Program Termination and Well Partial Orderings

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Abstract

The following observation may be useful in establishing program termination: if a transitive relation R is covered by finitely many well-founded relations $U_1, \ldots, U_n$ then R is well-founded. A question arises how to bound the ordinal height $|R|$ of the relation R in terms of the ordinals $\alpha_i = |U_i|$. We introduce the notion of the *stature* $\|P\|$ of a well partial ordering P and show that $|R| \leq \|\alpha_1 \times \cdots \times \alpha_n\|$ and that this bound is tight. The notion of stature is of considerable independent interest. We define $\|P\|$ as the ordinal height of the forest of nonempty bad sequences of P, but it has many other natural and equivalent definitions. In particular, $\|P\|$ is the supremum, and in fact the maximum, of the lengths of linearizations of P. And $\|\alpha_1 \times \cdots \times \alpha_n\|$ is equal to the natural product $\alpha_1 \otimes \cdots \otimes \alpha_n$.

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Contents

1 Introduction

A program π, possibly nondeterministic, is *terminating* if every computation of π from an initial state is finite. If there is a computation of π from state x to state y, we say that y is *reachable* from x and write $y \prec x$. A state y is *reachable* if it is reachable from an initial state. In practice termination is often established by means of ranking functions. A *ranking function* for π is an ordinal-valued function f on the reachable states of π such that $f(y) < f(x)$ whenever $y \prec x$. Clearly π is terminating if and only if the reachability relation $\prec$ over reachable states is well-founded if and only if π admits a ranking function. If π is terminating then the smallest ordinal α such that π admits a ranking function with values $< \alpha$ is the *ranking height* of π. The following observation [7, 4] may be helpful in establishing termination.

Lemma 1 (Covering Observation)**.** *Any transitive relation covered by finitely many well-founded relations is well-founded.*

In other words, if relations $U_1, \ldots, U_n$ are well-founded and $R \subseteq U_1 \cup \cdots \cup U_n$ is a transitive relation, then R is well-founded. The covering observation is proved by a straightforward application of Ramsey's theorem. The transitivity of R is essential here. If a, b are distinct elements then the relation $\{(a,b),(b,a)\}$ is covered by well-founded relations $\{(a,b)\}$ and $\{(b,a)\}$ but is not well-founded.

Example 2. Let π_1 be the program

```
while a ≤ 1000 ≤ b
   choose between
      a,b := a+2, b+1
      a,b := a-1, b-1
```

with integer variables a, b. Initially, a and b could be any integers. Since π_1's reachability relation $y \prec x$ is covered by well-founded relations $a_x < a_y \leq 1002$ and $999 \leq b_y < b_x$, π_1 terminates. Obviously the ranking height of π_1 isn't finite. In fact it is ω, the least infinite ordinal, because the function $|3b - 2a|$ is a ranking function for π_1. (The absolute value is used to guarantee that all values of the function are natural numbers.) Let π_2 be the following modification of π_1.

```
while a ≤ 1000 ≤ b
   choose between
      a := a+1
      a,b := arbitrary integer, b-1
```

Again, the covering observation applies, with the well-founded covering relations $a_x < a_y \leq 1001$ and $999 \leq b_y < b_x$. It is easy to see that the ranking height of π_2 is ω^2. In particular the function $\omega b + |1000 - a|$ is a ranking function for π_2. □

Example 3. Let π_3 be the program

```
while F ≠ ℕ²
   choose (a,b)∈ ℕ² − F
      F := F ∪ {(a′,b′)∈ ℕ³: a′ ≥ a and b′ ≥ b}
```

where $\mathbb{N}$ is the set of natural numbers, $F \subseteq \mathbb{N}^2$, and initially $F = \emptyset$. Think of F as the set of forbidden pairs. Initially, no pairs are forbidden, but once a pair becomes forbidden it remains so forever. As long as some pairs are not yet forbidden, the program nondeterministically chooses a non-forbidden pair (a, b) and forbids it and all pairs (a', b') such that $a' \geq a$ and $b' \geq b$. For every noninitial state x of π_3 let

$$A(x) = \min\{a : \ (a, b) \in F_x \text{ for some } b\}.$$

If x is the initial state, define $A(x) = \infty$. Define the function $B(x)$ similarly. Define $C(x)$ to be the number of points $(a, b) \in \mathbb{N}^2 - F_x$ such that $a \geq A(x)$ and $b \geq B(x)$. It is not hard to see that $C(x)$ is always finite. π_3's reachability relation $y \prec x$ is covered by the three well-founded relations $A(y) < A(x)$, $B(y) < B(x)$, and $C(y) < C(x)$. By the covering observation, π_3 is terminating. But this time the ranking height of the program is less obvious. It is $\omega^2 + 1$; the initial state has rank ω^2, and the rank of a noninitial state x is $\omega \cdot (A(x) + B(x)) + C(x)$. □

In § 2.2, we recall the definition of the ordinal height $|R|$ of a well-founded relation R as well as the definition of the ordinal height $|x|_R$ of any element x in the domain of R. If a program π is terminating then the ordinal height $|x|_\prec$ is a ranking function, and $|x|_\prec \leq f(x)$ for any ranking function f for π. It follows that the ranking height of π is the ordinal height $| \prec |$ of π's reachability relation $\prec$. Thus the ordinal height $|\prec|$ of the relation $\prec$ is a relevant measure. In this connection, our colleague, Byron Cook, working on a program-termination prover [4], asked the following question [3].

