On the Number of Near-Coherence Classes of Ultrafilters

Andreas Blass
Mathematics Department
University of Michigan
Ann Arbor, MI 48109–1109, U.S.A.
ablass@umich.edu

October 20, 2004

Abstract

We prove that the number of near-coherence classes of non-principal ultrafilters on the natural numbers is either finite or at least the larger of the dominating number and the minimum number of generators for such an ultrafilter.

1 Introduction

In this paper, “ultrafilter” means “non-principal ultrafilter on the set \( \omega \) of natural numbers” unless a different meaning is explicitly specified. If \( U \) is an ultrafilter and \( f : \omega \to \omega \) is any function, then \( f(U) \) is defined to be \( \{ X \subseteq \omega : f^{-1}(X) \in U \} \). This is an ultrafilter, except that it is principal if \( f \) is constant on a set in \( U \); we shall usually be concerned with finite-to-one functions, so that this exception will not arise. Two ultrafilters \( U \) and \( V \) are nearly coherent if \( f(U) = f(V) \) for some finite-to-one \( f : \omega \to \omega \). Near-coherence is an equivalence relation, introduced and extensively studied in [2]. It is natural to ask how many equivalence classes it has. Since the number of ultrafilters is \( 2^{2^{\aleph_0}} \) by a theorem of Pospíšil [11], the number of near-coherence classes of ultrafilters is obviously between 1 and \( 2^{2^{\aleph_0}} \), inclusive. Its
exact value, however, is independent of the usual (ZFC) axioms of set theory. The known consistency results are, in chronological order:

1. It is consistent relative to ZFC, and in fact it is a consequence of the continuum hypothesis (CH) or of Martin’s axiom (MA), that the number of near-coherence classes is $2^{2^\aleph_0}$.

2. It is consistent relative to ZFC that there is only one near-coherence class of ultrafilters.

3. It is consistent relative to ZFC that there are exactly two near-coherence classes of ultrafilters.

The first of these consistency results follows from the fact that among selective ultrafilters (those such that every function on $\omega$ is either constant or one-to-one on a set in the ultrafilter) near coherence is the same as isomorphism (via permutations of $\omega$). In particular, any selective ultrafilter is nearly coherent with only $2^{\aleph_0}$ others. On the other hand, CH (or just MA) implies that there are $2^{2^\aleph_0}$ selective ultrafilters, and therefore that there are $2^{2^\aleph_0}$ near-coherence classes. The history of this result is a bit obscure. Booth [6] writes that Galvin was the first to prove the existence of selective ultrafilters under CH, but doesn’t say that Galvin made the slight extension to get $2^{2^\aleph_0}$ selective ultrafilters. Booth himself weakens the hypothesis to MA and shows that the selective ultrafilters are dense in $\beta\omega - \omega$. Rudin [13] proves that CH yields $2^{2^\aleph_0}$ selective ultrafilters, but she describes the result as well known. In my thesis [1], the assumption here is reduced to MA (and in fact to the hypothesis there called FRH($\omega$) but nowadays expressed as $p = c$ or as MA($\sigma$-centered)), but the proof is essentially the same as under CH.

The statement that there is only one near-coherence class of ultrafilters is known as the principle of near coherence of filters (NCF) and is proved consistent in [5]. For more information about it, see [2, 3].

Finally, the consistency of the existence of exactly two near-coherence classes of ultrafilters is proved in [4], using another model constructed in [5].

For all cardinals $\kappa$ strictly between 2 and $2^{2^{\aleph_0}}$, it has remained an open question whether there could be exactly $\kappa$ near-coherence classes of ultrafilters. In this paper, we present the first negative result about this question, eliminating some $\kappa$’s, including $\aleph_0$, from the range of possibilities.

To state our result, we need two of the familiar cardinal characteristics of the continuum, $\mathfrak{d}$ and $\mathfrak{u}$. The dominating number, $\mathfrak{d}$, is defined as the smallest
cardinality of any family $D$ of functions $\omega \to \omega$ such that every $f : \omega \to \omega$ is eventually majorized by some member of $D$. The ultrafilter number, $u$, is defined as the smallest size of any base for an ultrafilter. It is easy to see that both of these cardinals are between $\aleph_1$ and $2^{\aleph_0}$, inclusive, but additional information, such as their relative order, is independent of ZFC.

**Theorem 1** If there are infinitely many near-coherence classes of ultrafilters, then there are at least $\max\{d, u\}$ of them.

