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A FAITHFUL MODAL INTERPRETATION OF PROPOSITIONAL ONTOLOGY

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Abstract. The propositional ontology introduced by Ishimoto is essentially the first-order universal fragment of Leśniewski's ontology. We give a faithful interpretation of this system in the modal logic K.

Introduction

Inoué [2] gave an interpretation of Ishimoto's propositional ontology [3] in the modal logic K. (The terminology used here will be defined below.) He showed that his interpretation is not faithful in general, although it is for a restricted class of formulas. In this note, we present another interpretation of propositional ontology in K, and we show that it is faithful.

In Section 1, we provide some known background information about Leśniewski's ontology and certain fragments of it. We include more in this section than is strictly needed in what follows, in order to place propositional ontology in its proper context. Section 2 contains the definitions of propositional ontology and K, and a discussion of their models. In Section 3, we describe our interpretation, compare it with Inoué's, and prove its correctness. Finally, Section 4 contains the proof that this interpretation is faithful.

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1. Ontology

Ontology is a logical system designed by Leśniewski [5] to formalize his understanding of names and the relation "is" between them. Names can be thought of as arbitrary nouns, both proper and common. Intuitively, "a is b" means that the name a applies to exactly one object and the name b also applies to that object (and possibly others as well). This intuition is formalized in the (original) axiom of ontology,

$$\begin{aligned} \varepsilon xy \iff & (\exists z)\, \varepsilon zx \wedge \\ & \forall z, w\, (\varepsilon zx \wedge \varepsilon wx \implies \varepsilon zw) \wedge \\ & \forall z\, (\varepsilon zx \implies \varepsilon zy). \end{aligned} \tag{1}$$

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Here we have used Leśniewski's symbol ε for "is", so εxy means "x is y," but we have used the customary notation for connectives and quantifiers as well as the convention that free variables in axioms are to be regarded as universally quantified.

Although the axiom (1) is formulated in first-order logic (without equality), Leśniewski intended it to be used in the context of a much richer logic, essentially equivalent to a type theory with arbitrary finite types over the base sets of names and truth values. The underlying logic incorporates axioms of comprehension (formulated as rules for defining new symbols) and extensionality for these higher types as well as for the base type of names. For more information on this underlying logic, see [6].

In the presence of this underlying logic, Leśniewski's axiom (1) is equivalent to each of the following, of which (2) is the form used by Inoué [2], (3) will be involved in our subsequent discussions, and (4) is mentioned because of its brevity. For proofs of the equivalences, see [7].

$$\varepsilon xy \iff \exists z\,(\varepsilon zx \wedge \varepsilon zy) \wedge \forall z, w\,(\varepsilon zx \wedge \varepsilon wx \implies \varepsilon zw). \tag{2}$$

$$\varepsilon xy \iff \exists z\,(\varepsilon zx \wedge \varepsilon zy) \wedge \forall z\,(\varepsilon zx \implies \varepsilon xz). \tag{3}$$

$$\varepsilon xy \iff \exists z\,(\varepsilon xz \wedge \varepsilon zy). \tag{4}$$

Of these, (2) is equivalent to (1) even in first-order logic, but (3) is slightly weaker and (4) much weaker in first-order logic. The implications from (4) to (3) and from (3) to (2) (or (1)) depend on the details of Leśniewski's rules for definitions; see [7].

We describe the models, in the sense of first-order logic, of (1), (2), and (3), leaving (4) to the reader since we shall have no use for that (somewhat more complicated) description. Henceforth, when we refer to models of a first-order sentence (like any of the axioms above) without specifying an underlying logic, first-order logic is to be understood; in particular, no comprehension principles need to be satisfied.

We begin with (3), since it is the axiom most directly relevant to propositional ontology. It is easy to deduce from (3) that

$$\varepsilon xx \iff \exists z\,(\varepsilon zx) \wedge \forall z\,(\varepsilon zx \implies \varepsilon xz), \tag{5}$$

$$\varepsilon xy \iff \varepsilon xx \wedge \exists z\,(\varepsilon zx \wedge \varepsilon zy), \tag{6}$$

and thence

$$\varepsilon xx \iff \exists y\,\varepsilon xy. \tag{7}$$

We call an element of a model *proper* if it satisfies εxx. (These elements correspond roughly to proper names, in that they apply to a unique object, in the intuitive meaning of ontology.) By (7), the domain of the ε relation in a model of (3) consists of exactly the proper elements.

The restriction of the ε relation to proper elements is reflexive (by definition of properness), symmetric (by the last part of (5)), and transitive (by the last part of (6)). So this restriction is an equivalence relation on the proper part of the model. Furthermore, the "rest" of ε, i.e., its restriction to proper first arguments and improper second arguments, respects this equivalence relation on the first argument. That is, if x and z are proper and ε-related and if εzy, then also εxy; this follows immediately from (6).

