Needed Reals and Recursion in Generic Reals*

Andreas Blass
Mathematics Department
University of Michigan
Ann Arbor, MI 48109–1109, U.S.A.
ablass@umich.edu

1 Introduction

Vojtáš [11] formalized the idea, implicit in earlier work of Fremlin [3] and Miller, that cardinal characteristics of the continuum are naturally associated with binary relations (usually Borel relations on the continuum). Subsequently, I hinted [1, 2] that the relations rather than the cardinals should be the primary objects of study. This is the point of view of the present paper, leading to a recursion-theoretic, rather than cardinality-based, way to describe the complexity of relations.

We shall be interested in triples $\mathbf{A} = (A_-, A_+, A)$ where A is a binary relation between the sets A_- and A_+. We often call such triples "relations." Call a subset $X \subseteq A_+$ *adequate* for $\mathbf{A}$ if

$$\forall y \in A_- \, \exists x \in X \; yAx.$$

Intuitively, we regard A_- as a set of challenges, A_+ as a set of possible responses, and A as the relation holding between a challenge and a successful response to it. Then an adequate set contains enough responses to successfully meet every challenge.

The theory of cardinal characteristics seeks to describe, for natural relations $\mathbf{A}$, the complexity needed in all their adequate sets. Originally, complexity was measured simply in terms of cardinality — how big must an

*Most of the material in this paper was presented at a set theory conference in Oberwolfach in December, 1999 adequate set be? The fundamental concept in this part of the theory is the *norm* of a relation

$$\|\mathbf{A}\| = \min\{|X| : X \text{ adequate for } \mathbf{A}\}.$$

A subtler measure of complexity was introduced by Pawlikowski and Recław [8]. Assuming $A_\pm$ are sets of reals, call a set S of reals *small* for $\mathbf{A}$ if it cannot be mapped onto an adequate set by any Borel function. It is shown in [8] that this notion of smallness, for various natural choices of $\mathbf{A}$, agrees with various notions of smallness studied in classical real analysis.

In this paper, we shall be concerned with yet another approach to the complexity of adequate sets, looking not at the complexity of the set as a whole but rather at the complexity of its individual members. In all that follows, we are concerned only with relations $\mathbf{A} = (A_-, A_+, A)$ in which $A_\pm \subseteq \mathbb{R}$. Here, as is customary in set theory, $\mathbb{R}$ ambiguously denotes the real line or any of the similar spaces ${}^\omega 2$, ${}^\omega\omega$, $[\omega]^\omega$, etc.

Definition 1 A real a is *needed* for a relation $\mathbf{A}$ if every set that is adequate for $\mathbf{A}$ contains a real in which a is recursive.

Inserting the definition of "adequate" into this definition, one easily sees that a is needed for $\mathbf{A}$ if and only if

$$\exists y \in A_- \, \forall x \in A_+ [yAx \implies a \leq_T x],$$

where $\leq_T$ denotes as usual the relation "recursive in," also called Turing reducibility. Thus, a is needed for $\mathbf{A}$ just in case there is a challenge y such that every successful response to it must have at least the complexity of a.

The primary purpose of this paper is to determine which reals are needed for certain natural relations, for example those corresponding to the cardinal characteristics in Cichoń's diagram. This leads naturally to questions about which ground-model reals are recursive in various sorts of generic reals, so we also study these questions for some of the most commonly used notions of forcing.

2 Dominating Reals

For functions $f, g : \omega \to \omega$, we write $f \leq g$ to mean that $f(n) \leq g(n)$ for all $n \in \omega$. We write $f \leq^* g$ to mean that $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. More generally, we use a star to indicate that finitely many exceptions are allowed. For example $X \subseteq^* Y$ means that $X - Y$ is finite.

Definition 2 $\mathbf{D}$ is the relation $({}^\omega\omega, {}^\omega\omega, \leq^*)$.

Thus a challenge is a function $\omega \to \omega$ and a succesful response is a function that almost everywhere majorizes the challenge. $\|\mathbf{D}\|$ is the *dominating number* $\mathfrak{d}$.