Question 4 (Covering Question)**.** If a transitive relation R is covered by well-founded relations $U_1, \ldots, U_n$, what is the best bound on $|R|$ in terms of the ordinals $\alpha_i = |U_i|$?

The covering question led us to investigate well partially ordered sets (in short, wpo sets). Recall that a sequence $\langle x_0, x_1, \ldots \rangle$, finite or infinite, of elements of a partially ordered set P is *bad* if there are no indices $i < j$ with $x_i \le x_j$ and that a partially ordered set P is wpo if every bad sequence in P is finite. In § 4, we introduce the key notion of this study, the *stature* $\|P\|$ of a wpo set P. We define $\|P\|$ as the ordinal height of the forest of nonempty bad sequences of P. We then give, in the same section, several alternative and equivalent definitions of $\|P\|$. In particular, $\|P\|$ is the height of the well-founded poset of proper ideals of P. In our view, the notion of stature is of central importance to the theory of wpo sets.

In § 5 we prove the following theorem to reduce the covering question to a question about the stature of the direct product $\alpha_1 \times \cdots \times \alpha_n$ of ordinals α_i. (By Corollary 18, the direct product of finitely many wpo sets is wpo.)

Theorem 5. *If $R, U_1, \ldots, U_n, \alpha_1, \ldots, \alpha_n$ are as in the covering question then*

$$|R| \le \|\alpha_1 \times \cdots \times \alpha_n\|$$

and this inequality is tight in the following sense. For any ordinals $\alpha_1, \ldots, \alpha_n$, there exist relations $R, U_1, \ldots, U_n$ such that R is transitive, $U_1, \ldots, U_n$ are well-founded, $R \subseteq U_1 \cup \cdots \cup U_n$, each $|U_i| = \alpha_i$, and $|R| = \|\alpha_1 \times \cdots \times \alpha_n\|$.

Applying this theorem to the reachability relation $\prec_{\pi_3}$ of program π_3 of Example 3, we get that $|\prec_{\pi_3}| \le \|\omega \times \omega \times \omega\|$. But what is $\|\alpha_1 \times \cdots \times \alpha_n\|$? This question is addressed in § 6.

Theorem 6. $\|\alpha_1 \times \cdots \times \alpha_n\| = \alpha_1 \otimes \cdots \otimes \alpha_n$.

Here $\alpha_1 \otimes \cdots \otimes \alpha_n$ is the natural product of the n ordinals. The natural sum and natural product of ordinals are recalled in § 2.4.

Corollary 7. *If $R, U_1, \ldots, U_n, \alpha_1, \ldots, \alpha_n$ are as in the covering question then*

$$|R| \le \alpha_1 \otimes \cdots \otimes \alpha_n$$

and this inequality is tight in the following sense. For any ordinals $\alpha_1, \ldots, \alpha_n$, there exist relations $R, U_1, \ldots, U_n$ such that R is transitive, $U_1, \ldots, U_n$ are well-founded, $R \subseteq U_1 \cup \cdots \cup U_n$, each $|U_i| = \alpha_i$, and $|R| = \alpha_1 \otimes \cdots \otimes \alpha_n$.

In the case of the program π_3 of Example 3, we have $| \prec_{\pi_3} | \leq \omega \otimes \omega \otimes \omega = \omega^3$. It is often convenient to generalize the notion of ranking function to allow such a function to have values in any well-ordered set. For each natural number ℓ, let $\mathbb{N}^\ell$ be the set of ℓ-tuples of natural numbers ordered lexicographically, so that the ordinal type of $\mathbb{N}^\ell$ is ω^ℓ.

Corollary 8. *Let $R, U_1, \ldots, U_n, \alpha_1, \ldots, \alpha_n$ be as in the covering question and assume that $\alpha_i \leq \omega^{\ell_i}$, so that U_i admits a ranking function with values in $\mathbb{N}^{\ell_i}$. Here each ℓ_i is a positive integer. Let $\ell = \ell_1 + \cdots + \ell_n$. Then R admits a ranking function with values in N^ℓ but may not admit a ranking function with values in $\mathbb{N}^{\ell-1}$.*

As we mentioned earlier, the notion of stature is of independent interest. § 7, the most involved section of this article, is devoted to a characterization of the stature of a wpo set P in terms of linearizations of P. Earlier, in § 4, we notice that every linearization (that is, every linear extension with the same underlying set) of a wpo set P is well-founded and of length (that is ordinal height) $\leq \|P\|$. In the process of proving Theorem 6, we construct a linearization of $\alpha_1 \times \cdots \times \alpha_n$ of length $\|\alpha_1 \times \cdots \times \alpha_n\|$.

Corollary 9. *The supremum of the lengths of linearizations of $\alpha_1 \times \cdots \times \alpha_n$ is $\|\alpha_1 \times \cdots \times \alpha_n\|$ and the supremum is attainable.*

It turns out that this corollary generalizes to all wpo sets.

Theorem 10. *The stature of any wpo set is the largest length of its linearizations.*

A part of our results on wpo sets has been known. In particular, De Jongh and Parikh proved that among the lengths of linearizations of any wpo set there is a largest. In § 8 we touch upon the involved history of the theory of wpo sets and other related work.