Conversely, it is easy to check that, if E is an equivalence relation on a set P and if $F \subseteq P \times I$ is a binary relation from P to a disjoint set I and if F respects E (in the sense that zEx and zFy together imply xFy), then the set $P \cup I$ with the binary relation $E \cup F$ is a model of (3). We thus have a complete description of all models of (3).

Equivalent (in the sense of ε) proper elements are indistinguishable in such a model; so are improper elements that have exactly the same proper elements ε-related to them. By identifying such indistinguishable elements, we can reduce every model of (3) to one in which the ε relation on proper elements is just equality and where each improper element is uniquely determined by the set of proper elements ε-related to it. Such a reduced model is therefore isomorphic to one of the form $\langle P \cup I, \varepsilon \rangle$ where I is a family of subsets of P and where ε is the union of the equality relation on P and the membership relation between P and I. (We assume that P is chosen so that no member of it is also a subset; so P and I are disjoint.) We call models of this simple form *normal* models of (3). Every model of (3) is obtainable from a normal one by replacing each element by some (non-zero) number of copies of itself. Furthermore, this replacement does not affect what first-order sentences (without equality) are true in the model.

The structure of models of (1) and (2) can be similarly analyzed, and the results are very similar to those obtained for (3). In fact, the models of (1) coincide with those of (2), and they differ from those of (3) only in that, for any improper element, the proper elements ε-related to it cannot constitute exactly one equivalence class. In terms of normal models, this means that I contains no singletons. (This restriction follows easily from (1) and from (2) in first-order logic; it is not a first-order consequence of (3), but it is a consequence of (3) in the presence of Leśniewski's comprehension and extensionality principles.)

By a *standard* model of ontology, we mean a normal model in which I consists of all the subsets of P except the singletons. Such a model satisfies all the first-order axioms above. If we interpret higher-type variables as ranging over all sets and functions of the appropriate types, then all of Leśniewski's axioms and rules are correct in such a model. In particular, any first-order statement that is provable in ontology (with Leśniewski's full logic) is true in every standard model.

Every normal model of (3) is, up to isomorphism, a submodel of some standard model. (This argument is due to Ishimoto [3].) Indeed, given a normal model, we can first enlarge it to make sure that I contains no singletons; just add one new proper element and add it as a member to any singletons in I. Then we can further enlarge the model by adding all non-singleton subsets of (the new) P to I. The result is clearly a standard model.

It follows from the preceding two paragraphs that any universal first-order sentence provable in ontology (with Leśniewski's full logic) is also deducible in first-order logic from (3). Indeed, if ϕ is such a sentence and if M is any model of (3), then M is (up to isomorphism) a submodel of a standard model M' in which ϕ, being provable in ontology, must be true. But universal sentences are preserved to substructures, so ϕ must hold in M.

2. Propositional Ontology

Not only does (3) imply (in first-order logic) all the universal first-order sentences provable in ontology, as shown at the end of the preceding section, but it is itself equivalent to such a sentence. That is, (3) is equivalent (in first-order logic) to a universal sentence. Indeed, our description of the models of (3) shows that any substructure of such a model is again one. (The same would not be true of (1).) The description also gives an explicit universal axiomatization, saying simply that ε is an equivalence relation on proper elements, that its domain consists only of proper elements, and that it respects equivalence in its first argument. In other words, (3) is equivalent (in first-order logic) to the following three axioms (where, as before, free variables are regarded as universally quantified).

$$\varepsilon xy \implies \varepsilon xx \tag{8}$$

$$\varepsilon xy \wedge \varepsilon yz \implies \varepsilon xz \tag{9}$$

$$\varepsilon xy \wedge \varepsilon yz \implies \varepsilon yx. \tag{10}$$

Here (8) ensures that the domain of ε consists of proper elements, (10) ensures symmetry of ε on proper elements (the variable z could be replaced with y, but we have chosen to follow the formulation in [3]), and (9) ensures transitivity of ε on proper elements and also requires ε between proper and improper elements to respect the equivalence.

Thus, (8), (9), and (10) constitute an axiomatization (in first-order logic) of the universal first-order part of Leśniewski's ontology. This result is due to Ishimoto [3], where it is formulated slightly differently. Ishimoto regards (8), (9), and (10) as quantifier-free formulas and shows that every quantifier-free formula provable in ontology is deducible by purely propositional reasoning (i.e., using tautologies and modus ponens) from instances of (8), (9), and (10). This variation is inessential, since it is well-known that, if a universal statement ϕ is first-order deducible from other universal first-order statements ψ_i, then the quantifier-free matrix of ϕ is propositionally deducible from instances of the matrices of the ψ_i. (This is a very special case of Herbrand's theorem; see for example the Théorème d'uniformité on page 21 of [4].) But the variation leads to one of the topics of this paper, *propositional ontology*, defined as the theory in classical propositional logic whose atomic propositions (sentential variables) are expressions εxy for certain (arbitrary but specified) symbols x, y, and whose axioms are all the instances (in this language) of (8), (9), and (10).

The other theory involved in our main result is the well-known modal logic K, the minimal normal modal logic [1]. Its language consists of primitive symbols (used as sentential variables but not to be confused with the sentential variables of propositional ontology), the usual propositional connectives, and the modal operator $\Box$. Its axioms are all tautologies (in this language) and

$$\Box(\phi \implies \psi) \implies (\Box\phi \implies \Box\psi),$$

and its rules of inference are modus ponens and necessitation (from ϕ infer $\Box\phi$). We record for future reference that the transitivity of necessary implication,

$$\Box(\phi \implies \psi) \wedge \Box(\psi \implies \chi) \implies \Box(\phi \implies \chi),$$

is easily provable in K.

A Kripke model (for K) is a set W (whose elements are called worlds) equipped with a binary relation $\prec$ (called accessibility) and with an interpretation $|x| \subseteq W$ for each primitive symbol x. Interpretations $|\phi| \subseteq W$ are assigned to all formulas ϕ by induction on their structure, using the obvious set-theoretic operations for the propositional connectives (complement for negation, union for disjunction, etc.) and using

$$|\Box\phi| = \{w \in W \mid \forall z\,(w \prec z \implies z \in |\phi|)\}.$$

It is well-known and trivial to prove that every theorem ϕ of K has $|\phi| = W$ in every Kripke model. (It is also well-known though not so trivial to prove that this characterizes the theorems of K.) For more information about K and its Kripke models, see Chapters 1 and 5 of [1].

3. The Interpretation

The definition of propositional ontology involved an unspecified set of symbols, to be used as the x and y in forming the atomic propositions εxy. Let us fix, from now on, a specific set of symbols for this purpose. Furthermore, let us use these same symbols as the primitive symbols (propositional variables) of K. To avoid confusion, we emphasize that, with these conventions, the atomic formulas in propositional ontology and K are quite different; in the latter they are the symbols x from the fixed set while in the former they have the form εxy for pairs of such symbols x, y.

In this context, Inoué [2] defines an interpretation of propositional ontology in K. He interprets εxy as $x \wedge \Box(x \iff y)$ and leaves all connectives unchanged. He shows that every theorem of propositional ontology becomes, under this translation, a theorem of K. Unfortunately, he also shows that the converse fails; there are formulas ϕ not provable in propositional ontology, whose translations are nevertheless provable in K. Much of his work in [2] is devoted to producing a reasonable class of formulas for which this does not occur.

We present here a modification of Inoué's interpretation that makes the converse (usually called faithfulness of the interpretation) true for all formulas. Specifically, we translate atomic formulas εxy of propositional ontology as

$$T(\varepsilon xy) := x \wedge \Box(x \implies y) \wedge [y \implies \Box(y \implies x)],$$

and we leave propositional connectives unchanged. Notice that, if we deleted the first occurrence of "$y \implies$" from our interpretation of εxy we would get Inoué's interpretation (up to provable equivalence in K).

We first verify that this is indeed an interpretation of propositional ontology in K.

Theorem 1. *If ϕ is provable in propositional ontology then $T(\phi)$ is provable in K.*

Proof. Since the only rule of inference in propositional ontology, namely modus ponens, is also a rule of inference in K, and since the translation of any tautology is another tautology (because T leaves connectives unchanged), it suffices to verify that the translation of every axiom of propositional ontology is provable in K. We consider the three axiom schemes (8), (9), and (10) in turn.

For (8), we note that $\Box(x \implies x)$ is provable in K and so $T(\varepsilon xx)$ implifies to just x. Thus the translation of (8) reduces to the form $(x \wedge \dots) \implies x$, a tautology.

For (9), we must prove in K an implication whose antecedent is the conjunction of the six formulas

$$x, \tag{11}$$

$$\Box(x \implies y), \tag{12}$$

$$y \implies \Box(y \implies x), \tag{13}$$

$$y, \tag{14}$$

$$\Box(y \implies z) \tag{15}$$

$$z \implies \Box(z \implies y), \tag{16}$$

and whose consequent is the conjunction of the three formulas

$$x, \tag{17}$$

$$\Box(x \implies z), \tag{18}$$

$$z \implies \Box(z \implies x). \tag{19}$$

Since (17) is identical with (11), and since (18) follows from (12) and (15) by transitivity of necessary implication, we consider (19). In the instance

$$\Box(z \implies y) \wedge \Box(y \implies x) \implies \Box(z \implies x) \tag{20}$$

of transitivity of necessary implication, the first conjunct in the antecedent follows tautologically from $z \wedge (16)$ and the second from $(13) \wedge (14)$. Thus, (13), (14), (16), and (20) together imply (19).

Finally, the translation of (10) is an implication whose antecedent includes conjuncts x, $\Box(x \implies y)$, $y \implies \Box(y \implies x)$, and y (and more). The consequent of the implication, consisting of conjuncts y, $\Box(y \implies x)$, and $x \implies \Box(x \implies y)$, is a tautological consequence. □

4. Faithfulness of the Interpretation

We prove the theorem that justifies introducing an interpretation more complicated than Inoué's.

Theorem 2. *The interpretation T is faithful. That is, if ϕ is a formula in the language of propositional ontology and if $T(\phi)$ is provable in K, then ϕ is provable in propositional ontology.*

Proof. Suppose ϕ were a counterexample. As ϕ is not provable in propositional ontology, we fix an assignment of truth values to the atomic formulas εxy making all instances of (8), (9), and (10) true but making ϕ false. This truth assignment defines the ε relation of a model of (3) whose universe is the set of symbols x occurring in the atomic formulas εxy of propositional ontology. So we have a model of (3) in which ϕ is false for certain values of the symbols x occurring in it. By identifying indistinguishable elements and replacing the model with an isomorphic copy, we may assume, without loss of generality, that the model is normal. Recall from Section 1 that this means that its universe is a disjoint union $P \cup I$ where I is a family of subsets of P and that ε is the union of the equality relation on P and the membership relation between P and I. Before we normalized the model, each of our symbols x was an element of the model; the image of that element, under the identification and the isomorphism involved in the normalization, will be called the value of x and written $\hat{x}$. Notice that a formula in the language of propositional ontology can be regarded as a first-order formula with the symbols as variables, and that it is true in our normal model with the indicated values for the variables if and only if it was true under the truth assignment we started with. In particular, our assignment of values to the symbols makes ϕ false in our normal model.

We construct a Kripke model as follows. The set W of worlds contains a world w_p for each $p \in P$ and one additional world $*$. The accessibility relation is defined so that $* \prec w_p$ for all p and nothing else is accessible (i.e., $w \prec w' \iff w = * \neq w'$). If $\hat{x} \in P$, then $|x|$ is defined to be $\{w_{\hat{x}}, *\}$. If $\hat{x} \in I$, then $|x|$ is defined to be the set of all w_p for $p \in \hat{x}$.

Lemma. *For any formula ψ of propositional ontology, $* \in |T(\psi)|$ if and only if ψ is true under our truth assignment.*

Proof of Lemma. We proceed by induction on the structure of ψ. The induction step for propositional connectives is trivial, so we need only consider the case that ψ is εxy.

If $\hat{x}$ is improper, then $* \notin |x|$ by definition and therefore $* \notin |T(\varepsilon xy)|$ since $T(\varepsilon xy)$ contains x as a conjunct. Furthermore, by definition of "improper," εxx has the value false, and so does εxy because the truth assignment satisfies (8). So the lemma is true in this case.

Henceforth, we assume $\hat{x}$ is proper. So $* \in |x|$. Suppose first that εxy is true. If $\hat{y}$ is also proper, then (since ε on P is equality) $\hat{x} = \hat{y}$, from which follow $|x| = |y|$ and thence $|\Box(x \implies y)| = |\Box(y \implies x)| = W$. Since we already had $* \in |x|$, it follows immediately that $* \in |T(\varepsilon xy)|$. If, on the other hand, $\hat{y}$ is improper, then $* \notin |y|$ and from the truth of εxy we infer $\hat{x} \in \hat{y}$. It follows that $* \in \Box(x \implies y)$, for among the worlds w_p accessible from $*$, only $w_{\hat{x}}$ is in $|x|$ and it is also in $|y|$. Since $*$ is in $|x|$ but not in $|y|$, we again have $* \in T(\varepsilon xy)$, as desired.

Now suppose εxy is false. If $\hat{y}$ is also proper, then falsity of εxy means that $\hat{x} \neq \hat{y}$, so $w_{\hat{x}}$ is in $|x|$ but not in $|y|$. Thus, a world accessible from $*$ is not in $|x \implies y|$ and so $*$ is not in $|\Box(x \implies y)|$ and a fortiori not in $|T(\varepsilon xy)|$. Finally, consider the case that $\hat{y}$ is improper. Falsity of εxy means $\hat{x} \notin \hat{y}$, so $w_{\hat{x}}$ is in $|x|$ but not in $|y|$. As before, it follows that $* \notin |T(\varepsilon xy)|$. $\Box$

Since we began with a truth assignment making ϕ false, the lemma tells us that $|T(\phi)|$ does not contain $*$ and is therefore certainly not all of W. But $T(\phi)$ was provable in K, so this contradicts the soundness of K for Kripke models. $\Box$

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