Our proof of this theorem proceeds differently in two cases, according to whether $u < d$ or $u \geq d$. The two cases are treated in Sections 2 and 3, respectively. In each case, we obtain somewhat more information than is required for the theorem. When $u < d$, we obtain $2^{\aleph_0}$ near-coherence classes, which may (consistently) be more than the $d$ required by the theorem. When $u \geq d$, we can omit the assumption that there are infinitely many near-coherence classes, as this actually follows from $u \geq d$.

**Remark 2** Mioduszewski [9, 10] has established that the number of near-coherence classes of ultrafilters is the same as the number of composants of the indecomposable continuum $\beta[0, \infty) - [0, \infty)$, the Stone-Čech remainder of a closed half-line. (See also [3] for a discussion and a proof with less machinery.) Thus, our result implies that, if $\beta[0, \infty) - [0, \infty)$ has infinitely many composants then it has at least $\max\{d, u\}$ composants. In particular, it cannot have exactly $\aleph_0$ composants.

## 2 When $u$ is small

In this section, we treat the case that $u < d$, and we obtain, in this case, a somewhat stronger inequality than is claimed in Theorem 1.

**Theorem 3** If $u < d$ and there are infinitely many near-coherence classes of ultrafilters, then there are at least $2^{\aleph_0}$ of them.

**Proof** Since $u < d$, there is an ultrafilter $U$ that is generated by fewer than $d$ sets. By a theorem of Ketonen [8], $U$ is a P-point. That is, whenever $\omega$ is partitioned into pieces that are not in $U$, then there is a set in $U$ that meets each piece in only a finite set. We shall need the following general fact about P-points only in the case where $X$ is $\beta\omega - \omega$, the Stone-Čech remainder of the
discrete space $\omega$, but the proof is no harder in the general case. (This result is in [1, Section 15] but may well be older; since [1] was never published, we include the proof here.)

**Lemma 4** If $U$ is a P-point and $f$ is a one-to-one map of $\omega$ into a compact Hausdorff space $X$, then there is a set $A \in U$ such that $f(A)$ is discrete and in fact has the following property. One can assign to each $a \in A$ a neighborhood $N_a$ of $f(a)$ in $X$ in such a way that these neighborhoods are pairwise disjoint.

**Remark 5** The property in the last sentence of the lemma is really a consequence of discreteness, because $f(A)$ is countable.

**Proof of Lemma** Because $X$ is compact, $f$ has a limit with respect to the ultrafilter $U$. That is, there is a point $x \in X$ such that $f^{-1}(G) \in U$ for every neighborhood $G$ of $x$. Since $f$ is one-to-one, it maps at most one number to $x$, so there is no loss of generality in supposing that $x \notin f(\omega)$. Inductively choose neighborhoods $G_0 \supseteq G_1 \supseteq \ldots$ of $x$ so that $G_0 = X$, $\overline{G_{n+1}} \subseteq G_n$, and $f(n) \notin G_{n+1}$. ($\overline{G}$ means the closure of $G$.) The sets $Y_n = f^{-1}(G_n - G_{n+1})$ constitute a partition of $\omega$, and, by our choice of $x$, none of them are in $U$. So there is a set $A \in U$ that meets each $Y_n$ only finitely often. Since $U$ is an ultrafilter, we can arrange in addition that $A$ meets $Y_n$ only for even $n$ or only for odd $n$. We assume the even case; the odd one is handled in exactly the same way.

For each even $n$, we can choose, since $X$ is a Hausdorff space, pairwise disjoint neighborhoods for the finitely many points in $f(A) \cap (G_n - G_{n+1})$. Intersecting these neighborhoods with the open set $G_n - G_{n+2}$, we ensure that they are also disjoint from the similarly constructed neighborhoods of the points in $f(A) \cap (G_{n'} - G_{n'+1})$ whenever $n'$ differs from $n$ by at least 2. Since we need only consider even $n$, we have the neighborhoods required in the lemma. \qed

Returning to the proof of the theorem, we invoke its hypothesis to fix, for the rest of the proof, a sequence of ultrafilters $(\mathcal{V}_n)_{n \in \omega}$, no two of which are nearly coherent. We consider ultrafilters as points in the compact space $\beta\omega - \omega$, and we consider the limits of the sequence $(\mathcal{V}_n)$ with respect to other ultrafilters $U$, particularly when $U$ is a P-point. These limits can be described combinatorially as

$$U\text{-lim}_n \mathcal{V}_n = \{A \subseteq \omega : \{n : A \in \mathcal{V}_n\} \in U\}.$$
Lemma 6 With \((V_n)_{n \in \omega}\) as above, if \(U\) and \(U'\) are two distinct P-points, then \(\text{U-lim}_n V_n\) and \(\text{U'-lim}_n V_n\) are not nearly coherent.

