal{S}_α would fail to split the infinite subset $e[X]$ of Y , contrary to what we proved above. Since the function $S \mapsto e^{-1}[S]$ is Borel (in fact continuous for the usual topology on the power set of ω) and maps \mathcal{S}_α onto \mathcal{S}^* , the theorem is proved. \square

COROLLARY 2.2. *The cofinality of \mathfrak{s} is at least \mathfrak{h} .*

Proof Otherwise, we could partition a splitting family of size \mathfrak{s} into fewer than \mathfrak{h} pieces, each smaller than \mathfrak{s} . But then the pieces would be too small to map onto a splitting family (by any map, let alone a Borel map), contrary to the theorem. \square

It seems to be unknown whether \mathfrak{s} can be singular.

COROLLARY 2.3. *Suppose \mathcal{X} can be mapped onto a splitting family by a Borel map, and let \mathcal{X} be partitioned into fewer than \mathfrak{h} pieces. Then one of the pieces can be mapped onto a splitting family by a Borel map.*

In other words, the sets that are small with respect to splitting form a $< \mathfrak{h}$ -complete ideal.

Proof Let f be a Borel function mapping \mathcal{X} onto a splitting family, and let \mathcal{X} be partitioned as $\mathcal{X} = \bigcup_{\alpha < \kappa} \mathcal{X}_\alpha$ with $\kappa < \mathfrak{h}$. Then the sets $f(\mathcal{X}_\alpha)$ cover the splitting family $f(\mathcal{X})$. Either shrinking them to make a partition, or noticing that the proof of the theorem works for covers as well as for partitions, we find that some $f(\mathcal{X}_\alpha)$ admits a Borel map, say g , onto a splitting family. But then $g \circ f$ is a Borel map of \mathcal{X}_α onto that splitting family. \square

3. Weak Bases for Ultrafilters

The *character* of an ultrafilter \mathcal{U} is defined to be the minimum cardinality of a base for \mathcal{U} . Thus, \mathfrak{u} is the minimum character of any ultrafilter on ω . It seems to be folklore that characters of ultrafilters on ω have uncountable cofinality. We show here that one argument for this result gives more information, once one introduces the notion of k -base for an ultrafilter.

THEOREM 3.1. *Let \mathcal{B} be a base for an ultrafilter \mathcal{U} on ω , and suppose \mathcal{B} is the union of an increasing sequence of subfamilies,*

$$\mathcal{B} = \bigcup_{n \in \omega} \mathcal{B}_n, \quad \mathcal{B}_0 \subseteq \mathcal{B}_1 \subseteq \dots$$

Then one of the \mathcal{B}_n is a 2-base for \mathcal{U} .

Proof We argue by contradiction, so assume that no \mathcal{B}_n is a 2-base for \mathcal{U} . Define a sequence of natural numbers k_n and a sequence of sets $X_n \in \mathcal{B}$ by simultaneous induction as follows. Begin with $X_0 = \omega$ and $k_0 = 0$. After X_n and k_n are defined, use the fact that \mathcal{B}_{k_n} is not a 2-base of \mathcal{U} to find some set $X \in \mathcal{U}$ that does not include the intersection of any two sets from \mathcal{B}_{k_n} . By shrinking X we may require in addition that $X \subseteq X_n$, that $n \notin X$, and that $X \in \mathcal{B}$. Then let X_{n+1} be this shrunken X . Finally, choose k_{n+1} so that $X_{n+1} \in \mathcal{B}_{k_{n+1}}$.

Notice that $k_0 < k_1 < k_2 < \dots$ and $X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$. In addition, having required $n \notin X_{n+1}$, we have $\bigcap_n X_n = \emptyset$. Thus, we can define a function $\nu : \omega \rightarrow \omega$ by

$$\nu(p) = \text{the smallest } n \text{ with } p \notin X_n.$$

As \mathcal{U} is an ultrafilter, it must contain either $E = \{p : \nu(p) \text{ is even}\}$ or its complement. Suppose it contains E ; the contrary case is handled the same way with “even” replaced by “odd”. So there is a set $A \in \mathcal{B}$ with $A \subseteq E$. Choose an even $n > 0$ so large that $A \in \mathcal{B}_{k_n}$ (note that any sufficiently large even n will do). By definition of k_n , we have $X_n \in \mathcal{B}_{k_n}$. Thus, $X_n \cap A$ is the intersection of two sets from \mathcal{B}_{k_n} and is therefore not a subset of X_{n+1} , by our definition of X_{n+1} .

So let p be an element of $X_n \cap A$ that is not in X_{n+1} . The definition of ν gives that $\nu(p) = n + 1$. But as n is even, this contradicts the fact that $p \in A \subseteq E$. \square

COROLLARY 3.2. *The character of an ultrafilter on ω has uncountable cofinality. In particular, $cf(\mathfrak{u}) > \omega$.*

REMARK 3.3. One can omit “on ω ” from the statement of the corollary. If the ultrafilter in question is countably incomplete, say with $A_n \in \mathcal{U}$ and $\bigcap_n A_n = \emptyset$, then the proof of the theorem still works if we replace the requirement “ $n \notin X_{n+1}$ ” with “ $X_n \subseteq A_n$.” On the other hand, if \mathcal{U} is countably complete, then one of the \mathcal{B}_n is a base, for if there were sets $A_n \in \mathcal{U}$ such that A_n has no subset in \mathcal{B}_n , then $\bigcap_n A_n$ would have no subset in \mathcal{B} , so it would not be a member of \mathcal{U} , and so \mathcal{U} would be countably incomplete.

REMARK 3.4. Since we deal with countable partitions in this section, it may be worth pointing out that finite partitions are trivial here. If a base for an ultrafilter \mathcal{U} is partitioned into finitely many pieces, then one of the pieces is a base for \mathcal{U} . The proof is just like the countably complete case of the preceding remark, with “countable” replaced by “finite.”