We attempt to make this article self-contained. In § 2 we give some preliminary information on partially ordered sets, well-founded partially ordered sets, wpo sets, ordinal arithmetic, and infinite combinatorics.

In § 3 we introduce games that allow us to compare ordinal heights of well-founded sets. The game criterion for height inequalities proved to be very useful. It may be known, but we have not found an explicit statement of it in the literature.

Remark 11. The notations $|P|$ and $\|P\|$ have many different uses in the literature. But they are convenient for our purposes and so, with some apprehension, we use them.

2 Preliminaries

We recall various definitions and facts and use this occasion to fix terminology and notation.

2.1 Partially Ordered Sets

A binary relation R can be viewed as a set of pairs of elements. A directed graph, in short digraph, is a pair (X, R) where X is a set and $R \subseteq X \times X$; the set X is the *domain* of the digraph. The smallest set X such that $R \subseteq X \times X$ will be called the *domain* of R.

A *poset* is a partially ordered set. In other words, a poset is a digraph where the relation is a partial order. Let P be a poset. The relation $<_P$ (resp. $\leq_P$) is the strict (resp. non-strict) version of the partial order of P. If $x <_P y$, we say that x is lower than y in P and that y is higher than x in P. In this and other similar cases, the subscript may be omitted when it is clear.

A poset Q *extends* P if $\leq_P \subseteq \leq_Q$, so that the digraph $(\mathrm{Dom}(Q), Q)$ may have more elements as well as more relationships than the digraph $(\mathrm{Dom}(P), P)$. A *linearization* of P is a linearly ordered set with the same domain that extends P.

An element $x \in \mathrm{Dom}(P)$ is the *top* of P if $x \geq_P$ every element of P.

A subset A of (the domain of) P is an *antichain* of P if the elements of A are pairwise incomparable. A subset F of P is a *filter* of P if it is upward closed, so that $y \in F$ if $x \leq y$ for some $x \in F$. If X is a subset of P then $\mathrm{Min}(X)$ is the antichain of minimal elements of X and

$$\mathrm{Filter}_P(X) = \{y \ : \ y \geq_P x \text{ for some } x \in X\}.$$

$\mathrm{Filter}(X)$ is the smallest filter that includes X. If A is an antichain then $A = \mathrm{Min}(\mathrm{Filter}(A))$.

A subset D of P is an *ideal* if it is downward closed, so that $x \in D$ if $x \leq y$ for some $y \in D$. Ideals are the complements of filters in P, and the other way round. If $X \subseteq \mathrm{Dom}(P)$ then the $\mathrm{Ideal}_P(X)$, or simply $\mathrm{Ideal}(X)$, is the ideal $\mathrm{Dom}(P) - \mathrm{Filter}(X)$. In other words,

$$\mathrm{Ideal}(X) = \{y \in \mathrm{Dom}(P) \ : \ (\forall x \in X)\ x \not\leq_P y\}.$$

An ideal D of P is *proper* if does not contain all elements of P.

Warning 12. $\mathrm{Ideal}(X)$ is the largest ideal that avoids X, rather than — which is more usual — the smallest ideal that includes X. We will not use the latter notion while the first one will play a role in this paper. Note also that many authors require filters to be not only upward closed but also downward directed (and dually for ideals). Such authors use terminology like "order-filter" or "up-set" where we use "filter", and they use "order-ideal" or "down-set" where we use "ideal".

The sets $\mathrm{Filter}_P(X)$ and $\mathrm{Ideal}_P(X)$ inherit partial orderings from P and thus give rise to posets that are also called $\mathrm{Filter}_P(X)$ and $\mathrm{Ideal}_P(X)$ respectively. If $x \in \mathrm{Dom}(P)$ then

$$\mathrm{Filter}_P(x) = \mathrm{Filter}_P(\{x\}) = \{y \in \mathrm{Dom}(P) : \ y \geq_P x\},$$
$$\mathrm{Ideal}_P(x) = \mathrm{Ideal}_P(\{x\}) = \{y \in \mathrm{Dom}(P) : \ y \not\geq_P x\}$$

Given finitely many posets $P_1, \ldots, P_n$, we can form the direct (or Cartesian) product $P_1 \times \cdots \times P_n$ with domain $\mathrm{Dom}(P_1) \times \cdots \times \mathrm{Dom}(P_n)$ where the n-tuples are ordered componentwise. The direct product operation generalizes to infinitely many components but we will not need the generalization.

2.2 Well-Founded Partially Ordered Sets

A poset is *well-founded* if it has no infinite descending sequence. A *well-ordered* set is a well-founded, linearly ordered set. Fix a well-founded poset P. If F is a filter of P then $F = \mathrm{Filter}(\mathrm{Min}(F))$.

Each element x of P has an ordinal *height* $|x|_P$ defined by the recursion

$$|x| = \min\{\text{ordinal } \alpha \ : \ \alpha > |y| \text{ for all } y <_P x\}.$$

The *height* $|P|$ of the poset P itself is the smallest ordinal $> |x|_P$ for all $x \in \mathrm{Dom}(P)$. If a poset Q is obtained from P by adding a new top element ∞ to P, then $|\infty|_Q = |P|$. The ordinal height of a well-ordered set is also called its *length*.

