Purity and Reid's Theorem

Andreas Blass*
Mathematics Department
University of Michigan
Ann Arbor, MI 48109–1109, U.S.A.
ablass@umich.edu

John Irwin
Professor Emeritus
Mathematics Department
Wayne State University
Detroit, MI 48202

Abstract

We give conditions under which an abelian group is the sum of two free subgroups, one or both of which are pure.

1 Introduction

Reid [1] proved that every torsion-free abelian group of infinite rank is the sum of two free subgroups. In this paper, we consider the problem of arranging for one or both of those free subgroups to be pure.

Theorem 1.1 *For any torsion-free abelian group G of infinite rank κ, the following two statements are equivalent.*

1. *G is the sum of two free subgroups, at least one of which is pure in G.*
2. *G has a pure free subgroup of rank κ.*

Moreover, any subgroup as in 2 can serve as one of the subgroups in 1.

With a stronger condition on the rank, we get the same result with both summands pure.

*Partially supported by NSF grant DMS–9505118

Theorem 1.2 *For any torsion-free abelian group G of uncountable rank κ, the following two statements are equivalent.*

1. *G is the sum of two pure free subgroups.*

2. *G has a pure free subgroup of rank κ.*

Moreover, any subgroup as in 2 can serve as one of the subgroups in 1.

The uncountability of the rank is really needed for the implication from 2 to 1 in Theorem 1.2. We shall construct a group of countable rank where 2 holds but 1 fails.

2 Proof of Theorem 1.1

Throughout this section and the next, G is a torsion-free abelian group of rank κ.

To prove the implication from 1 to 2 in Theorem 1.1, suppose $G = E+F$ with both E and F free and with E pure in G. If E has rank κ then we have the desired conclusion 2. So assume E has rank $\mu < \kappa$. Of course then F must have rank κ. Being a subgroup of E, the intersection $E \cap F$ is freely generated by a set X of cardinality at most μ.

Since F is free, fix a basis for it, and express all elements of X in terms of this basis. Since $\mu < \kappa$, fewer than κ basis elements occur in these expressions. Let F_1 be the subgroup of F generated by these basis elements and F_2 the subgroup generated by the rest of our basis. So $F = F_1 \oplus F_2$, $E \cap F \subseteq F_1$, and F_2 has rank κ.

To complete the verification of 2, it suffices to show that F_2 is pure in G. In fact, we show that it is a direct summand of G. Since $G = E + F = E + F_1 + F_2$, it suffices to show that $(E + F_1) \cap F_2 = (0)$. Suppose, therefore, that $e \in E$, $f_1 \in F_1$, $f_2 \in F_2$, and $e + f_1 = f_2$. Then $e = f_2 - f_1 \in E \cap F \subseteq F_1$. So $e + f_1 = f_2 \in F_1 \cap F_2 = (0)$. This completes the proof that 1 implies 2.

The converse implication and the "Moreover" statement in the theorem are proved just like Reid's theorem [1], but we include the proof here for the sake of completeness.

Since G has infinite rank κ and is torsion-free, it has cardinality κ, so we can enumerate it as $G = \{g_\alpha : \alpha < \kappa\}$, where we use the customary identification of a cardinal κ with the initial ordinal of that cardinality. Let E be a pure free subgroup of G of rank κ. We shall define a certain function $e : \kappa \to E$, set $f_\alpha = g_\alpha + e(\alpha)$, and let F be the subgroup of G generated by $\{f_\alpha : \alpha < \kappa\}$. No matter how we choose e, we shall have $G = E + F$, because each element g_α of G is $-e(\alpha) + f_\alpha$ with $-e(\alpha) \in E$ and $f_\alpha \in F$. So our task is to choose e so that F is free. In fact, we shall arrange that the f_α's are independent.