The characterization of the reals needed for $\mathbf{D}$ is essentially contained in work of Jockusch [4] and Solovay [10], which we now summarize.

Theorem 3 (Jockusch, Solovay) *The following five statements are equivalent, for any real a.*

1. *There exist $f \in {}^\omega\omega$ and $e \in \omega$ such that, for every $g \geq f$, a is recursive in g with index e.*
2. *There exists $f \in {}^\omega\omega$ such that a is recursive in every $g \geq f$.*
3. *There exists $e \in \omega$ such that every infinite $X \subseteq \omega$ has an infinite subset Y in which a is recursive with index e.*
4. *Every infinite $X \subseteq \omega$ has an infinite subset Y in which a is recursive.*
5. *a is hyperarithmetic.*

Proof Notice first that the only difference between (1) and (2) is the uniformity required in (1): the same index e works for all relevant g. Similarly, (3) differs from (4) only by requiring uniformity. So trivially, (1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (4).

To show (1) $\Longrightarrow$ (3), suppose f and e are as in (1), and let an arbitrary infinite $X \subseteq \omega$ be given. Then X has an infinite subset Y whose increasing enumeration is a function g majorizing f. Then a is recursive in g with index e and is therefore also recursive in Y with an index e' easily (and uniformly) obtainable from e.

The same argument establishes (2) $\Longrightarrow$ (4).

(4) $\Longrightarrow$ (5) is Theorem 2.3 of [10]; Solovay's notion of "recursively encodable" is precisely (4).

Finally, (5) $\Longrightarrow$ (1) follows from Theorem 6.8 of [4]. That theorem characterizes the hyperarithmetic sets as those Turing reducible to uniformly majorreducible sets. Here a set $B \subseteq \omega$ is called uniformly majorreducible if there is an e such that B is recursive with index e in every function $g : \omega \to \omega$ that majorizes the increasing enumeration of B. That clearly implies (1) with the increasing enumeration of B as f and with e modified to include also the computation of a from B. $\square$

Corollary 4 *A real is needed for* $\mathbf{D}$ *if and only if it is hyperarithmetic.*

Proof Notice that, in clause (2) of the theorem, we can write $g \geq^* f$ in place of $g \geq f$. This is because a finite change in g will not affect whether a is recursive in it. (Analogously, we could modify (4) by requiring only that $Y \subseteq^* X$. But we cannot similarly modify (1) or (3), since a finite modification in g or Y must be compensated for by the index of the computation.) The modified (2) says precisely that a is needed for $\mathbf{D}$. $\square$

The characterization of the reals needed for $\mathbf{D}$ gives some information, though less than a full characterization, for certain other relations. The connection is given by the notion of a morphism (in the terminology of [1, 2]; except for the direction of the arrow, it's the same as a generalized Galois-Tukey connection in the terminology of [11]).

Definition 5 A *morphism* ξ from one relation $\mathbf{A} = (A_-, A_+, A)$ to another $\mathbf{B} = (B_-, B_+, B)$ consists of two functions $\xi_- : B_- \to A_-$ and $\xi_+ : A_+ \to B_+$ such that, for all $b \in B_-$ and $a \in A_+$,

$$\xi_-(b) A a \implies b B \xi_+(a).$$

Morphisms were originally introduced to describe the constructions used in the proofs of many inequalities between cardinal characteristics. If $\xi : \mathbf{A} \to \mathbf{B}$ and X is adequate for $\mathbf{A}$, then $\xi_+(X)$ is adequate for $\mathbf{B}$, and therefore $\|\mathbf{A}\| \geq \|\mathbf{B}\|$. Nicely behaved (Borel, or even continuous) morphisms also play an important role also in the Pawlikowski-Recław theory of small sets of reals [8]. We shall use them to relate the needed reals for different relations, as follows.

Theorem 6 *Suppose* $\xi : \mathbf{A} \to \mathbf{B}$ *is a morphism whose "plus" component* ξ_+ *is continuous with a recursive code. Then every real needed for* $\mathbf{B}$ *is also needed for* $\mathbf{A}$.