Lemma 13. *For every $\alpha < |P|$, there is an element x with $|x|_P = \alpha$. If $\alpha < |y|_P$, then there is an element $x < y$ with $|x|_P = \alpha$.*

Proof. The second claim follows from the first: consider the sub-poset given by the set $\{z : z < y\}$. To prove the first claim, notice that elements of height $\geq \alpha$ form a nonempty filter; any minimal element x of that filter is of height α. □

Definition 14. A binary relation R is *well-founded* if there is no infinite sequence $\langle x_0, x_1, \ldots \rangle$ such that $x_{n+1} R x_n$ holds for all n. In the obvious way, the definition of height generalizes to well-founded relations.

A well-founded relation does not have to be transitive. For example the successor relation on natural numbers is well-founded but not transitive. However the transitive closure of a well-founded relation is well-founded as well.

2.3 Well Partially Ordered Sets

A good reference for this subsection is [14].

Definition 15. Let P be a poset. A sequence $\langle x_0, x_1, \ldots \rangle$ of elements of P, finite or infinite, is *bad* if there are no indices $i < j$ with $x_i \leq x_j$. A poset P is *well partially ordered*, or *wpo*, if all bad sequences in P are finite.

Remark 16. Admittedly, the terminology is not very good. But it is accepted. We discuss the issue in § 8.

There are many equivalent characterizations of the wpo sets.

Lemma 17. *Let P be a poset. The following are equivalent characterizations of the wpo property. In other words, each of the following claims is equivalent to the claim that P is wpo.*

1. *Every infinite sequence $\langle x_0, x_1, \ldots \rangle$ of elements of P includes an infinite weakly increasing subsequence.*

2. *P is well-founded, and all its antichains are finite.*

3. *For every filter F of P, there is a finite antichain A such that $F =$ Filter(A).*

4. *For every filter F of P, the antichain Min(F) is finite and F = Filter(Min(F)).*

5. *For every ideal D of P, there is a finite antichain A such that D = Ideal(A).*

6. *For every ideal I of P, the antichain $A = Min(Dom(P) - I)$ is finite and $D = Ideal(A)$.*

The proof is straightforward, using Ramsey's theorem, Theorem 20, for items 1 and 2.

Corollary 18. *The direct product of finitely many wpo set is wpo.*

Proof. Use the first equivalent characterization of the wpo property in the preceding lemma. □

2.4 Ordinal Arithmetic

A good reference for this subsection is the book [17], particularly Section 5.11.

In set theory, an ordinal α is defined as the set $\{\xi : \xi < \alpha\}$ of smaller ordinals. In particular, the first infinite ordinal ω is the set of natural numbers. Every well-ordered set P is isomorphic to a unique ordinal, namely the length of P. The *cardinality* of a set X is the least ordinal α such that there is a bijection between X and α; ordinals that arise in this way are *cardinals*.

A *limit ordinal* is an ordinal $\alpha > 0$ not of the form $\beta + 1$ for any β. A set X of ordinals is *cofinal* in a limit ordinal α if α is the supremum of X and $\alpha \notin X$. Thus X is cofinal in α if and only if $X \subseteq \alpha$ and for every $\beta < \alpha$ there is an element of X that is $> \beta$. The *cofinality* of a limit ordinal α is the least cardinal κ such that α has a cofinal subset of cardinality κ. Alternatively and equivalently, the cofinality of a limit ordinal α can be defined as the least ordinal κ such that there is a (strictly) increasing sequence $s = \langle \beta_\xi : \xi < \kappa \rangle$ of ordinals whose range is cofinal in α.

A cardinal is *regular* if it is equal to its own cofinality. It is easy to see that, for every limit ordinal α, the cofinality of α is a regular cardinal.

For any ordinal α, the ordinal ω^α is the length of the following well-ordered set P. Dom(P) is the set of functions $f : \alpha \to \omega$ such that the support $\{\xi < \alpha : f(\xi) > 0\}$ is finite. The order is reverse lexicographic. That is, $f_1 <_P f_2$ if and only if $f_1(\xi) < f_2(\xi)$ for the largest ξ with $f_1(\xi) \neq f_2(\xi)$.

Such a largest ξ exists, whenever f_1 and f_2 are distinct, because the supports of f_1, f_2 are finite.

Any ordinal number α can be written in Cantor normal form (with base ω),

$$\alpha = \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_n}$$

for a unique finite sequence $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$. (There is an alternative version where the terms have the form $\omega^{\alpha_i} m_i$ with integer coefficients and where the exponents are strictly decreasing. The two are obviously equivalent.)

The *natural sum* $\alpha \oplus \beta$ of two ordinals α and β is obtained by adding their Cantor normal forms as if they were polynomials (i.e., as if ω were an indeterminate), arranging the terms in non-increasing order of exponents. It is well known and easy to check that the natural sum is strictly increasing in each of its arguments. (In fact, there is an equivalent definition of $\oplus$ by recursion: $\alpha \oplus \beta$ is the smallest ordinal strictly above $\alpha' \oplus \beta$ and $\alpha \oplus \beta'$ for all $\alpha' < \alpha$ and $\beta' < \beta$.)

The natural product, $\alpha \otimes \beta$, of two ordinals is defined by multiplying their Cantor normal forms as if they were polynomials in the indeterminate ω, using natural addition for the exponents, and arranging the resulting terms in non-increasing order.