It is natural to ask whether Theorem 3.1 can be improved by strengthening “2-base” to “base” in the conclusion. Taking the question literally, we have a negative answer, but a reasonable reinterpretation of the question leads to more interesting results. First, here is the negative answer.

PROPOSITION 3.5. *Any ultrafilter base \mathcal{B} on ω can be expressed as the union of an increasing sequence \mathcal{B}_n such that no \mathcal{B}_n is a base.*

Proof Let \mathcal{B}_n consist of those elements of \mathcal{B} having a member $\leq n$. Since \mathcal{B}_n contains no subset of $\{m \in \omega : m > n\}$, it cannot be a base for a non-principal ultrafilter. \square

This proof is, in a sense, silly because it depends crucially on finite parts of sets in \mathcal{B}_n . If we closed any \mathcal{B}_n from the proof under finite modifications, it would be a base, namely all of \mathcal{B} . And when dealing with non-principal ultrafilters and their bases, it is natural to work modulo finite sets. We therefore make the following definitions to facilitate a “mod finite” formulation of the question.

DEFINITION 3.6. A subfamily \mathcal{B} of an ultrafilter \mathcal{U} on ω is a *base mod finite* (or *k-base mod finite*, or *subbase mod finite*) if for every $X \in \mathcal{U}$ there is $Y \subseteq^* X$ such that Y is in \mathcal{B} (or is the intersection of k sets from \mathcal{B} , or is the intersection of finitely many sets from \mathcal{B} , respectively).

In terms of these “mod finite” notions, we can formulate several reasonable questions about strengthenings of Theorem 3.1. The answers will involve two special classes of ultrafilters, whose definitions we recall next.

DEFINITION 3.7. An ultrafilter \mathcal{U} on ω is a *P-point* if, whenever \mathcal{X} is a countable subfamily of \mathcal{U} then there is $A \in \mathcal{U}$ such that $A \subseteq^* X$ for all $X \in \mathcal{X}$. \mathcal{U} is a *weak P-point* if there do not exist countably many ultrafilters $\mathcal{V}_n \neq \mathcal{U}$ such that every set in \mathcal{U} is also in at least one \mathcal{V}_n .

It is easy to see that every P-point is a weak P-point. In terms of the usual topology on the space ω^* of non-principal ultrafilters on ω , the Stone-Ćech remainder of the discrete space ω , P-points are those points for which every countable intersection of open neighborhoods includes an open neighborhood. Weak P-points are those points that are not in the closure of any countable set of other points.

The existence of P-points cannot be proved in ZFC (see [20]) but it follows from the continuum hypothesis (see [18]). In contrast, the existence of weak P-points is provable in ZFC (see [13]).

The following theorem shows that the question answered negatively by Proposition 3.5 becomes more interesting if “base” is changed to “base mod finite”; it then provides a characterization of P-points.

THEOREM 3.8. *An ultrafilter \mathcal{U} on ω is a P-point if and only if, whenever a base mod finite \mathcal{B} for \mathcal{U} is expressed as the union of a countable increasing sequence of subfamilies \mathcal{B}_n , some \mathcal{B}_n is also a base mod finite for \mathcal{U} .*

Proof Suppose \mathcal{U} is a P-point, $\bigcup_n \mathcal{B}_n$ is a base mod finite for \mathcal{U} , but no \mathcal{B}_n is a base mod finite for \mathcal{U} . The last means that there is, for each n , some $X_n \in \mathcal{U}$ with no almost subset in \mathcal{B}_n . As \mathcal{U} is a P-point, it contains an A almost included in all the X_n 's. But then A has no almost subset in $\bigcup_n \mathcal{B}_n$, contrary to the assumption that this union is a base mod finite for \mathcal{U} . This contradiction proves the “only if” half of the theorem.

For the “if” half, suppose \mathcal{U} is not a P-point. So there are sets $X_n \in \mathcal{U}$, for $n \in \omega$, such that no set in \mathcal{U} is almost included in all the X_n . Let

$$\mathcal{B}_n = \{A \in \mathcal{U} : A \not\subseteq^* X_n\}.$$

Each \mathcal{B}_n fails to be a base mod finite for \mathcal{U} , for it fails to contain an almost subset of X_n . Yet $\bigcup_n \mathcal{B}_n$ is not only a base mod finite for \mathcal{U} , it is all of \mathcal{U} . \square

Another question naturally arising from Theorem 3.1 is whether we really need the \mathcal{B}_n to form an increasing sequence. What if they were a partition of \mathcal{B} instead? Once again, there is a trivial negative answer to the question, even if we weaken the conclusion from “2-base” to “subbase.”

PROPOSITION 3.9. *Any ultrafilter base \mathcal{B} can be partitioned into countably many pieces \mathcal{B}_n , none of which is a subbase.*

Proof Let \mathcal{B}_n consist of those sets in \mathcal{B} whose smallest member is n . The intersections of finitely many sets from \mathcal{B}_n all contain n , so they cannot form a base for a non-principal ultrafilter. \square

But as before, the question becomes interesting if we work mod finite. In the first place, the proof of Theorem 3.8 did not depend on the nesting of the \mathcal{B}_n , so we immediately have the following consequence.

COROLLARY 3.10. *An ultrafilter \mathcal{U} on ω is a P-point if and only if, whenever a base mod finite \mathcal{B} for \mathcal{U} is partitioned into countably many subfamilies \mathcal{B}_n , then some \mathcal{B}_n is also a base mod finite for \mathcal{U} .*

If we weaken the conclusion from “base mod finite” to “subbase mod finite” then we get a characterization of weak P-points.