Proof Suppose a is needed for $\mathbf{B}$, and fix $y \in B_-$ such that

$$\forall x \in B_+[yBx \implies a \leq_T x].$$

Then $\xi_-(y) \in A_-$ and

$$\forall x \in A_+[\xi_-(y)Ax \implies a \leq_T \xi_+(x)]$$

because $\xi_-(y)Ax \implies yB\xi_+(x)$. Because ξ_+ is continuous with recursive code, $\xi_+(x) \leq_T x$. So

$$\forall x \in A_+[\xi_-(y)Ax \implies a \leq_T x].$$

This shows that a is needed for $\mathbf{A}$. $\square$

We apply this result to some known morphisms into $\mathbf{D}$.

Definition 7 $\mathbf{Cof}(\mathcal{B})$ is the relation $(\mathcal{B}, \mathcal{B}, \subseteq)$ where $\mathcal{B}$ is the family of meager F_σ subsets of $\mathbb{R}$. Similarly, $\mathbf{Cof}(\mathcal{L})$ is $(\mathcal{L}, \mathcal{L}, \subseteq)$ where $\mathcal{L}$ is the family of measure zero G_δ subsets of $\mathbb{R}$. Whenever it is important that our relations be on reals, we identify F_σ and G_δ subsets with reals coding them.

Corollary 8 *Every hyperarithmetic real is needed for* $\mathbf{Cof}(\mathcal{B})$ *and* $\mathbf{Cof}(\mathcal{L})$.

Proof The proofs of the cardinal invariant inequalities $\mathfrak{d} \leq \mathbf{cof}(\mathcal{B}), \mathbf{cof}(\mathcal{L})$ produce continuous morphisms from each of $\mathbf{Cof}(\mathcal{B})$ and $\mathbf{Cof}(\mathcal{L})$ into $\mathbf{D}$; see [8]. The codes are recursive, so Theorem 6 applies. $\square$

We shall prove later that the reals needed for $\mathbf{Cof}(\mathcal{B})$ are exactly the hyperarithmetic reals. The corresponding result for $\mathbf{Cof}(\mathcal{L})$, which I conjectured in the talk on which this paper is based, is established in the paper [7] by Mildenberger and Shelah.

3 Cohen Reals

In contrast to the results of the preceding section, we shall show next that, for certain natural relations, the only needed reals are the recursive ones. (Of course, all recursive reals are trivially needed for any $\mathbf{A}$, as long as $A_- \neq \emptyset$.) The results of this section depend on the following proposition, which seems to be folklore. We give a proof for the sake of completeness (and because there has been at least one incorrect proof that nearly got published).

Proposition 9 *Suppose c is a real Cohen-generic over a model M of set theory. Suppose further that $a \in M$ is a real recursive in c. Then a is recursive.*

Proof Working in M, fix a condition p, a finite restriction of c, that forces "$\check{a}$ is the real recursive in $\dot{c}$ with index e" for a specific natural number e. We describe an algorithm for computing a.

On input n, start running the oracle algorithm with index e on input n. If it queries its oracle about a number $k \in \mathrm{Dom}(p)$, then let the query be answered in accordance with p (which will be correct information about c). If it queries its oracle about a number $k \notin \mathrm{Dom}(p)$, then begin parallel (or interleaved) subcomputations, one for each possible answer to the query. Further branching into subcomputations occurs whenever there is a query not answered in p or in previous decisions along a branch.

As soon as any of the subcomputations halts with an answer, give that answer as the output of the whole computation. This completes the description of our algorithm; we must verify that it produces the right answer, $a(n)$.

The "true" answers given by c are used along one branch in this computation, and they will eventually produce the correct answer $a(n)$. Therefore, our algorithm eventually produces an answer. But the answer it produces may come from another branch, using oracle information different from c. Nevertheless, the answer will be correct, because it uses an extension of p, which forces e to give answers in agreement with a. $\square$

Definition 10 Say that a subset S of ω *splits* an infinite subset X of ω if both $X \cap S$ and $X - S$ are infinite. Let $\mathbf{S} = ([\omega]^\omega, \mathcal{P}(\omega), \text{is split by})$.

