

# POWER-DEDEKIND FINITENESS

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At the May, 2012, Finite Model Theory conference at Les Houches, Mikołaj Bojańczyk described “orbit-finiteness” in permutation models and proposed it as a substitute for finiteness in many computer-science contexts. This note collects some observations about this concept.

**Definition 1.** Work in a Fraenkel-Mostowski permutation model  $M$  determined by a group  $G$  of permutations of a set  $A$  of atoms and finite supports. A set  $X \in M$  is *orbit-finite* if, for every (not necessarily minimal) finite support  $E$  of  $X$ , the pointwise stabilizer in  $G$  of  $E$ , which I call  $\text{Fix}(E)$  and which obviously acts on  $X$ , has only finitely many orbits in  $X$ .

**Proposition 2.**  $X$  is orbit-finite if and only if its power set (in the sense of  $M$ )  $\mathcal{P}^M(X)$  is Dedekind-finite in  $M$ .

*Proof.* Recall (for example from [1, Proposition 2.1], but it must be much older) that a set  $Y$  in a permutation model is Dedekind finite there iff each group fixing  $Y$  in the filter defining  $M$  fixes only finitely many elements of  $Y$ . In support models, this means that each support is the support of only finitely many elements. But it is easy to see that a subset  $S$  of  $X$  is (in  $M$  and) supported by  $E$  iff it is a union of  $\text{Fix}(E)$ -orbits in  $X$ .  $\square$

**Definition 3.** Call a set  $X$  *PD-finite* if its power set is Dedekind-finite.

So orbit-finiteness amounts to PD-finiteness in permutation models, but PD-finiteness can be studied more generally, in the context of ZF or ZFA.

PD-finiteness has some nice preservation properties.

**Proposition 4.** *All of the following are PD-finite.*

- (1) *Any subset of a PD-finite set*
- (2) *Any quotient of a PD-finite set*
- (3) *The union of any two PD-finite sets*
- (4) *Any indexed union  $\bigcup_{i \in I} X_i$  where both the index set  $I$  and all the constituents  $X_i$  are PD-finite*

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(5) *The product of any two PD-finite sets*

*Proof.* Recall that subsets of Dedekind-finite sets are Dedekind-finite. That gives items (1) and (2) because if  $Y$  is either a subset or a quotient of  $X$ , then  $\mathcal{P}(Y)$  is equinumerous with a subset of  $\mathcal{P}(X)$ . Items (3) and (5) are special cases of (4), so it remains only to prove (4).

Suppose, toward a contradiction, that the assumptions in (4) hold but we have a one-to-one map  $f : \omega \rightarrow \mathcal{P}(\bigcup_{i \in I} X_i)$ . To each  $x \in \bigcup_{i \in I} X_i$ , we associate a real  $g(x) \in \mathcal{P}(\omega)$  by  $g(x) = \{n \in \omega : x \in f(n)\}$ . At this point, we invoke the following lemma, which will be proved after the rest of the proof of the proposition is complete.

**Lemma 5.** *Every PD-finite set of reals is finite.*

By the lemma and part (2) of the present proposition, already proved above, we find that, for each  $i \in I$ , the image  $g[X_i]$  of the PD-finite set  $X_i$  is finite. Since the reals admit a canonical linear ordering, every finite set of reals can be regarded as an ordered tuple of reals (by listing the elements in increasing order) and can therefore be coded as a single real in some standard way. Let  $h(i)$  be the code for the finite set  $g[X_i]$  of reals. Invoking the lemma and part (2) again, we see that the image  $h[I]$  of the PD-finite index set  $I$  is finite. So

$$g[X] = \bigcup_{i \in I} g[X_i]$$

is, despite the apparent indexing by  $I$ , really a union of finitely many finite sets, hence finite.

Each  $n \in \omega$  gives rise to a subset  $j(n) = \{z \in g[X] : n \in z\}$  of  $g[X]$ , and, since  $g[X]$  has only finitely many subsets, there must be distinct  $m, n \in \omega$  with  $j(m) = j(n)$ . That is,  $m$  and  $n$  belong to precisely the same sets of the form  $g(x)$  for  $x \in X$ . By definition of  $g$ , this means  $f(m) = f(n)$ , contrary to the assumption that  $f$  is one-to-one. This completes the proof except for the lemma.  $\square$

*Proof of Lemma.* It is notationally convenient to take the set  ${}^\omega 2$  of  $\omega$ -sequences of 0's and 1's as the reals. Suppose  $S$  is an infinite set of reals; we shall show that  $S$  cannot be PD-finite.

Define a sequence  $\vec{b} = \langle b_i : i \in \omega \rangle \in {}^\omega 2$  so that, for each  $n \in \omega$ , each initial segment  $\vec{b} \upharpoonright n = \langle b_i : i < n \rangle$  of  $\vec{b}$  is also an initial segment of infinitely many members of  $S$ . It is trivial to define such a  $\vec{b}$  by induction. The rest of the proof splits into two cases.

*Case 1:* For only finitely many  $n \in \omega$  is the sequence

$$c_n = (\vec{b} \upharpoonright n) \frown \langle 1 - b_n \rangle$$

an initial segment of a member of  $S$ . Choose  $N < \omega$  larger than all those finitely many  $n$ 's. Then every member  $x$  of  $S$  that has  $\vec{b} \upharpoonright N$  as an initial segment must also have, as initial segments,  $\vec{b} \upharpoonright n$  for all  $n > N$ . Indeed, if  $x$  were a counterexample, then consider the first  $n$  such that  $\vec{b} \upharpoonright n$  is not an initial segment of  $x$ . Since  $x$  extends  $\vec{b} \upharpoonright (n-1)$  but not  $\vec{b} \upharpoonright n$ , it must extend  $c_{n-1}$ , contrary to our choice of  $N$ . But then every  $x \in S$  with  $\vec{b} \upharpoonright N$  as an initial segment is equal to  $\vec{b}$ . That contradicts the definition of  $\vec{b}$ , which says that there must be infinitely many distinct such  $x$ 's. So Case 1 cannot occur.

*Case 2:* For infinitely many  $n$ , the set of extensions in  $S$  of  $c_n$  is nonempty. These sets, for different  $n$ , are disjoint, as no  $c_n$  is an extension of another. So the nonempty ones are distinct subsets of  $S$ . Thus, we have a countably infinite collection of distinct elements of  $\mathcal{P}(S)$ ; that is,  $S$  is not PD-finite.  $\square$

#### REFERENCES

- [1] David Blair, Andreas Blass, and Paul Howard, "Divisibility of Dedekind finite sets," *J. Math. Logic* 5 (2005) 49–85.

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