THEOREM 3.11. *An ultrafilter \mathcal{U} on ω is a weak P-point if and only if, whenever a base mod finite \mathcal{B} for \mathcal{U} is partitioned into countably many subfamilies \mathcal{B}_n , then some \mathcal{B}_n is a subbase mod finite for \mathcal{U} .*

Proof Suppose first that \mathcal{U} is not a weak P-point, and fix a sequence of ultrafilters $\mathcal{V}_n \neq \mathcal{U}$ such that every set in \mathcal{U} is in at least one \mathcal{V}_n . Thus, the sets

$$\mathcal{B}_n = \{X \in \mathcal{U} : n \text{ is the smallest integer with } X \in \mathcal{V}_n\}$$

constitute a partition of \mathcal{U} . Each \mathcal{B}_n , being a subfamily of \mathcal{V}_n , cannot be a subbase for \mathcal{U} .

Conversely, suppose \mathcal{U} is a weak P-point and suppose some base mod finite \mathcal{B} for \mathcal{U} is partitioned into countably many pieces \mathcal{B}_n , none of which is a subbase mod finite for \mathcal{U} . Fix, for each n , a set $A_n \in \mathcal{U}$ that does not almost include any intersection of finitely many sets from \mathcal{B}_n . Then the family $\mathcal{B}_n \cup \{\omega - A_n\}$ has the strong finite intersection property, so it can be extended to an ultrafilter \mathcal{V}_n . Since \mathcal{U} is a weak P-point, let $A \in \mathcal{U}$ be such that A belongs to none of the \mathcal{V}_n . But then, as $\mathcal{B}_n \subseteq \mathcal{V}_n$, A cannot almost include any set from \mathcal{B}_n for any n . This contradicts the fact that $\bigcup_n \mathcal{B}_n = \mathcal{B}$ is a base mod finite for \mathcal{U} . \square

Instead of working modulo finite, we can circumvent the negative result of Proposition 3.9 by looking not at single pieces of the partition but at unions of two pieces.

THEOREM 3.12. *Let \mathcal{B} be a base for an ultrafilter \mathcal{U} on ω , and let \mathcal{B} be partitioned into countably many pieces \mathcal{B}_n . Then among these pieces there are two whose union is a 2-base for \mathcal{U} .*

Proof Suppose the hypotheses of the theorem hold but the conclusion fails. Consider the sets $\mathcal{C}_n = \bigcup_{i \leq n} \mathcal{B}_i$. They form an increasing sequence with union \mathcal{B} . So by Theorem 3.1 we can fix an n such that \mathcal{C}_n is a 2-base for \mathcal{U} .

For each $i \leq j \leq n$, since we are assuming $\mathcal{B}_i \cup \mathcal{B}_j$ is not a 2-base for \mathcal{U} , we can choose a set $A_{ij} \in \mathcal{U}$ with no subset that is an intersection of two sets from $\mathcal{B}_i \cup \mathcal{B}_j$. But then $\bigcap_{i \leq j \leq n} A_{ij}$ is in \mathcal{U} and has no subset that is an intersection of two sets from \mathcal{C}_n . This contradicts the fact that \mathcal{C}_n is a 2-base for \mathcal{U} . \square

4. Weak Dominating

The notion of k -dominating family has already been used in the literature, though not given a name there. One occurrence is connected with the notion of superperfectness.

DEFINITION 4.1. Consider the tree of all finite, non-decreasing sequences of natural numbers. A subtree T of this tree is *superperfect* if it is nonempty and every node of T has an extension with infinitely many immediate extensions in T . A subset of $\omega \nearrow \omega$ is called *superperfect* if it is the set of all (unions of) paths through a superperfect tree.

The following proposition is well known, and a proof can be assembled from results in Laflamme's papers [14, 15], but we give a direct proof for the reader's convenience.

PROPOSITION 4.2. *Every superperfect subset of $\omega \nearrow \omega$ is 2-dominating.*

Proof Let T be a superperfect tree of finite, non-decreasing sequences of natural numbers, and let $f \in \omega \nearrow \omega$. We shall construct two paths, g_0 and g_1 , through T in such a way that $f(n) \leq \max\{g_0(n), g_1(n)\}$ for all sufficiently large n . We shall obtain the g_i 's by defining successive approximations,

$$g_0 \upharpoonright k_0 \subset g_0 \upharpoonright k_2 \subset g_0 \upharpoonright k_4 \subset \dots$$

with union g_0 and

$$g_1 \upharpoonright k_1 \subset g_1 \upharpoonright k_3 \subset g_1 \upharpoonright k_5 \subset \dots$$

with union g_1 . Here the lengths of these approximations are interleaved $k_0 < k_1 < k_2 < k_3 < \dots$. Furthermore, each of our approximations $g_0 \upharpoonright k_{2n}$ and $g_1 \upharpoonright k_{2n+1}$ will be a *split node* of T , i.e., a node t having infinitely many immediate successors $t \frown(j)$ in T . Remember that, being superperfect, T has split nodes beyond any given node.

Begin by defining $g_0 \upharpoonright k_0$ to be an arbitrarily chosen split node and defining $g_1 \upharpoonright k_1$ to be an arbitrarily chosen split node at a higher level in T than $g_0 \upharpoonright k_0$, i.e., $k_0 < k_1$.

Now suppose that the approximations have been defined up to and including $g_0 \upharpoonright k_{2n-2}$ and $g_1 \upharpoonright k_{2n-1}$. Because the former is a split node, it has an immediate successor $g_0 \upharpoonright k_{2n-2} \frown(j)$ with $j > f(k_{2n-1} - 1)$. In view of the monotonicity of f , this means that j is larger than all values of $f \upharpoonright k_{2n-1}$. Using the superperfectness of T , define $g_0 \upharpoonright k_{2n}$ to be split node of T that extends $g_0 \upharpoonright k_{2n-2} \frown(j)$ and is at a higher level than $g_1 \upharpoonright k_{2n-1}$, i.e., $k_{2n} > k_{2n-1}$. Notice that, by monotonicity of the finite sequences that are nodes of T , $g_0(x) > f(x)$ for all x in the range $k_{2n-2} \leq x < k_{2n-1}$.

Next, since $g_1 \upharpoonright k_{2n-1}$ is a split node, it has an immediate successor $g_1 \upharpoonright k_{2n-1} \frown(j)$ with $j > f(k_{2n} - 1)$. In view of the monotonicity of f , this means that j is larger than all values of $f \upharpoonright k_{2n}$. Using the superperfectness of T , define $g_1 \upharpoonright k_{2n+1}$ to be split node of T that extends $g_1 \upharpoonright k_{2n-1} \frown(j)$ and is at a higher level than $g_0 \upharpoonright k_{2n}$,

i.e., $k_{2n+1} > k_{2n}$. Notice that, by monotonicity of the finite sequences that are nodes of T , $g_1(x) > f(x)$ for all x in the range $k_{2n-1} \leq x < k_{2n}$.

This completes the inductive construction of g_0 and g_1 . The ranges on which one or the other dominates f are all the intervals $[k_n, k_{n+1})$, and these cover all natural numbers $\geq k_0$. So $\max\{g_0, g_1\}$ eventually dominates f as required. \square

Kechris [12] and Saint-Raymond [19] independently proved that every unbounded analytic subset of $\omega \nearrow \omega$ has a superperfect subset. Thus, we have the following result, which suggests that, although 2-dominating is defined as a variant of dominating and leads to the same cardinal characteristic \mathfrak{d} , it is surprisingly connected with \mathfrak{b} .

PROPOSITION 4.3. *Every unbounded analytic set in $\omega \nearrow \omega$ is 2-dominating.*

Notice that the use of $\omega \nearrow \omega$ rather than ${}^\omega\omega$ is essential in this proposition. In ${}^\omega\omega$, the functions f that satisfy $f(n) = 0$ for all even n form a closed, unbounded, but not finitely dominating family.

The proposition would also become false if we omitted “analytic” from the hypothesis. The counterexamples showing this will be needed again later, so we introduce a name and notation for them and for some related functions.

DEFINITION 4.4. If \mathcal{F} is a filter on ω , we say that a function $f : \omega \rightarrow \omega$ is \mathcal{F} -bounded if $\{n \in \omega : f(n) \leq n\} \in \mathcal{F}$. We write $\mathcal{B}(\mathcal{F})$ for the collection of all non-decreasing \mathcal{F} -bounded functions.

DEFINITION 4.5. If A is any infinite subset of ω then $\text{next}(A, -)$ is the function on ω defined by

$$\text{next}(A, n) = \text{the smallest element of } A \text{ that is } \geq n.$$

The following lemma, whose straightforward verification we leave to the reader, gives the connection between these definitions.

LEMMA 4.6. *Let \mathcal{F} be a filter on ω . The functions $\text{next}(A, -)$ for $A \in \mathcal{F}$ are \mathcal{F} -bounded, and every \mathcal{F} -bounded function in $\omega \nearrow \omega$ is majorized by $\text{next}(A, -)$ for some $A \in \mathcal{F}$.*

We return to the discussion of Proposition 4.3 minus the hypothesis of analyticity.

PROPOSITION 4.7. *If \mathcal{U} is an ultrafilter then $\mathcal{B}(\mathcal{U})$ is an unbounded subfamily of $\omega \nearrow \omega$ that is not finitely dominating.*

Proof The pointwise maximum of finitely many \mathcal{U} -bounded functions is again \mathcal{U} -bounded. So if $\mathcal{B}(\mathcal{U})$ were finitely dominating then it would be dominating. But in fact it fails to dominate the function $n \mapsto n + 1$.

To see that $\mathcal{B}(\mathcal{U})$ is unbounded, suppose not, and let $f \in \omega \nearrow \omega$ eventually majorize $\text{next}(A, -)$ for all $A \in \mathcal{U}$. Partition ω into consecutive finite intervals $[e_n, e_{n+1})$ such that $f(e_n) < e_{n+1}$. For any $A \in \mathcal{U}$ we have, once n is large enough,

$$\text{next}(A, e_n) \leq f(e_n) < e_{n+1},$$

which means that A meets the interval $[e_n, e_{n+1})$. But then every set $A \in \mathcal{U}$ meets both $\bigcup_{n \text{ even}} [e_n, e_{n+1})$ and its complement, which is absurd as \mathcal{U} is an ultrafilter. \square

Weak domination occurs, without being named, in [16, Theorem 2.2], where a direct proof is given of the result from [9] that $\text{cf}(\mathfrak{d}) \geq \mathfrak{g}$. That proof is based on showing that, whenever a dominating family in ${}^\omega\omega$ is partitioned into fewer than \mathfrak{g} pieces, one of the pieces is finitely dominating. In fact, Mildenerger's argument shows that one of the pieces must be 3-dominating. This can be improved to 2-dominating by slightly modifying the proof; we obtain this result as a corollary to the following slightly more general (but less elegant) result about partitioning k -dominating families. The proof is still based on the same idea as Mildenerger's proof.

THEOREM 4.8. *If a k -dominating family $\mathcal{D} \subseteq {}^\omega\omega$ is partitioned into fewer than \mathfrak{g} pieces, then the union of some k of the pieces is $2k$ -dominating.*

In fact, the proof will establish a bit more about those k pieces, namely that, given any $f \in {}^\omega\omega$, we can choose 2 functions from each of these k pieces in such a way that the maximum of the $2k$ chosen functions eventually majorizes f .

Proof Let the pieces be indexed as \mathcal{D}_i for i ranging over some cardinal $\kappa < \mathfrak{g}$. For each k -tuple $\vec{i} = (i_1, \dots, i_k)$, define

$$\mathcal{G}_{\vec{i}} = \{A \in [\omega]^\omega : \forall f_1 \in \mathcal{D}_{i_1} \forall f_2 \in \mathcal{D}_{i_2} \dots \forall f_k \in \mathcal{D}_{i_k} \\ \exists^\infty n \text{ next}(A, n) \geq \max\{f_1(n), f_2(n), \dots, f_k(n)\}\}.$$

If there were a set $A \in \bigcap_{\vec{i}} \mathcal{G}_{\vec{i}}$ then $\text{next}(A, n)$ would not be eventually majorized by the maximum of any k functions from $\bigcup_i \mathcal{D}_i = \mathcal{D}$, contrary to the hypothesis that \mathcal{D} is k -dominating. So the intersection of all the $\mathcal{G}_{\vec{i}}$ is empty. Since there are fewer than \mathfrak{g} of these families $\mathcal{G}_{\vec{i}}$, fix an $\vec{i} = (i_1, \dots, i_k)$ such that $\mathcal{G}_{\vec{i}}$ is not groupwise dense. Since \mathcal{G}_i obviously satisfies the first clause in the definition of groupwise dense, it must violate the second clause. So fix a partition Π of ω into finite intervals such that no union of infinitely many of these intervals is in $\mathcal{G}_{\vec{i}}$. Notice for future reference that this property of Π would persist if we coarsened Π by merging some consecutive intervals.

Now let an arbitrary $h \in {}^\omega\omega$ be given. We intend to complete the proof by finding two functions from each of the k sets \mathcal{D}_{i_r} , such that the maximum of these $2k$ functions eventually majorizes h .

By merging intervals of Π , we may assume that Π consists of intervals $[p_j, p_{j+1})$ such that $h(n) < p_{j+1}$ for all $n < p_j$, i.e., $h(n)$ is at most one interval beyond n . Now consider the union of every fourth interval of Π ,

$$A = \bigcup_{j \in \omega} [p_{4j}, p_{4j+1}).$$

Being a union of infinitely many intervals from Π , this A cannot be in $\mathcal{G}_{\vec{i}}$. So fix functions $f_1 \in \mathcal{D}_{i_1}, \dots, f_k \in \mathcal{D}_{i_k}$ such that $f = \max\{f_1, \dots, f_k\}$ eventually majorizes $\text{next}(A, -)$. Notice that $\text{next}(A, -)$ majorizes h on all intervals of the form $[p_{4j+1}, p_{4j+3})$. Indeed, for n in such an interval, $\text{next}(A, n) = p_{4j+4}$, while $h(n)$, being at most one interval beyond n , is $< p_{4j+4}$. Thus, we have that f eventually majorizes h on $\bigcup_j [p_{4j+1}, p_{4j+3})$.

Repeat the argument with everything shifted two intervals to the right. That is, let

$$A' = \bigcup_{j \in \omega} [p_{4j+2}, p_{4j+3}),$$

and obtain functions $f'_1 \in \mathcal{D}_{i_1}, \dots, f'_k \in \mathcal{D}_{i_k}$ whose maximum f' eventually majorizes h on $\bigcup_{j>0} [p_{4j-1}, p_{4j+1})$. Then

$$\max\{f, f'\} = \max\{f_1, \dots, f_k, f'_1, \dots, f'_k\}$$

eventually majorizes h , as required. \square

The special case $k = 1$, a very slight improvement of Mildenerger's result from [16] (replacing 3 with 2), is considerably more pleasant than the theorem for general k , so we state it separately.

COROLLARY 4.9. *If a dominating family in ${}^\omega\omega$ is partitioned into fewer than \mathfrak{g} pieces, then one of the pieces is 2-dominating.*

Could the corollary be improved by replacing 2 with 1, i.e., will there always be a dominating piece? The answer is independent of ZFC, because of the following easy observation.

PROPOSITION 4.10. *The smallest cardinal κ such that some dominating family can be partitioned into κ non-dominating pieces is \mathfrak{b} .*

Proof If \mathcal{B} is an unbounded family of size \mathfrak{b} , then we can partition ${}^\omega\omega$ into \mathfrak{b} non-dominating pieces, namely

$$\mathcal{X}_g = \{f \in {}^\omega\omega : f \not\leq^* g\}$$

for $g \in \mathcal{B}$.

Conversely, suppose a dominating family \mathcal{D} is partitioned into κ non-dominating pieces \mathcal{D}_i . Choose, for each i , a function g_i that is not eventually majorized by any member of \mathcal{D}_i . If there were a function f eventually majorizing all the g_i , then f would not be eventually majorized by any member of \mathcal{D} . Since \mathcal{D} is dominating, there is no such f , and therefore the g_i constitute an unbounded family of size (at most) κ . Thus $\kappa \geq \mathfrak{b}$. \square

This proposition implies that the 2 in Corollary 4.9 can be improved to 1 if and only if $\mathfrak{g} \leq \mathfrak{b}$. This inequality is consistent with ZFC, holding for example in all models of Martin's axiom as well as in models obtained by adjoining uncountably many Cohen reals (to any model). But it is also consistent that $\mathfrak{g} > \mathfrak{b}$. This holds for example in the iterated superperfect forcing model (the Miller model); see [5]. Thus, this improved version of Corollary 4.9 is independent of ZFC.

The second half of the proof of the proposition can be routinely extended to k -dominating families as follows; we leave the details to the reader.