Thus $\|\mathbf{S}\|$ is the splitting number $\mathfrak{s}$, the smallest number of subsets of ω needed to split all infinite subsets of ω.

Definition 11 $\mathbf{Non}(\mathcal{B}) = (\mathcal{B}, \mathbb{R}, \not\ni)$ and $\mathbf{Non}(\mathcal{L}) = (\mathcal{L}, \mathbb{R}, \not\ni)$

Thus $\|\mathbf{Non}(\mathcal{B})\|$ is the minimum size of a non-meager set of reals, and $\|\mathbf{Non}(\mathcal{L})\|$ is the minimum size of a set of positive outer measure.

Theorem 12 *Only recursive reals are needed for $\mathbf{S}$ and for $\mathbf{Non}(\mathcal{B})$.*

Proof We consider first the case of $\mathbf{Non}(\mathcal{B})$. Suppose a is needed for $\mathbf{Non}(\mathcal{B})$, and let Y be a meager F_σ set such that a is recursive in every real not in Y. The statement "a is recursive in every real not in Y" is a Π^1_1 statement about a and the code y for Y. So it is absolute to any forcing extension of the universe, in particular to the extension $V[c]$ obtained by adding a Cohen-generic real c to the universe V. Being Cohen-generic, c is not in any meager set coded in the ground model; in particular it is not in the set coded by y in the extension. Therefore a is recursive in c. By Proposition 9, it follows that a is recursive.

The result for $\mathbf{S}$ can be proved similarly, using the fact that a Cohen-generic subset of ω splits every infinite ground-model subset of ω. Alternatively, it can be deduced from the result for $\mathbf{Non}(\mathcal{B})$ via Theorem 6, once one checks that the usual proof of $\mathfrak{s} \leq \mathbf{non}(\mathcal{B})$ provides a continuous morphism with recursive code. $\square$

Using the continuous morphisms provided, for example, in [8], and checking that the codes can be taken to be recursive, one can extend the theorem to all the (relations associated to) cardinal characteristics in the left half of Cichoń's diagram, i.e., the additivities of measure and category, the covering number for measure, and the bounding number.

4 Random Reals

Theorem 13 *Only recursive reals are needed for* $\mathbf{Non}(\mathcal{L})$.

The proof is exactly like the proof of the corresponding result for $\mathbf{Non}(\mathcal{B})$, once we have the analog of Proposition 9 for measure. That analog was established, in pre-forcing times and therefore with different terminology, in [6].

Proposition 14 (de Leeuw, Moore, Shannon, Shapiro) *Suppose r is a real random over a model M of set theory. Suppose further that $a \in M$ is a real recursive in r. Then a is recursive.*

Proof We take the forcing conditions for random forcing to be Borel sets of positive measure in ${}^\omega 2$. For a finite partial function $p : \omega \rightharpoonup 2$, we write $[p]$ for the set of all $x \in {}^\omega 2$ that agree with p on $\mathrm{Dom}(p)$. Thus the measure of $[p]$ is $2^{-|\mathrm{Dom}(p)|}$.

Working in M, fix an index e and a condition q from the generic filter defining r, forcing "$\check{a}$ is recursive in $\dot{r}$ with index e." By the Lebesgue density theorem, fix a finite partial function $p : \omega \rightharpoonup 2$ such that the measure of $q \cap [p]$ is at least $\frac{2}{3}$ of the measure of $[p]$.

Now proceed as in the proof of Proposition 9 to define an algorithm which, on input n, applies the oracle algorithm with index e, using p as the source of oracle answers whenever the query is in $\mathrm{Dom}(p)$, and branching whenever the answer to the query is unavailable. In contrast to the proof of Proposition 9, however, do not stop and accept the first answer given on any branch. Instead, continue the computation (until a time to be specified later), recording all answers obtained along any branches, and also recording the probabilities conditional on $[p]$. That is, if an answer v is produced on a branch where, in addition to oracle answers from p, an additional m queries were asked (so there were m branchings on the path leading to v), then conditional probability 2^{-m} is assigned to this answer.