Proof Suppose the contrary, and let \(f : \omega \rightarrow \omega\) be a finite-to-one function such that \(f(\text{U-lim}_n V_n) = f(\text{U'-lim}_n V_n)\). Because the action of \(f\) on ultrafilters is continuous with respect to the topology of \(\beta\omega - \omega\), we have

\[\text{U-lim}_n f(V_n) = \text{U'-lim}_n f(V_n);\]
call this ultrafilter \(W\). Because the ultrafilters \(V_n\) are pairwise not nearly coherent and because \(f\) is finite-to-one, we know that the map \(\omega \rightarrow \beta\omega - \omega\) is one-to-one. So we can apply Lemma 4 to find sets \(A \in U\) and \(A' \in U'\) such that \(Z = \{f(V_n) : n \in A\}\) and \(Z' = \{f(V_n) : n \in A'\}\) are discrete. Since \(U\) and \(U'\) are distinct ultrafilters, we can arrange that \(A\) and \(A'\) are disjoint, and therefore so are \(Z\) and \(Z'\). Since \(W = \text{U-lim}_n f(V_n)\) and \(A \in U\), we have \(W \in Z\), and similarly \(W \in Z'\). This situation, a point \(W \in \beta\omega - \omega\) lying in the closures of two disjoint, countable, discrete subsets of \(\beta\omega - \omega\), is impossible by a result of Rudin [12, Lemma 2], which says that a point in the closures of two countable, discrete subsets of \(\beta\omega - \omega\) is also in the closure of their intersection. This contradiction completes the proof of the lemma.

To complete the proof of the theorem, it suffices, in view of the lemma, to find \(2^{\aleph_0}\) distinct P-points. We saw, at the beginning of the proof, that the assumption \(u < d\) gives us one P-point \(U\). We can get additional P-points, by applying arbitrary one-to-one functions \(f : \omega \rightarrow \omega\) to \(U\), and these isomorphic images of \(U\) will be distinct as long as the corresponding \(f\)'s are equal only on a set not in \(U\). But it is well known that there are \(2^{\aleph_0}\) one-to-one functions \(f : \omega \rightarrow \omega\), any two of which agree only on a finite set. For example, assign to each \(S \subseteq \omega\) the function \(f_S\) defined by letting \(f_S(n)\) code, in some standard way, the set \(S \cap \{0, \ldots, n\}\). Thus, we have \(2^{\aleph_0}\) distinct P-points, and so the proof of the theorem is complete.

\[\square\]

3 When \(u\) is large

In this section, we finish the proof of Theorem 1 by treating the case that \(u \geq d\). Of course, the \(\max\{u, d\}\) of that theorem is simply \(u\) in the present context. Notice that, in contrast with Theorem 1, we do not need to assume
here that there are infinitely many near-coherence classes of ultrafilters; the proof will show that this is a consequence of \( u \geq d \).

**Theorem 7** If \( u \geq d \), then there are at least \( u \) near-coherence classes of ultrafilters.

**Proof** Suppose, toward a contradiction, that there are exactly \( \kappa \) near-coherence classes of ultrafilters, with \( \kappa < u \). Let \( F \) be a family of \( \kappa \) ultrafilters consisting of one representative from each near-coherence class. We shall obtain a contradiction by constructing an ultrafilter \( V \) that is not nearly coherent with any \( U \in F \).

The construction is a modification of the argument for Theorem 14 in [2]. (That theorem includes the special case \( \kappa = 1 \) of the present argument.) As in that argument, we use a family of \( \delta \) finite-to-one functions \( (f_\alpha : \omega \to \omega)_{\alpha < \delta} \) such that, whenever two ultrafilters are nearly coherent then they have the same image under some \( f_\alpha \). The existence of such a family was established in [2, Lemma 10]. So our objective is to construct \( V \) so that \( f_\alpha(V) \neq f_\alpha(U) \) for all \( \alpha < \delta \) and all \( U \in F \).

The essential part of our construction of \( V \) will be an induction of length \( \delta \), ensuring at stage \( \alpha \) that \( V \) behaves as desired with respect to \( f_\alpha \). More precisely, we inductively define an increasing sequence of filters \( (V_\alpha)_{\alpha \leq \delta} \) (note that we have added a final stage to the induction) satisfying:

1. \( V_0 \) is the filter of cofinite subsets of \( \omega \).
2. \( V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha \) for limit ordinals \( \lambda \).
3. \( V_{\alpha + 1} \) is generated by \( V_\alpha \) plus at most \( \kappa \) additional subsets of \( \omega \).
4. For each \( U \in F \) and each \( \alpha < \delta \), there exists \( A \in V_{\alpha + 1} \) with \( f_\alpha(A) \notin f_\alpha(U) \).

If we can carry out this induction, then any ultrafilter extending the filter \( V_\delta \) will serve as our desired \( V \). Indeed, condition (4) ensures that, for each \( U \in F \) and each \( \alpha < \delta \), there is a set \( f_\alpha(A) \in f_\alpha(V) \) with \( f_\alpha(A) \notin f_\alpha(U) \). So \( f_\alpha(V) \neq f_\alpha(U) \), as required.

It remains only to carry out the inductive construction of the filters \( V_\alpha \). In view of requirements (1) and (2), we need only describe the successor stages of the induction. So suppose \( V_\alpha \) is given, and that it and the previous \( V_\beta \)'s have been built in accordance with our requirements. We must find \( \kappa \)}
or fewer sets to add to $V_\alpha$ (according to requirement (3)) so as to satisfy requirement (4). It would suffice to choose, for each $U \in \mathcal{F}$, some $A_U$ such that $f_\alpha(A_U) \notin f_\alpha(U)$ and such that all the sets $A_U$ can be adjoined to $V_\alpha$ so as to generate a filter (i.e., so that the finite intersection property is maintained). The rest of the proof is devoted to showing that such a choice is possible.

Requirements (1), (2), and (3) imply that $V_\alpha$ is generated by a family of at most $|\alpha| \cdot \kappa + \aleph_0$ sets. Since $\alpha < \delta \leq u$ and $\kappa < u$ and $\aleph_0 < u$, we have that $V_\alpha$ and therefore also $f_\alpha(V_\alpha)$ are generated by strictly fewer than $u$ sets and are therefore not ultrafilters. (The definition of applying a function to a filter is the same as for ultrafilters; it is easy to check that the image filter is generated by the images of any basis for the original filter.) Furthermore, $f_\alpha(V_\alpha)$ will not become an ultrafilter if finitely many new generators are adjoined.

In terms of $\beta\omega - \omega$ and its topology, the situation in the preceding paragraph admits the following equivalent description. The ultrafilters that extend the filter $f_\alpha(V_\alpha)$ constitute a nonempty closed set $C$, and it has no isolated points. In particular, $C$ is infinite. It is well known that any infinite closed set in $\beta\omega - \omega$ contains a copy of $\beta\omega$ (see for example the combination of Lemmas 3.3(b), 14.16, and 16.15(b) in [7]) and therefore has cardinality $2^{2^{\aleph_0}}$ by [11]. In particular, $C$ has cardinality strictly greater than $\kappa$ (which is $< u \leq 2^{\aleph_0}$). So we can fix an ultrafilter $W \in C$ distinct from $f_\alpha(U)$ for all $U \in \mathcal{F}$. Choose, for each $U \in \mathcal{F}$, a set $B_U \in W - f_\alpha(U)$. Set $A_U = f_\alpha^{-1}(B_U)$. So $A_U \notin U$. To verify that the sets $A_U$ are what we needed, two paragraphs ago, to complete the construction of $V_{\alpha+1}$, it remains only to check that adjoining them to $V_\alpha$ does not destroy the finite intersection property. That is, given any $Z \in V_\alpha$ and any finitely many of the $A_U$’s, we must show that their intersection is nonempty. In view of the definition of the $A_U$’s, this amounts to showing that $f_\alpha(Z)$ meets the intersection of the corresponding finitely many $B_U$’s. But $f_\alpha(Z) \in f_\alpha(V_\alpha) \subseteq W$, and all the $B_U$’s are also in $W$. So the fact that $W$ is an ultrafilter gives what we need to complete the proof.

\[ \square \]

4 Additional Results and Questions

We point out some additional results that can be obtained by essentially the same proof as Theorem 3. Observe that the proof of Theorem 3 used the hypothesis $u < \delta$ only to ensure the existence of a P-point. Furthermore,
the estimate $2^\aleph_0$ for the number of near-coherence classes came from the fact that, once there is a P-point, there are automatically $2^\aleph_0$ of them. Thus, the same proof immediately gives the following stronger result.

**Corollary 8** Assume that there are infinitely many near-coherence classes of ultrafilters. If there is a P-point, then there are at least $2^\aleph_0$ near-coherence classes of ultrafilters. If there are $\kappa > 2^\aleph_0$ P-points, then there are at least $\kappa$ near-coherence classes of ultrafilters.