Lemma 19. *Natural multiplication is commutative and associative, and it distributes over natural addition. It is a strictly increasing function of either argument as long as the other argument isn't zero.*

2.5 Infinite Combinatorics

We recall Ramsey's theorem for pairs and one generalization of it. If S is a set then $[S]^2$ is the collection of two-element subsets of S.

Theorem 20 (Ramsey [16])**.** *If S is an infinite set and $[S]^2$ is partitioned into finitely many pieces, $S_1, \ldots, S_m$, then there exists an infinite $T \subseteq S$ such $[T]^2 \subseteq S_i$ for some i.*

Theorem 21 (Dushnik and Miller [6])**.** *If κ is a regular cardinal and $[\kappa]^2$ is partitioned into two pieces, S_1 and S_2, then either there exists a κ-element $T_1 \subseteq \kappa$ with $[T_1]^2 \subseteq S_1$ or else there exists an infinite $T_2 \subseteq \kappa$ with $[T_2]^2 \subseteq S_2$.*

Theorem 20 easily follows from the special case where $m = 2$ and $S = \omega$, which is the special case $\kappa = \omega$ of Theorem 21. The proof of Theorem 21 is very similar to a standard argument for Ramsey's theorem. (Theorem 21 also holds for singular κ, but the proof, due to Erdős, is more complicated.)

3 Games

We give a useful way to compare the heights of well-founded posets.

Given two posets, P and Q, define a game $\Gamma(P, Q)$ between two players, called 1 and 2, played as follows. Player 1 (resp. 2) has a pebble which, at each noninitial stage of the game is at some element of P (resp. Q). Initially, the pebble is off the poset. Think about the initial position of the pebble being above all elements of the poset, at the *summit position* of P (resp. Q). This allows us to pretend that, at each stage, including the initial stage, the pebble occupies some position in the poset. Define the *height* of the summit position to be the height of the poset.

The players move alternately, with 1 moving first. A player's move shifts his pebble to a position lower than the current one. In particular, the first move puts the pebble at any element of the poset. If a player is unable to move, he loses the game, and his opponent wins. We say that a player *wins* the game if he has a winning strategy, i.e., a strategy by which he wins no matter how the opponent plays.

Proposition 22 (Game Criterion)**.** *Let P and Q be posets.*

- *Suppose that P is well-founded. Then player* 1 *wins* $\Gamma(P, Q)$ *if and only if Q is well-founded and* $|P| > |Q|$*.*
- *Suppose that Q is well-founded. Then player* 2 *wins* $\Gamma(P, Q)$ *if and only if P is well-founded and* $|P| \leq |Q|$*.*

Proof. Q is well-founded if P is well founded and 1 wins the game. Indeed, if Q is not well-founded then it has an infinite descending sequence, and 1 cannot win a play where 2 moves his pebble along the descending sequence. Similarly, P is well-founded if Q is well-founded and 2 wins the game. In the rest of the proof, we may assume that both P and Q are well-founded. It remains to prove the following two claims.

1. Player 1 wins $\Gamma(P, Q)$ if and only if $|P| > |Q|$.
2. Player 2 wins $\Gamma(P, Q)$ if and only if $|P| \leq |Q|$.

The two right-to-left implications are easy because, in each case, the winning strategy is to move your pebble to a position whose height is at least as great as that of the other player's pebble. Once you have both right-to-left implications, the left-to-right ones follow, because at most one player can have a winning strategy and at least one of the ordinal inequalities must hold. □

A map f from a poset P to a poset Q is *monotone* if $f(x) <_Q f(y)$ whenever $x <_P y$.

Corollary 23. *If there is a monotone map from a poset P to a well-founded poset Q then P is well-founded and $|P| \le |Q|$.*

Proof. Player 2 has a winning strategy in $\Gamma(P, Q)$: whenever 1 moves to a position x, move to position $f(x)$. Now use Proposition 22. □

Remark 24. The game is even more natural for posets with a distinguished element (pointed posets). Then we don't need the summit position; initially the pebbles are at the distinguished elements. The proposition, appropriately adjusted, remains valid. The corollary, also properly adjusted, remains valid if we require that the monotone map takes the distinguished element to the distinguished element. We skip the details of adjustment as we will not use pointed posets.

Remark 25. The game, the proposition and the corollary generalize in a straightforward way to the case when P, Q are directed graphs.

4 Stature of a WPO Set

First we give a number of equivalent definitions of stature. Then we prove some useful facts related to statures and natural sums of ordinals.

4.1 Definition and Equivalent Characterizations

Fix a wpo set P.

Definition 26. $\mathcal{B}(P)$ is the poset of nonempty bad sequences of P. The ordering is reverse extension: $s \le_{\mathcal{B}} t$ if and only if t is an initial segment of s.

Thus $s \leq_{\mathcal{B}} t$ if and only if s is an end extension of t. It is easy to see that $\mathcal{B}(P)$ is a downward-growing forest. By the definition of wpo sets, $\mathcal{B}$ is well-founded.

Definition 27. The *stature* $\|P\|$ of P is the height $|\mathcal{B}(P)|$ of $\mathcal{B}(P)$.

Remark 28. We could have defined $\mathcal{B}(P)$ to be the tree of all bad sequences of P including the empty sequence. The empty sequence would be the root and the top element of $\mathcal{B}(P)$. Then $\|P\|$ could be defined as the height of the empty sequence in $\mathcal{B}(P)$.

We will give several equivalent characterizations of $\|P\|$. To this end, we define some useful posets.