If the same answer is obtained on several branches, add the conditional probabilities. Continue computing until one answer has accumulated a probability $> \frac{1}{2}$. Then give that answer as the final result. This completes the description of the algorithm; we must verify that it produces the right answer, $a(n)$.

Consider, for each branch that produces an answer, the extension of p to $p' : \omega \rightharpoonup 2$ obtained by adjoining all the oracle answers used along that branch. (So $|\mathrm{Dom}(p') - \mathrm{Dom}(p)|$ is the m in our description of the algorithm.) Let C be the union of the sets $[p']$ corresponding to computations yielding the correct answer $a(n)$. Then $q \cap [p] - C$ has measure 0. To see this, notice that, if $q \cap [p] - C$ had positive measure, it would be a condition forcing that "the computation of the oracle algorithm with index e and oracle $\dot{r}$ on input n does not produce the answer $\check{a}(n)$." But this is absurd, because this condition is an extension of q, which forces "$\check{a}$ is recursive in $\dot{r}$ with index e."

So the measure of C is at least that of $q \cap [p]$ and thus at least $\frac{2}{3}$ of the measure of $[p]$. In our algorithm therefore, the sum of the conditional probabilities associated with the correct answer $a(n)$ would, if the algorithm were allowed to run forever, approach at least $\frac{2}{3}$. So this sum will exceed $\frac{1}{2}$ after a finite time, and the algorithm will then terminate with the correct answer $a(n)$. (Of course the algorithm cannot terminate earlier with a wrong answer, because all the conditional probabilities add up to at most 1, so only one answer can ever accumulate a total conditional probability $> \frac{1}{2}$.) $\square$

5 Refining and Cofinalities

In this section, we complete the description of needed reals for the relations corresponding to Cichoń's diagram as well as splitting and refining. Two of the deepest results here are from [7] (motivated by the talk on which this paper is based); the rest will be proved here.

Definition 15 $\mathbf{R}$ is the relation $(\mathcal{P}(\omega), [\omega]^\omega, \text{does not split})$.

Thus $\|\mathbf{R}\|$ is the refining number $\mathfrak{r}$, the smallest number of infinite subsets of ω such that every subset of ω almost (modulo finite sets) includes one of them or is disjoint from one of them.

Theorem 16 *Every real needed for* $\mathbf{R}$ *is hyperarithmetic.*

Proof Suppose a is needed for $\mathbf{R}$, and let Y be a subset of ω such that a is recursive in every infinite subset of Y and in every infinite subset of $\omega - Y$. Then every infinite subset X of ω has an infinite subset Z included in or disjoint from Y and therefore $\geq_T a$. By the implication (4) $\Longrightarrow$ (5) in Theorem 3, it follows that a is hyperarithmetic. $\square$

It is shown in [7] that the converse of this theorem is not provable in ZFC. As far as I know, it is open whether the converse is refutable in ZFC.

At the end of Section 2 we showed that all hyperarithmetic reals are needed for $\mathbf{Cof}(\mathcal{B})$ and for $\mathbf{Cof}(\mathcal{L})$. We now prove the converse of this for the category case. The proof depends on a theorem about Hechler reals analogous to Propositions 9 and 14 about Cohen and random reals, but with a crucial difference: The conclusion is "hyperarithmetic" rather than "recursive."

Proposition 17 *Suppose h is a real Hechler-generic over a model M of set theory. Suppose further that $a \in M$ is a real recursive in h. Then a is hyperarithmetic.*

Proof Working in M, fix an index $e \in \omega$ and fix a Hechler condition (p, f) in the generic filter defining h, forcing "$\check{a}$ is recursive in $\dot{h}$ with index e."