PROPOSITION 4.11. *If a k -dominating family is partitioned into fewer than \mathfrak{b} pieces, then the union of some k of the pieces is k -dominating.*

5. Weak Domination If $2^{\aleph_0} = \aleph_1$

Theorem 4.8 suggests several questions about possible improvements. Can we get by with one piece instead of the union of k pieces? Can we improve the conclusion from $2k$ -dominating to k -dominating, so as to match the hypothesis?

For that matter, is there really a difference between the notions of k -dominating for different values of $k > 1$? There is a difference between $k = 1$ and larger k , since superperfect sets are 2-dominating but not always dominating. There is also a difference in general as long as we work in ${}^\omega\omega$. To see this, partition ω into k infinite pieces A_i and let \mathcal{X}_i consist of those functions that are identically zero on

all pieces except A_i (but are arbitrary on A_i). Then $\bigcup_i \mathcal{X}_i$ is k -dominating but not $k-1$ -dominating. Also, it is covered by k subsets \mathcal{X}_i , no union of $k-1$ of which is even finitely dominating.

To avoid simple examples like this, we work primarily in $\omega \nearrow \omega$, the subset of ${}^\omega\omega$ consisting of the non-decreasing functions. Notice that the \mathcal{X}_i example in the preceding paragraph relied on the availability of non-monotone functions; we shall see that nothing similar can be done in $\omega \nearrow \omega$.

In this section, we investigate questions like those posed above, under the assumption of the continuum hypothesis. (Actually, our results work under considerably weaker hypotheses; see below.) The results we obtain here say that the situation is complicated in every respect. In the next section we shall consider a strong form of the negation of CH, namely the hypothesis $\mathfrak{u} < \mathfrak{g}$, and show that it leads to a much simpler picture.

Our results under CH really need only certain weak consequences of CH, which we now review.

DEFINITION 5.1. If f is a function $\omega \rightarrow \omega$ and \mathcal{U} is an ultrafilter on ω then $f(\mathcal{U})$ is the ultrafilter $\{X \subseteq \omega : f^{-1}(X) \in \mathcal{U}\}$. We shall use this construction only when f is finite-to-one; then $f(\mathcal{U})$ is non-principal, so we do not violate our convention that principal ultrafilters are excluded from consideration.

DEFINITION 5.2. Two ultrafilters \mathcal{U} and \mathcal{V} on ω are *nearly coherent* if there is a finite-to-one function $f : \omega \rightarrow \omega$ such that $f(\mathcal{U}) = f(\mathcal{V})$.

CH (or Martin's axiom) implies that there is a family of $2^{2^{\aleph_0}}$ ultrafilters on ω , no two of which are nearly coherent. Indeed, it is well known that under CH or MA there are $2^{2^{\aleph_0}}$ pairwise non-isomorphic selective ultrafilters. No two of these can be nearly coherent, for the only finite-to-one images of a selective ultrafilter (or indeed of any Q-point) are its isomorphic copies.

On the other hand, it is consistent relative to ZFC that every two ultrafilters on ω are nearly coherent. See [5] and the references there.

Our first result under CH shows that the concept of k -dominating gets strictly weaker as k increases. We confine attention to $\omega \nearrow \omega$ to avoid the simple (ZFC) example described above.

THEOREM 5.3. *Assume that there are $k+1$ ultrafilters on ω no two of which are nearly coherent. Then there is a $k+1$ -dominating family in $\omega \nearrow \omega$ that is not k -dominating.*

Proof Let $\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_k$ be $k+1$ ultrafilters such that no two of them are nearly coherent. Let

$$\mathcal{X} = \{f \in \omega \nearrow \omega : \text{For at least } k \text{ values of } i, \{n : f(n) \leq n\} \in \mathcal{U}_i\}.$$

Thus, \mathcal{X} is the union of the $k+1$ sets $\mathcal{B}(\mathcal{F}_i)$ where the filter \mathcal{F}_i is defined as the intersection of all the \mathcal{U}_j except \mathcal{U}_i .

We check first that \mathcal{X} is not k -dominating; in fact, the maximum of k functions f_1, \dots, f_k from \mathcal{X} cannot eventually strictly majorize the identity function. To see this, notice that, for each of our k functions $f_j \in \mathcal{X}$, there is at most one i , call it $i(j)$ if it exists, such that $\{n : f_j(n) \leq n\} \in \mathcal{U}_i$. Since there are $k+1$ possible values for i but only k for j , there is some i that does not occur as $i(j)$ for any j . Then all our f_j satisfy $\{n : f_j(n) \leq n\} \in \mathcal{U}_i$. Therefore, the set $\{n : \max\{f_1(n), \dots, f_k(n)\} \leq n\}$

is the intersection of k sets from \mathcal{U}_i , so it is itself in \mathcal{U}_i and in particular is infinite. This completes the verification that \mathcal{X} is not k -dominating.

It remains to prove that \mathcal{X} is $k+1$ -dominating. Let $h \in \omega \nearrow \omega$ be arbitrary; we shall find $k+1$ functions in \mathcal{X} whose maximum eventually majorizes h . Partition ω into intervals $[p_r, p_{r+1})$ in such a way that $h(p_r) < p_{r+1}$ for all r ; it follows, since h is monotone, that $h(n)$ is always at most one interval beyond n .