Definition 29.

- $\mathcal{A}(P)$ is the set of nonempty antichains of P with partial order

$$A \leq_{\mathcal{A}} B \iff (\forall b \in B)(\exists a \in A)\, a \leq_P b.$$

- $\mathcal{I}(P)$ is the set of proper ideals of P ordered by inclusion.
- A *pointed ideal* is a pair (D, d) where D is an ideal and d is a maximal element of D. $\mathcal{P}(P)$ is the set of pointed ideals of P with partial order

$$(D, d) <_{\mathcal{P}} (E, e) \iff D \subseteq E - \{e\}.$$

We take the liberty of omitting the argument of $\mathcal{A}, \mathcal{B}, \mathcal{I}, \mathcal{P}$ when it is clear from the context.

Proposition 30. *$\mathcal{A}$, $\mathcal{I}$ and $\mathcal{P}$ are well-founded, and* $|\mathcal{A}| = |\mathcal{I}| = |\mathcal{P}| = |\mathcal{B}| = \|P\|$.

Proof. We split the proposition into a number of claims and repeatedly use the game criterion of § 3.

$\mathcal{P}$ is well-founded and $|\mathcal{P}| \leq |\mathcal{B}|$. By the game criterion, it suffices to construct a winning strategy for player 2 in game $\Gamma(\mathcal{P}, \mathcal{B})$. When player 1 has just played (D, x), extend your previous bad sequence by appending x. This strategy always provides a legal move, so it wins.

$\mathcal{I}$ is well-founded and $|\mathcal{I}| \leq |\mathcal{P}|$. We construct a winning strategy for player 2 in game $\Gamma(\mathcal{I}, \mathcal{P})$. Let $D_0 = \mathrm{Dom}(P)$. The i^{th} move of 1 is some ideal $D_i \subsetneq D_{i-1}$. Choose any $y_i \in D_{i-1} - D_i$ and play (E_i, y_i) where $E_i = \text{Ideal}(\{y_1, \dots, y_i\}) \cup \{y_i\} = \{x \in \text{Dom}(P) \ : \ x \not> y_i$ and, for all $j < i,\ x \not\geq y_j\}$. (The key is to play y_i in the second component; the first component E_i is chosen as big as possible subject to the requirement that y_i be maximal in it and that $E_i \subseteq E_j - \{y_j\}$ for all $j < i$.) This strategy always provides a legal move, so it wins.

$\mathcal{A}$ is well-founded and $|\mathcal{A}| = |\mathcal{I}|$, because $\mathcal{A}$ and $\mathcal{I}$ are isomorphic. An isomorphism from $\mathcal{I}$ to $\mathcal{A}$ is given by $D \mapsto \text{Min}(\text{Dom}(P) - D)$, and its inverse is $X \mapsto \text{Ideal}(X)$).

$|\mathcal{B}| \leq |\mathcal{I}|$. We construct a winning strategy for player 2 in game $\Gamma(\mathcal{B}, \mathcal{I})$. When player 1 has just played a bad sequence $\langle x_1, \dots, x_\ell \rangle$, reply with $\text{Ideal}\{x_1, \dots, x_\ell\}$. This strategy always provides a legal move, so it wins.

To summarize, we established that $\mathcal{A}, \mathcal{I}, \mathcal{P}$ are well-founded and that $|\mathcal{A}| = |\mathcal{I}| \leq |\mathcal{P}| \leq |\mathcal{B}| \leq |\mathcal{I}|$. It follows that $|\mathcal{A}| = |\mathcal{I}| = |\mathcal{P}| = |\mathcal{B}|$. It remains to recall that, by the definition of stature, $\|P\| = |\mathcal{B}|$. □

For future reference, we record the following fact about linearizations of P.

Proposition 31. *Every linearization of a wpo P is well-founded and has length $\leq \|P\|$.*

Proof. Let A be a linearization of P. By the definition of stature in § 4, $\|P\| = |\mathcal{B}|$. By the game criterion of § 3, it suffices to construct a winning strategy for player 2 in $\Gamma(A, \mathcal{B})$. 1's moves up to stage ℓ form a decreasing sequence s_ℓ of length ℓ in A. Use s_ℓ as your reply at stage ℓ. (A direct proof that A is well-founded is to observe that every decreasing sequence in A is a bad sequence in P.) □

4.2 Statures and Natural Sums

Recall that any ordinal α has a unique Cantor normal form

$$\alpha = \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_n}$$

where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, and that the natural sum $\alpha \oplus \beta$ of α and β is obtained by adding their Cantor normal forms as if they were polynomials.

Lemma 32. *Let α have the Cantor normal form exhibited above, let $\beta < \alpha$, and let $\gamma < \omega^{\alpha_n}$. Then $\beta \oplus \gamma < \alpha$.*

Proof. Increasing β if necessary, we may assume that it has the form

$$\beta = \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_{n-1}} + \delta$$

where $\delta < \omega^{\alpha_n}$ (and where the exponents $\alpha_1, \ldots, \alpha_{n-1}$ are the same as in the normal form of α). As both γ and δ are $< \omega^{\alpha_n}$, their Cantor normal forms involve only exponents $< \alpha_n$. So the same is true of their natural sum. But

$$\beta \oplus \gamma = \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_{n-1}} + (\delta \oplus \gamma),$$

and so the required inequality follows. □

Corollary 33. *An ordinal of the form ω^β exceeds the natural sum of any two strictly smaller ordinals.*

The corollary is a bit stronger than the analogous assertion without "natural", since $\xi + \eta \leq \xi \oplus \eta$ and the inequality can be strict. We will use the following obvious consequence of Corollary 33.