Define an oracle algorithm, with oracle f, to compute a as in the proof of Proposition 9, using the finite part p of (p, f) as the known information to respond to queries, and branching whenever a query asks for unknown information. But when branching to answer a query of the form "What is $h(k)$?" use only answers $\geq f(k)$. (In other words, pretend you know that $h \geq f$ except where p says otherwise.) As in the proof of Proposition 9 accept the first answer you get from any branch and use it as your final answer.

The branch corresponding to answers from h will produce an answer, since a is recursive in h with index e. So our algorithm will produce an answer, but it may come from another branch, using incorrect information about h. Nevertheless, the answer will be correct because the information used to produce it is consistent with the condition (p, f), which forces the generic real to produce the correct a via algorithm e.

This shows that a is recursive in f. Furthermore, it shows that a is recursive in any $g \geq f$, since all we used about f was that (p, f) forces a certain statement, and (p, g), an extension of (p, f), will force the same statement.

Therefore, a satisfies statement (2) in Theorem 3. By the implication (2) $\Longrightarrow$ (5) from that theorem, a is hyperarithmetic. $\square$

We shall also need the following hyperarithmetic analog of Proposition 9.

Proposition 18 *Suppose c is a real Cohen-generic over a model M of set theory. Suppose further that $a \in M$ is a real hyperarithmetic in c. Then a is hyperarithmetic.*

Proof Fix a condition p, a finite part of c, forcing that $\check{a}$ satisfies a particular Δ^1_1 description in terms of $\dot{c}$. Then the fact that p forces this is itself a Δ^1_1 description of a in the ground model. The reason is that Cohen forcing conditions can be coded as natural numbers, and so the definition of forcing adds only number quantifiers to the complexity of the statement being forced. $\square$

Theorem 19 *Every real needed for* $\mathbf{Cof}(\mathcal{B})$ *is hyperarithmetic.*

Proof Suppose a is needed for $\mathbf{Cof}(\mathcal{B})$ with witness y. That is, if x codes a meager F_σ superset of the meager F_σ set coded by y, then $a \leq_T x$. Being a Π^1_2 statement about a and y, this remains true in any forcing extension of the universe V, in particular in the extension $V[c, h]$ obtained by first adjoining a real c Cohen-generic over V and then a real h Hechler-generic over $V[c]$. It is known (see for instance [2]) that $V[c, h]$ contains a code x for a meager F_σ

set that includes all meager F_σ sets coded in the ground model V. In fact, x is recursive in the Turing join $c \oplus h$. Thus, we have $a \leq_T x \leq_T c \oplus h$.

Applying Proposition 17 relativized to c and with $V[c]$ as the ground model M, we find that a is hyperarithmetic in c. Then applying Proposition 18 (unrelativized and with V as the ground model), we obtain that a is hyperarithmetic. □

The analogous result for $\mathbf{Cof}(\mathcal{L})$ is considerably more difficult and is established in [7].

Summarizing our results and those from [7] we can describe the needed reals for (relations corresponding to) all the cardinals in Cichoń's diagram as follows.

Corollary 20 *Exactly the hyperarithmetic reals are needed for* $\mathbf{D}$*,* $\mathbf{Cof}(\mathcal{B})$*, and* $\mathbf{Cof}(\mathcal{L})$*. Exactly the recursive reals are needed for the rest of the relations in Cichoń's diagram: the additivities, uniformities, coverings, and bounding.*

In addition, the reals needed for splitting are exactly the recursive ones, while those needed for refining are at most the hyperarithmetic ones and can consistently be fewer.

6 Recursion in Generic Reals

We have proved several propositions (9, 14, 17) describing the ground-model reals recursive in certain sorts of generic reals. In this section, we do the same for several other sorts of generic reals. Before stating our results, we need a convention and a distinction.

The convention is that we work in a context where plenty of generic filters are available. For any notion of forcing and any condition in it, there should be a generic filter containing that condition. For example, we could work with a countable ground model. Alternatively, we could take the whole universe as our ground model but pass to Boolean-valued extensions to get the required generic objects.