For this purpose, we work in the divisible hull $\bar{G}$ of G, and we define $e(\alpha)$ by transfinite recursion on α. So assume that $e(\beta)$ and therefore f_β are already defined for all $\beta < \alpha$ and that the f_β's are linearly independent (over $\mathbb{Q}$). We must choose $e(\alpha)$ so as to preserve this independence when f_α is adjoined. That is, we must choose $e(\alpha)$ so that $f_\alpha = g_\alpha + e(\alpha)$ is not in the $\mathbb{Q}$-span of $\{f_\beta : \beta < \alpha\}$. Equivalently, $e(\alpha)$ should not be in the affine subspace of $\bar{G}$ spanned by $\{f_\beta - g_\alpha : \beta < \alpha\} \cup \{-g_\alpha\}$. That affine subspace has dimension strictly smaller than κ because $\alpha < \kappa$. So it cannot include all of the rank κ group E. Therefore appropriate choices for $e(\alpha)$ exist (in great profusion). This completes the recursive construction and thus the proof of Theorem 1.1.

3 Proof of Theorem 1.2

In view of Theorem 1.1, we need only prove the implication from 2 to 1 and the "Moreover" assertion, on the assumption that the rank κ of G is uncountable. (As before, G is a torsion-free abelian group.) As in the corresponding part of the proof of Theorem 1.1, we begin with a pure free subgroup E of rank κ and an enumeration $\{g_\alpha : \alpha < \kappa\}$ of G, and we let F be generated by $\{f_\alpha : \alpha < \kappa\}$ where $f_\alpha = g_\alpha + e(\alpha)$ for a certain inductively defined $e : \kappa \to E$. As before, we trivially have $G = E + F$. As before, we must ensure that F is free, but in addition we must ensure that F is pure in G.

For this purpose, we shall arrange that the f_α are linearly independent modulo every prime p. In more detail, this means the following. For each prime number p, each finitely many (say k) ordinals $\alpha_1 < \alpha_2 < \cdots < \alpha_k < \kappa$, and each k integers $c_1, \ldots, c_k$, if $c_1 f_{\alpha_1} + \cdots + c_k f_{\alpha_k}$ is divisible by p in G then all the c_i are divisible by p in $\mathbb{Z}$.

If we achieve this, then it will follow that F is free and in fact the f_α are linearly independent. Indeed, any non-trivial dependence relation between them, over $\mathbb{Z}$, would also be a non-trivial dependence relation modulo any sufficiently large prime.

Furthermore, it will follow that F is pure in G. Indeed, linear independence modulo p clearly implies that, if $c_1 f_{\alpha_1} + \cdots + c_k f_{\alpha_k}$ is divisible by p in G, then it is divisible by p in F also.

So the proof will be complete if we define $e : \kappa \to E$ so that the f_α are linearly independent modulo every prime. We do this by recursion on α. Suppose we have already defined $e(\beta)$ for all $\beta < \alpha$ and that the resulting f_β's are independent modulo every prime. We wish to define $e(\alpha)$ so that this independence persists when f_α is included. This means that, for each prime p, the images $\bar{f}_\beta$ of f_β in G/pG should be linearly independent over $\mathbb{Z}/p\mathbb{Z}$ for $\beta \leq \alpha$. Since they're already independent for $\beta < \alpha$, we just have to make sure that $\bar{f}_\alpha$ is not a $\mathbb{Z}/p\mathbb{Z}$-linear combination of the earlier $\bar{f}_\beta$'s. Since $\alpha < \kappa$, there are (strictly) fewer than κ such linear combinations. Each is a coset of pG in G, and we must ensure that f_α is in none of these cosets. We use uncountability of κ to observe that, even for all primes p together, there are still fewer than κ cosets to be avoided.

Fix a basis for the free group E, and notice that the difference $b - b'$ of two basis elements is divisible by no primes p, in E and therefore in G as E is pure. Thus, none of the cosets that we must avoid can contain $g_\alpha + b$ for two different basis elements b. Thus, the fewer than κ cosets that we must avoid prohibit fewer than κ basis elements from serving as $e(\alpha)$. Since E has rank κ, the basis has κ elements, so some (in fact most) of them can serve as $e(\alpha)$.

4 A Counterexample

In this section, we show that the hypothesis of uncountable rank is needed in Theorem 1.2. We construct a group G and a pure, free, proper subgroup E, both of rank $\aleph_0$, such that every element of $G \setminus E$ is divisible by infinitely many primes. This clearly implies that every pure free subgroup of G is included in E, so G cannot be a sum of such subgroups.

Let $\bar{G}$ be a rational vector space with basis $\{e_0, e_1, e_2, \dots\} \cup \{g\}$, and let $\bar{E}$ be the subspace generated by $\{e_0, e_1, e_2, \dots\}$. (These will be the divisible hulls of our G and E.) Let E be the $\mathbb{Z}$-span of $\{e_0, e_1, e_2, \dots\}$, a free abelian group of rank $\aleph_0$. We shall obtain G as the union $\bigcup_{n=0}^\infty G_n$ of an increasing sequence of subgroups G_n, each generated by E together with some finite subset of $\bar{G}$. Notice that, if we ensure that E is pure in each G_n, then it will also be pure in G.

List the countable set $\bar{G} \setminus \bar{E}$ in a sequence $(v_k)_{k \in \mathbb{N}}$ in such a way that each element of $\bar{G} \setminus \bar{E}$ occurs infinitely often in the list. We define G_n by induction on n, starting with G_0 generated by $E \cup \{g\}$.

Suppose G_n has been defined, that it is a subgroup of $\bar{G}$ generated by E plus a finite set, and that E is pure in it. (This is clearly true for $n = 0$.) We shall define G_{n+1} having the same properties. If $v_n \notin G_n$ then do nothing, i.e., set $G_{n+1} = G_n$. Now suppose $v_n \in G_n$. Since $v_n \notin \bar{E}$, we have $\bar{G} = \bar{E} \oplus \mathbb{Q}v_n$. Let $\pi : \bar{G} \to \mathbb{Q}v_n$ be the projection homomorphism with kernel $\bar{E}$. Because G_n is generated by E plus a finite set, $\pi(G_n)$ is a finitely generated subgroup of $\mathbb{Q}v_n$. It contains v_n because G_n does. So

$$\pi(G_n) = \frac{1}{m}\mathbb{Z}v_n$$

for some positive integer m. Let p be a prime larger than both m and n, and let G_{n+1} be generated by $G_n \cup \{v_n/p\}$. Thus, v_n is divisible by p in G_{n+1} and therefore in the final group G.

This completes the definition of G. We check that it and E have the desired properties. It is clear that they have rank $\aleph_0$, that E is free, and that E is a proper subgroup of G (as $g \in G_0 \setminus E$). Also, if $v \in G \setminus \bar{E}$ then $v \in G_k$ for some k and there are infinitely many $n \geq k$ with $v_n = v$. For each such n, the construction of G_{n+1} ensured that v is divisible in G by a prime $p > n$. So v is divisible by infinitely many primes.

It remains to check that E is pure in G; this will also ensure that the preceding remarks about $v \in G \setminus \bar{E}$ apply to all $v \in G \setminus E$. As pointed out above, it suffices to check that E is pure in each G_n, and we shall do this by induction on n. The result is clear for $n = 0$, so assume E is pure in G_n and we have

$$k \in \mathbb{Z}, \quad k \geq 1, \quad x \in G_{n+1}, \quad \text{and} \quad kx \in E.$$

We must show $x \in E$. By definition of G_{n+1}, we have

$$x = y + \frac{r}{p} v_n, \quad r \in \mathbb{Z}, \quad y \in G_n,$$

where p is as in the definition of G_{n+1}. Using the notation of that definition, we have, since $kx \in E$,

$$y + \frac{r}{p} v_n = x \in \bar{E} = \text{kernel}(\pi),$$

so

$$\frac{r}{p} v_n = \pi(\frac{r}{p} v_n) = -\pi(y) \in \pi(G_n) = \frac{1}{m} \mathbb{Z} v_n.$$

Thus

$$\frac{r}{p} v_n = \frac{t}{m} v_n$$

for some integer t. Therefore $rm = pt$. But p is a prime not dividing m (recall p was chosen $> m$). So p divides r, and we have

$$\frac{r}{p} v_n \in \mathbb{Z} v_n \subseteq G_n.$$

Therefore $x \in G_n$. This and $kx \in E$ and the induction hypothesis that E is pure in G_n entail $x \in E$, as required.

References

[1] J. Reid, *A note on torsion free abelian groups of infinite rank*, Proc. Amer. Math. Soc. **13** (1962), 222–225.