Corollary 34. *An ordinal of the form ω^β exceeds the natural sum of any finitely many strictly smaller ordinals.*

Our next goal is to relate the statures of a wpo set and its subsets. It is clear that $\|P\| \leq \|Q\|$ whenever $P \subseteq Q$; indeed the copycat strategy of player 2 wins $\Gamma(P, Q)$. On the other hand, we shall see that when two subsets of a wpo set P cover P their statures add (in the sense of $\oplus$) to at least the stature of P. For this, as well as for other purposes later, we need the following information about statures.

Recall that $\text{Filter}_P(x)$ is the smallest filter that contains x, $\text{Ideal}_P(x)$ is the largest ideal that avoids x, and that the corresponding posets (with partial orders inherited from P) are also denoted $\text{Filter}_P(x)$ and $\text{Ideal}_P(x)$ respectively.

Lemma 35. *The stature $\|P\|$ of a wpo set P is the smallest ordinal strictly above $\|Ideal_P(v)\|$ for all $v \in P$.*

Proof. Since $\|P\|$ is the height of the forest $\mathcal{B}(P)$, it is the smallest ordinal strictly greater than the heights of all the roots $\langle v \rangle$ of the trees that constitute the forest $\mathcal{B}(P)$. The height of any $\langle v \rangle$ can be computed in the tree of which $\langle v \rangle$ is the root, and we shall complete the proof by checking that this height is exactly $\|\mathrm{Ideal}_P(v)\|$.

The tree with $\langle v \rangle$ as root consists of all the bad sequences of the form $\langle v \rangle^\frown s$. For such a sequence to be bad means that s is bad and that no term in s is $\geq_P v$. In other words, s must be a bad sequence in $\mathrm{Ideal}_P(v)$. Thus, the tree with root v is isomorphic to $\mathcal{B}(\mathrm{Ideal}_P(v))$ with a top element added, and so the proof is complete. □

Lemma 36. *Let (the domain of) a wpo set Q be the union of (the domains of) finitely many sub-posets $Q_1, \dots, Q_n$. For any sub-poset P of Q, we have $\|P\| \leq \|Q_1\| \oplus \cdots \oplus \|Q_n\|$.*

Proof. We proceed by induction on the stature $\|P\|$ of the focal poset P, so assume the lemma holds for all cases where the focal poset is of strictly smaller stature. In view of the preceding lemma, it suffices to prove that, for each $x \in P$, $\|\mathrm{Ideal}_P(x)\| < \|Q_1\| \oplus \cdots \oplus \|Q_n\|$. For this purpose, fix an arbitrary $x \in P$; assume without loss of generality that $x \in Q_1$ so that $\mathrm{Ideal}_{Q_1}(x)$ is a proper ideal of Q_1. If $i > 1$ and $x \in Q_i$ then $\mathrm{Ideal}_{Q_i}(x)$ is a well-defined proper ideal of Q_i; otherwise the notation $\mathrm{Ideal}_{Q_i}(x)$ has not yet been assigned a meaning but it is convenient to adopt the convention that $\mathrm{Ideal}_{Q_i}(x) = Q_i$.

By induction hypothesis, we have

$$\|\mathrm{Ideal}_P(x)\| \leq \|\mathrm{Ideal}_{Q_1}(x)\| \oplus \cdots \oplus \|\mathrm{Ideal}_{Q_n}(x)\|.$$

Now since $x \in Q_1$, we have that

$$\|\mathrm{Ideal}_{Q_1}(x)\| < \|Q_1\|.$$

For Q_i with $i > 1$ we cannot use the same argument, since we don't necessarily have $x \in Q_i$, but we still have

$$\|\mathrm{Ideal}_{Q_i}(x)\| \leq \|Q_i\| \text{ for all } i > 1.$$

Combining the three displayed inequalities and the strict monotonicity of $\oplus$, we get $\|\mathrm{Ideal}_P(x)\| < \|Q_1\| \oplus \cdots \oplus \|Q_n\|$ as required. □

5 Reduction of Covering Question

We return to the covering question discussed in the introduction and prove Theorem 5. Assume that $U_1, \dots, U_n$ are well-founded relations and that R is a transitive relation included in $U_1 \cup \cdots \cup U_n$. Let $X = \mathrm{Dom}(R)$. Without loss of generality, each $\mathrm{Dom}(U_i) \subseteq X$. By an argument using Ramsey's theorem, Theorem 20, R is well-founded. We seek to bound its ordinal height in terms of the ordinals $\alpha_i = |U_i|$. The direct product $\alpha_1 \times \cdots \times \alpha_n$ of the ordinals $\alpha_1, \dots, \alpha_n$ can be seen as a poset where the n-tuples are ordered componentwise. That poset is wpo by Corollary 18.

Proposition 37. *Under the assumptions above,* $|R| \le \|\alpha_1 \times \cdots \times \alpha_n\|$.

Proof. According to Definition 27, we need to prove that $|R| \le |\mathcal{B}(\alpha_1 \times \cdots \times \alpha_n)|$. According to the game criterion of § 3, it suffices to prove that player 2 has a winning strategy in game

$$\Gamma\big((X,R), \mathcal{B}(\alpha_1 \times \cdots \times \alpha_n)\big).$$

The desired strategy is simple. Whenever player 1 moves his pebble to a new point $x \in X$, extend the current bad sequence (or the empty sequence if this is the first move) by appending the element $(|x|_{U_1}, \dots, |x|_{U_n})$ of $\alpha_1 \times \cdots \times \alpha_n$. We need to check only that this preserves badness of the sequence. Since the opponent's current move x is R-below all his previous moves y (thanks to transitivity of R) and since $R \subseteq U_1 \cup \cdots \cup U_n$, we have, for each earlier y, that $x\, U_i\, y$ and therefore $|x|_{U_i} < |y|_{U_i}$ hold for some i. Thus, the n-tuple $(|x|_{U_1}, \dots, |x|_{U_n})$ cannot be $\ge$ the earlier n-tuple $(|y|_{U_1}, \dots, |y|_{U_n})$, so badness persists. □

Proposition 38. *The bound in the previous proposition is tight. That is, given any ordinals* $\alpha_1, \dots, \alpha_n$*, we can find well-founded relations* $U_1, \dots, U_n$ *of at most these heights and we can find a transitive* $R \subseteq U_1 \cup \cdots \cup U_n$ *with* $|R| = \|\alpha_1 \times \cdots \times \alpha_n\|$.

Proof. Let (X, R) be the poset $\mathcal{B}(\alpha_1 \times \cdots \times \alpha_n)$ of nonempty bad sequences in $\alpha_1 \times \cdots \times \alpha_n$. If s, t are nonempty bad sequences with last members $(\xi_1, \dots, \xi_n)$ and $(\eta_1, \dots, \eta_n)$ respectively, set $tU_is \iff \eta_i < \xi_i$. Obviously, every U_i is well-founded and $|U_i| = \alpha_i$. If tRs holds, then t is an end-extension of s. By the definition of bad sequences, $(\xi_1, \dots, \xi_n) \not\le (\eta_1, \dots, \eta_n)$, and so tU_is holds for some i. Thus R is covered by the relations $U_1, \dots, U_n$. □

Theorem 5 follows from the two propositions.

6 Stature of Direct Product of Several Ordinals

The goal of this section is to prove Theorem 6. The theorem asserts that $\|\alpha_1 \times \cdots \times \alpha_n\| = \alpha_1 \otimes \cdots \otimes \alpha_n$ for any natural number n and any ordinals $\alpha_1, \ldots, \alpha_n$. In § 6.1 we will prove that $\|\alpha_1 \times \cdots \times \alpha_n\| \geq \alpha_1 \otimes \cdots \otimes \alpha_n$, and in § 6.2 will prove that $\|\alpha_1 \times \cdots \times \alpha_n\| \leq \alpha_1 \otimes \cdots \otimes \alpha_n$. But first we state a characterization of natural sums for future reference.

Recall that the natural sum $\alpha \oplus \beta$ of two ordinals is defined by adding their Cantor normal forms as if they were polynomials [§ 2.4]. It is clear from this definition that a well-ordered set of length $\alpha \oplus \beta$ can be partitioned into two subsets of length α and β.

Lemma 39. *$\alpha \oplus \beta$ is the largest ordinal that admits a partition into two subsets of lengths α and β.*

Though this is well-known, we point out that it follows also from Lemmas 36 and 44.

6.1 The Natural Product is Small Enough

In this section, we prove that the stature of $\alpha_1 \times \cdots \times \alpha_n$ is at least as large as natural product of the n ordinals. We start with the case $n = 2$.

Lemma 40. *Let α and β be arbitrary ordinals.*

1. *There is a linearization of $\alpha \times \beta$ of length $\alpha \otimes \beta$.*
2. *$\|\alpha \times \beta\| \geq \alpha \otimes \beta$.*

Proof. In virtue of Proposition 31, the first claim implies the second. So it suffices to prove claim 1. In the rest of the proof we construct the desired linearization. We do that by induction on $\alpha \otimes \beta$. The zero case is trivial. The induction step splits into two cases.

Case 1: At least one of α and β is not a power of ω. Without loss of generality, suppose it is α. Let the Cantor normal forms of α and β be

$$\alpha = \omega^{\mu_1} + \cdots + \omega^{\mu_m},$$
$$\beta = \omega^{\nu_1} + \cdots + \omega^{\nu_n},$$

where $\mu_1 \geq \cdots \geq \mu_m$ and $\nu_1 \geq \cdots \geq \nu_n$ and $m > 1$. Concerning n, we know only that $n \geq 1$. The Cantor normal form of $\alpha \otimes \beta$ has the form

$$\alpha \otimes \beta = \omega^{\pi_1} + \cdots + \omega^{\pi_{mn}}$$

where $\pi_1 \geq \cdots \geq \pi_{mn}$ and every $\pi_p = \mu_i \oplus \nu_j$ for some $1 \leq i \leq m$ and $1 \leq j \leq n$, arranged in non-increasing order. This gives rise to a bijection

$$f: \{1, \ldots, m\} \times \{1, \ldots, n\} \to \{1, \ldots, mn\}$$

such that $\pi_{f(i,j)} = \mu_i \oplus \nu_j$. