POWER-DEDEKIND FINITENESS

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At the May, 2012, Finite Model Theory conference at Les Houches, Mikołaj Bojańczyk described "orbit-finiteness" in permutation models and proposed it as a substitute for finiteness in many computer-science contexts. This note collects some observations about this concept.

Definition 1. Work in a Fraenkel-Mostowski permutation model M determined by a group G of permutations of a set A of atoms and finite supports. A set $X \in M$ is *orbit-finite* if, for every (not necessarily minimal) finite support E of X, the pointwise stabilizer in G of E, which I call $\mathrm{Fix}(E)$ and which obviously acts on X, has only finitely many orbits in X.

Proposition 2. *X is orbit-finite if and only if its power set (in the sense of M) $\mathcal{P}^M(X)$ is Dedekind-finite in M.*

Proof. Recall (for example from [1, Proposition 2.1], but it must be much older) that a set Y in a permutation model is Dedekind finite there iff each group fixing Y in the filter defining M fixes only finitely many elements of Y. In support models, this means that each support is the support of only finitely many elements. But it is easy to see that a subset S of X is (in M and) supported by E iff it is a union of $\mathrm{Fix}(E)$-orbits in X. □

Definition 3. Call a set X *PD-finite* if its power set is Dedekind-finite.

So orbit-finiteness amounts to PD-finiteness in permutation models, but PD-finiteness can be studied more generally, in the context of ZF or ZFA.

PD-finiteness has some nice preservation properties.

Proposition 4. *All of the following are PD-finite.*

(1) *Any subset of a PD-finite set*
(2) *Any quotient of a PD-finite set*
(3) *The union of any two PD-finite sets*
(4) *Any indexed union $\bigcup_{i\in I} X_i$ where both the index set I and all the constituents X_i are PD-finite*

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(5) *The product of any two PD-finite sets*

Proof. Recall that subsets of Dedekind-finite sets are Dedekind-finite. That gives items (1) and (2) because if Y is either a subset or a quotient of X, then $\mathcal{P}(Y)$ is equinumerous with a subset of $\mathcal{P}(X)$. Items (3) and (5) are special cases of (4), so it remains only to prove (4).

Suppose, toward a contradiction, that the assumptions in (4) hold but we have a one-to-one map $f : \omega \to \mathcal{P}(\bigcup_{i\in I} X_i)$. To each $x \in \bigcup_{i\in I} X_i$, we associate a real $g(x) \in \mathcal{P}(\omega)$ by $g(x) = \{n \in \omega : x \in f(n)\}$. At this point, we invoke the following lemma, which will be proved after the rest of the proof of the proposition is complete.

Lemma 5. *Every PD-finite set of reals is finite.*

By the lemma and part (2) of the present proposition, already proved above, we find that, for each $i \in I$, the image $g[X_i]$ of the PD-finite set X_i is finite. Since the reals admit a canonical linear ordering, every finite set of reals can be regarded as an ordered tuple of reals (by listing the elements in increasing order) and can therefore be coded as a single real in some standard way. Let $h(i)$ be the code for the finite set $g[X_i]$ of reals. Invoking the lemma and part (2) again, we see that the image $h[I]$ of the PD-finite index set I is finite. So

$$g[X] = \bigcup_{i\in I} g[X_i]$$

is, despite the apparent indexing by I, really a union of finitely many finite sets, hence finite.

Each $n \in \omega$ gives rise to a subset $j(n) = \{z \in g[X] : n \in z\}$ of $g[X]$, and, since $g[X]$ has only finitely many subsets, there must be distinct $m, n \in \omega$ with $j(m) = j(n)$. That is, m and n belong to precisely the same sets of the form $g(x)$ for $x \in X$. By definition of g, this means $f(m) = f(n)$, contrary to the assumption that f is one-to-one. This completes the proof except for the lemma. □

Proof of Lemma. It is notationally convenient to take the set ${}^{\omega}2$ of ω-sequences of 0's and 1's as the reals. Suppose S is an infinite set of reals; we shall show that S cannot be PD-finite.

Define a sequence $\vec{b} = \langle b_i : i \in \omega\rangle \in {}^{\omega}2$ so that, for each $n \in \omega$, each initial segment $\vec{b}\restriction n = \langle b_i : i < n\rangle$ of $\vec{b}$ is also an initial segment of infinitely many members of S. It is trivial to define such a $\vec{b}$ by induction. The rest of the proof splits into two cases.

Case 1: For only finitely many $n \in \omega$ is the sequence

$$c_n = (\vec{b}\restriction n)^\frown\langle 1 - b_n\rangle$$

an initial segment of a member of S. Choose $N < \omega$ larger than all those finitely many n's. Then every member x of S that has $\vec{b} \restriction N$ as an initial segment must also have, as initial segments, $\vec{b} \restriction n$ for all $n > N$. Indeed, if x were a counterexample, then consider the first n such that $\vec{b} \restriction n$ is not an initial segment of x. Since x extends $\vec{b} \restriction (n-1)$ but not $\vec{b} \restriction n$, it must extend c_{n-1}, contrary to our choice of N. But then every $x \in S$ with $\vec{b} \restriction N$ as an initial segment is equal to $\vec{b}$. That contradicts the definition of $\vec{b}$, which says that there must be infinitely many distinct such x's. So Case 1 cannot occur.

Case 2: For infinitely many n, the set of extensions in S of c_n is nonempty. These sets, for different n, are disjoint, as no c_n is an extension of another. So the nonempty ones are distinct subsets of S. Thus, we have a countably infinite collection of distinct elements of $\mathcal{P}(S)$; that is, S is not PD-finite. □

References

[1] David Blair, Andreas Blass, and Paul Howard, "Divisibility of Dedekind finite sets," *J. Math. Logic* 5 (2005) 49–85.

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