For each pair of the given $k+1$ ultrafilters, say \mathcal{U}_i and \mathcal{U}_j where $0 \leq i < j \leq k$, use the fact that they are not nearly coherent to find two sets $A_i^j \in \mathcal{U}_i$ and $A_j^i \in \mathcal{U}_j$ such that no interval $[p_r, p_{r+1})$ and in fact no union $[p_r, p_{r+2})$ of two consecutive such intervals meets both of the sets A_i^j and A_j^i . (The notation is chosen so that the subscript of an A indicates which ultrafilter it is in and the superscript tells which one it avoids.) To obtain such sets, we proceed as follows. First consider the finite-to-one function f whose value at all points of $[p_r, p_{r+1})$ is $\lfloor r/2 \rfloor$. So f is constant on intervals of the form $[p_r, p_{r+2})$ with r even. Non-near-coherence implies that $f(\mathcal{U}_i)$ and $f(\mathcal{U}_j)$ are distinct ultrafilters, so we can find a set X in the first whose complement is in the second. Then $f^{-1}(X) \in \mathcal{U}_i$ and $\omega - f^{-1}(X) \in \mathcal{U}_j$ do not both meet $[p_r, p_{r+2})$ for any even r . Now repeat this argument with the function g whose value on $[p_r, p_{r+1})$ is $\lceil r/2 \rceil$ to get Y such that $Y \in g(\mathcal{U}_i)$ and $\omega - Y \in g(\mathcal{U}_j)$. Then $g^{-1}(Y) \in \mathcal{U}_i$ and $\omega - g^{-1}(Y) \in \mathcal{U}_j$ do not both meet $[p_r, p_{r+2})$ for any odd r . Therefore,

$$A_i^j = f^{-1}(X) \cap g^{-1}(Y) \in \mathcal{U}_i \quad \text{and} \quad A_j^i = (\omega - f^{-1}(X)) \cap (\omega - g^{-1}(Y)) \in \mathcal{U}_j$$

serve as the required sets.

Now define, for each i , the set $A_i \in \mathcal{U}_i$ as the intersection of the A_i^j for all $j \neq i$. Thus, the intervals $[p_r, p_{r+1})$ that meet A_i are distinct from and not adjacent to those that meet A_j for any $j \neq i$.

Define $f_i : \omega \rightarrow \omega$ for each i by

$$f_i(n) = \text{next} \left(\bigcup_{j \neq i} A_j, n \right).$$

Notice that $f_n = n$ for all $n \in \bigcup_{j \neq i} A_j$ and that this union is in \mathcal{U}_j for all $j \neq i$. Thus $f_i \in \mathcal{X}$. To complete the proof, we need only show that the maximum of the $k+1$ functions f_i eventually majorizes h .

For any $n \in \omega$, the interval $[p_r, p_{r+1})$ containing n and the next interval $[p_{r+1}, p_{r+2})$ together meet at most one A_i . For this i (or for any i if $[p_r, p_{r+2})$ doesn't meet any A_i) we have that $\bigcup_{j \neq i} A_j$ contains no point in $[p_r, p_{r+2})$. This means that $f_i(n) \geq p_{r+2}$. But the intervals were chosen so that $h(n) < p_{r+2}$. So $f_i(n) > h(n)$, as required. \square

We next show, using the same construction as in the preceding proof, that in the conclusion of Theorem 4.8 the number k of pieces is optimal; it cannot be reduced (in ZFC) to $k-1$, even if we weaken the rest of the conclusion by changing $2k$ -dominating to finitely dominating. We again work in $\omega \nearrow \omega$ to avoid trivial examples.

THEOREM 5.4. *Assume that there are $k+1$ ultrafilters on ω no two of which are nearly coherent. Then in $\omega \nearrow \omega$ there is a $k+1$ -dominating family that can be partitioned into $k+1$ pieces, no union of k of which is finitely dominating.*

Proof The required family is the same \mathcal{X} as in the preceding proof. We showed there that it is $k + 1$ -dominating. We also pointed out there, immediately after its definition, that \mathcal{X} is the union of the sets $\mathcal{B}(\mathcal{F}_i)$, where each \mathcal{F}_i is a filter obtained by intersecting all but one of the ultrafilters \mathcal{U}_j . So all we need to prove is that $\mathcal{B}(\mathcal{F}_i)$ is not finitely dominating. But this is easy; $\mathcal{B}(\mathcal{F}_i)$ is obviously closed under finite maxima since \mathcal{F}_i is a filter, and it obviously fails to eventually majorize the function $n \mapsto n + 1$. \square

We state the case $k = 1$ separately for emphasis. It shows that the conclusion of Theorem 4.8 can't be improved (in ZFC) to get a single piece, even if we merely ask that the piece be finitely dominating, and even if the partition is into just two (rather than merely fewer than \mathfrak{g}) pieces. Furthermore, its hypothesis, the negation of the principle of Near Coherence of Filters (see [4, 5]) is a very weak consequence of CH.

COROLLARY 5.5. *Assume that there are two ultrafilters on ω that are not nearly coherent. Then in $\omega \nearrow \omega$ there is a 2-dominating family that can be partitioned into two pieces neither of which is finitely dominating.*

REMARK 5.6. In connection with the results of this section, it is natural to ask about the number of near-coherence classes of ultrafilters on ω . We already mentioned above that the values 1 and $2^{2^{\aleph_0}}$ are consistent. It is shown in [9] that 2 is also consistent. As far as I am aware, these are the only consistency results known concerning this number. The question is of some interest for general topology, since this number also counts the composants in the Stone-Ćech remainder of a half-line.

6. Weak Domination If $\mathfrak{u} < \mathfrak{g}$

In this section, we consider the opposite extreme from the CH results of the preceding section. We assume the combinatorial principle $\mathfrak{u} < \mathfrak{g}$, which was proved to be consistent (relative to ZFC) and to imply near coherence of all ultrafilters on ω in [8] (see also [5]). Our results depend heavily on Laflamme's classification in [14] of subfamilies of $\omega \nearrow \omega$. This classification is relative to a suitable notion of equivalence, which we review next.