The assumptions here can be weakened further. The only property of P-points that was used in the proof is Lemma 4, and that only for the space $X = \beta\omega - \omega$. This property is not limited to P-points. For example, if $U$ and all $V_n$ are P-points, then $U\text{-}\lim_n V_n$ also has this discreteness property, for all Hausdorff spaces $X$.

**Question 9** Can one prove in ZFC that there is an ultrafilter $U$ such that, whenever $f : \omega \to \beta\omega - \omega$ is one-to-one, then $f(A)$ is discrete for some $A \in U$?

It is important in this question that we care only about maps $f$ into $\beta\omega - \omega$. There may be no ultrafilters that satisfy the same requirement for maps into the real line, even if one weakens “discrete” to “nowhere dense”; this is proved in [14].

The following corollary uses an alternative approach to the discreteness needed in the proof of Theorem 3.

**Corollary 10** Assume that there are infinitely many P-points no two of which are nearly coherent. Then there are $2^\aleph_0$ near-coherence classes of ultrafilters.

**Proof** We need two well-known facts about P-points. First, if $U$ is a P-point, then so is $f(U)$ for any $f : \omega \to \omega$ that is not constant on any set in $U$ (in particular, for any finite-to-one $f$). This follows immediately from the definition of P-points (quoted in the proof of Theorem 3 above), by applying $f^{-1}$ to the pieces of a partition of $\omega$. Second, any countable set of P-points is a discrete subset of $\beta\omega - \omega$. To see this, let the P-points be $U_0$ and temporarily concentrate on one of them, say $U_0$. For each $n \neq 0$, let $A_n$ be a set that is in $U_0$ but not in $U_n$. Changing each $A_n$ by only a finite amount,
we arrange in addition that \( \bigcap_{n \in \omega - \{0\}} A_n = \emptyset \). Then \( \omega \) is partitioned by the sets \( B_k = \bigcap_{0 \neq n < k} A_n - A_k \); and none of these sets is in \( U_0 \) (because \( B_k \) is disjoint from \( A_k \in U_0 \)). As \( U_0 \) is a P-point, it contains a set \( C \) that has finite intersection with each \( B_k \). For each \( m \neq k \), we have \( \omega - A_m \subseteq \bigcup_{k \leq m} B_k \), and so the set \( \omega - A_m \), which is in \( U_m \), has only a finite intersection with \( C \). In particular, \( C \notin U_m \). The set of ultrafilters that contain \( C \) is therefore a neighborhood of \( U_0 \) in \( \beta \omega - \omega \) that contains no other \( U_m \); this makes \( U_0 \) an isolated point of \( \{U_m : m \in \omega \} \). The same argument applies with any other \( U_m \) in place of \( U_0 \), and so the set \( \{U_m : m \in \omega \} \) is discrete.

With these preliminary facts available, the corollary is proved by the same argument as Theorem 3. Let \( (\mathcal{V}_n)_{n \in \omega} \) be a sequence of P-points, no two of which are nearly coherent. We consider the limits of this sequence, \( \lim_{n} \mathcal{V}_n \), for arbitrary ultrafilters \( \mathcal{U} \). Since there are, by [11], \( 2^{2^{\aleph_0}} \) ultrafilters \( \mathcal{U} \), we need only prove that no two of these limits are nearly coherent. For this purpose, we follow the proof of Lemma 6. The only difference is that, instead of invoking Lemma 4 to obtain \( A \in \mathcal{U} \) and \( A' \in \mathcal{U} \) such that \( Z = \{f(\mathcal{V}_n) : n \in A\} \) and \( Z' = \{f(\mathcal{V}_n) : n \in A'\} \) are discrete, we can now obtain the same properties by taking arbitrary \( A \in \mathcal{U} \) and \( A' \in \mathcal{U} \). Indeed, the first of our preliminary facts says that the ultrafilters \( f(\mathcal{V}_n) \) are P-points, and then the second preliminary fact says that \( Z \) and \( Z' \) are discrete. \( \square \)

Finally, we repeat the central open question motivating this work.

**Question 11** What cardinals can consistently be the number of near-coherence classes of ultrafilters?

Particular cases left open by our work and all previous results include the following.

**Question 12** Can the number of near-coherence classes of ultrafilters be finite but greater than two? Can it be \( 2^{2^{\aleph_0}} \)? Can it be \( \max\{d, u\} \) if this is smaller than \( 2^{\aleph_0} \)?

**References**