The distinction is between universal and existential quantification over generic reals. It is possible for a ground model real to be recursive in some but not all generic reals of a certain sort. So there are really two questions for each type of forcing: Which ground model reals are recursive in some generic real? And which are recursive in all generic reals?

The answers are given in the table below, in which the first column lists the types of forcing we shall consider, the second tells which ground model reals are recursive in some generic real of these types, and the third tells which ground model reals are recursive in all generic reals of these types.

Forcing	Some	All
Cohen	recursive	recursive
random	recursive	recursive
Hechler	hyperarithmetic	hyperarithmetic
Sacks	arbitrary	recursive
Miller	arbitrary	recursive
Laver	arbitrary	hyperarithmetic
Mathias	arbitrary	hyperarithmetic

Propositions 9 and 14 immediately imply the first two lines of this table. As for the third, we already have

$$\begin{aligned}\text{recursive in all Hechler reals} &\implies \text{recursive in some Hechler real}\\ &\implies \text{hyperarithmetic}\end{aligned}$$

by Proposition 17. The following proposition closes the cycle.

Proposition 21 *Every hyperarithmetic real is recursive in every generic real that is $\geq^*$ all ground model functions. In particular, every hyperarithmetic real is recursive in every Hechler real, every Laver real, and every Mathias real.*

Proof If a is hyperarithmetic, then by Theorem 3 there is $f \in {}^\omega\omega$ in the ground model such that a is recursive in every real $\geq f$. As finite changes in a real don't affect its Turing degree, a is recursive in every real $\geq^* f$. This statement, being Π^1_1, is absolute, so it remains true in any forcing extension. $\square$

This completes the verification of the first three rows of the table. The next proposition gives the four "arbitrary" entries in the rest of the "Some" column.

Proposition 22 *Every real in the ground model is recursive in some Sacks real, some Miller real, some Laver real, and some Mathias real.*

Proof Let an arbitrary $a \in {}^{\omega}2$ be given. The tree

$$T = \{s \in {}^{<\omega}2 : \left(\forall n < \frac{\text{length}(s)}{2}\right) s(2n) = a(n)\}$$

is a Sacks condition. Since every path through it codes a (as every second bit), T forces that the canonical generic real adjoined by Sacks forcing is $\geq_T a$.

If, in the definition of T, we change ${}^{<\omega}2$ to ${}^{<\omega}\omega$, then the same argument works for Miller forcing.

If, in addition, we change the equality $s(2n) = a(n)$ to congruence modulo 2, then T is a Laver condition and the same argument still works.

Finally, to handle Mathias forcing, fix a bijection from ${}^{<\omega}2$ to ω, and let B be the image, under this bijection, of the set of finite initial segments of a. Notice that a is recursive in every infinite subset of B and that this fact is absolute. So the Mathias condition $(\emptyset, B)$ forces the canonical generic real to be $\geq_T a$. □

Finally, we work our way down the remaining four entries in the "All" column of the table. We already know that every recursive real is recursive in every Sacks or Miller real (trivially) and every hyperarithmetic real is recursive in every Laver or Mathias real (by Proposition 21). It remains to establish the converses. The two "hyperarithmetic" cases are easy consequences of Theorem 3.

Proposition 23 *If a real a is not hyperarithmetic, then there exist Laver reals and Mathias reals in which a is not recursive.*

Proof According to the implication (4) $\implies$ (5) in Theorem 3, we can fix an infinite subset X of ω in the ground model such that

$$a \text{ is not recursive in any } Y \subseteq X.$$

This displayed fact is Π^1_1 and therefore absolute. So we need only find a Laver real and a Mathias real that are subsets of X. But this is trivial: The tree of initial segments of subsets of X is a Laver condition, and $(\emptyset, X)$ is a Mathias condition, and they force the respective generic reals to be as required. □

Proposition 24 *If a real a is not recursive, then there exist Sacks reals in which a is not recursive.*

Proof Consider (in the ground model) the set $N = \{x \in {}^{\omega}2 : a \not\leq_T x\}$. If N is countable, then it gets no new members in any forcing extension, because the Π^1_1 fact that a particular real codes all the members of N is absolute. In particular, if we adjoin a Cohen real to our ground model, it will not be in (the set in the extension defined like) N. So a will be recursive in this Cohen real. But, by Proposition 9, this contradicts the hypothesis that a is not recursive.

So N is uncountable. Since it is obviously a Borel set, it has a perfect subset, the set of paths through some perfect tree $T \subseteq {}^{<\omega}2$. Then T is a Sacks condition forcing the canonical generic real to be in (the set in the extension defined like) N and therefore not to be $\geq_T a$. $\square$

Proposition 25 *If a real a is not recursive, then there exist Miller reals in which a is not recursive.*

Proof Proceeding analogously to the previous proof, consider (in the ground model) the set $N = \{x \in {}^{\omega}\omega : a \not\leq_T x\}$ (note: ${}^{\omega}\omega$, not ${}^{\omega}2$). If there is (in the ground model) a function $f \in {}^{\omega}\omega$ that is $\geq^*$ all members of N, then this fact remains true in any forcing extension. In particular, if we add a Cohen generic subset of ω then its increasing enumeration will not be in (the set in the extension defined like) N, since it is not majorized by any ground model function. So a is recursive in this Cohen real. By Proposition 9, this contradicts the hypothesis that a is not recursive.

So N is unbounded in ${}^{\omega}\omega$. Since it is obviously a Borel set, a theorem of Kechris [5] and Saint-Raymond [9] provides a superperfect tree T such that all paths through T are in N. Then T is a Miller condition forcing the canonical generic real to be in (the set in the extension defined like) N and therefore not to be $\geq_T a$. $\square$

References

[1] A. Blass, "Questions and answers: A category arising in linear logic, complexity theory, and set theory," in *Advances in Linear Logic*, ed. by J.-Y. Girard, Y. Lafont, and L. Regnier, London Math. Soc. Lecture Notes 222 (1995) 61–81.

[2] A. Blass, "Reductions between cardinal characteristics of the continuum," in *Set Theory: Annual Boise Extravaganza in Set Theory (BEST)*

Conference, 1992–1994, ed. by T. Bartoszyński and M. Scheepers, Contemporary Mathematics 192, Amer. Math. Soc. (1996) 31–49.

[3] D. Fremlin, "Cichoń's diagram," in *Séminaire Initiation à l'Analyse*, ed. G. Choquet, M. Rogalski, and J. Saint-Raymond, Publications Mathématiques de l'Université Pierre et Marie Curie, Paris (1984) 5-01–5-13.

[4] C. G. Jockusch, Jr., "Uniformly introreducible sets," J. Symbolic Logic 33 (1968) 521–536.

[5] A. S. Kechris, "On a notion of smallness for subsets of the Baire space," Trans. Amer. Math. Soc. 229 (1977) 191–207.

[6] K. de Leeuw, E. F. Moore, C. E. Shannon, and N. Shapiro, "Computability by probabilistic machines," in *Automata Studies*, ed. by C. E. Shannon and J. McCarthy, Princeton Univ. Press (1956) 183–212.

[7] H. Mildenberger and S. Shelah, "Needed reals" [Shelah's paper 725], preprint.

[8] J. Pawlikowski and I. Recław, "Parametrized Cichoń's diagram and small sets," Fundamenta Mathematicae 147 (1995) 135–155.

[9] J. Saint-Raymond, "Approximation des sous-ensembles analytiques par l'interieur," C. R. Acad. Sci. Paris, Ser. A–B 281 (1975) A85–A87.

[10] R. M. Solovay, "Hyperarithmetically encodable sets," Trans. Amer. Math. Soc. 239 (1978) 99–122.

[11] P. Vojtáš, "Generalized Galois-Tukey connections between explicit relations on classical objects of real analysis," in *Set Theory of the Reals*, ed. by H. Judah, Israel Mathematical Conference Proceedings 6, Amer. Math. Soc. (1993) 619–643