DEFINITION 6.1. Let \mathcal{X} and \mathcal{Y} be subsets of $\omega \nearrow \omega$. Then $\mathcal{X} \preceq \mathcal{Y}$ means

$$\exists r \in \omega \nearrow \omega \forall x \in \mathcal{X} \exists y \in \mathcal{Y} x \leq^* y \circ r.$$

We call \mathcal{X} and \mathcal{Y} *equivalent*, written $\mathcal{X} \sim \mathcal{Y}$, if each is \preceq the other.

This ordering was introduced in [11] and studied further in [6, 14]. Laflamme [14] proved the following classification, up to equivalence, under the assumption $\mathfrak{u} < \mathfrak{g}$. (The names of the cases are from Laflamme's paper.)

THEOREM 6.2 (Laflamme [14]). *Assume $\mathfrak{u} < \mathfrak{g}$. Then every family $\mathcal{X} \subseteq \omega \nearrow \omega$ not bounded by a constant is equivalent to one of the following.*

- $\omega \nearrow \omega$ [the dominating case]
- $\{f \in \omega \nearrow \omega : \exists^\infty n f(n) \leq n\}$ [the high case]
- $\{f \in \omega \nearrow \omega : \{n \in \omega : f(n) \leq n\} \in \mathcal{U}\} = \mathcal{B}(\mathcal{U})$ for some (or every) ultrafilter \mathcal{U} [the ultrafilter case]
- $\{f \in \omega \nearrow \omega : \forall^\infty n f(n) \leq n\} = \mathcal{B}(\mathcal{F})$ for the cofinite filter \mathcal{F} [the low case]
- $\{f \in \omega \nearrow \omega : \exists c \in \omega \forall n f(n) < c\}$ [the bounded case]

In the ultrafilter case, the sets obtained from different ultrafilters are equivalent, because the ultrafilters are nearly coherent.

In order for Laflamme's classification to be useful in our investigation of weak forms of dominating, we need to know that these weak forms are invariant under the equivalence relation \sim used by Laflamme. That is the purpose of the following lemma.

LEMMA 6.3. *If $\mathcal{X} \subseteq \omega \nearrow \omega$ is k -dominating and $\mathcal{X} \preceq \mathcal{Y}$ then \mathcal{Y} is also k -dominating.*

Proof Fix an $r \in \omega \nearrow \omega$ as in the definition of $\mathcal{X} \preceq \mathcal{Y}$. Define $r^+(n)$ to be the smallest value of r strictly above $r(n)$, i.e., $\text{next}(\text{Range}(r), r(n) + 1)$. (We are using that the range of r is infinite, which is a trivial consequence of the fact that \mathcal{X} is k -dominating.)

Let an arbitrary $g \in \omega \nearrow \omega$ be given; we seek k elements of \mathcal{Y} whose maximum eventually majorizes g . By assumption, we have k elements of \mathcal{X} , say x_1, \dots, x_k , whose maximum eventually majorizes $g \circ r^+$. By our choice of r , we have functions $y_1, \dots, y_k \in \mathcal{Y}$ such that $x_i \leq^* y_i \circ r$ for each i . So the maximum y of the y_i 's satisfies

$$g \circ r^+ \leq^* \max_i x_i \leq^* y \circ r.$$

That is, $g(r^+(n)) \leq y(r(n))$ for all sufficiently large n . But for any sufficiently large $m \in \omega$, we can find n such that $r(n) \leq m \leq r^+(n)$, and then we have, since all the functions under consideration are monotone,

$$g(m) \leq g(r^+(n)) \leq y(r(n)) \leq y(m).$$

Thus y , the maximum of k functions from \mathcal{Y} , eventually majorizes g , as required. \square

COROLLARY 6.4. *If $\mathcal{X} \subseteq \omega \nearrow \omega$ is finitely dominating and $\mathcal{X} \preceq \mathcal{Y}$ then \mathcal{Y} is also finitely dominating.*

Proof Either repeat the proof of the lemma with k variable, or combine the lemma with Lemma 1.7. \square

THEOREM 6.5. *Assume $\mathfrak{u} < \mathfrak{g}$. Then any finitely dominating subset of $\omega \nearrow \omega$ is 2-dominating.*

Proof In view of the lemma and its corollary, it suffices to check this for the specific subsets of $\omega \nearrow \omega$ listed in Laflamme's classification theorem. The first, $\omega \nearrow \omega$ itself, is of course dominating (and therefore 2-dominating). The second, the high case $\{f \in \omega \nearrow \omega : \exists^\infty n f(n) \leq n\}$, is also 2-dominating. This follows from Proposition 4.3 because this set is an unbounded analytic (in fact G_δ) set, but it is also easy to verify directly by an argument similar to but easier than the proof of Proposition 4.2. The third case, the ultrafilter case $\mathcal{B}(U)$, is not finitely dominating by Proposition 4.7, and the remaining two are not even unbounded, much less finitely dominating. \square

THEOREM 6.6. *Assume $\mathfrak{u} < \mathfrak{g}$. If a 2-dominating family is partitioned into $< \mathfrak{g}$ pieces, then one of the pieces is 2-dominating.*

Proof By Theorem 4.8, the union of some two of the pieces is 4-dominating. By the proof of Theorem 6.5, that union of two pieces is in the dominating or the high case of Laflamme’s theorem, and we need only show that one of the two pieces is in the dominating or high case. So it suffices to show that, if two sets are in the ultrafilter case or lower, then so is their union.

Suppose therefore that we have \mathcal{X} and \mathcal{Y} both $\preceq \mathcal{B}(\mathcal{U})$, witnessed by r_1 and r_2 , respectively. Then clearly $r = \max\{r_1, r_2\}$ witnesses that $\mathcal{X} \cup \mathcal{Y} \preceq \mathcal{B}(\mathcal{U})$. \square

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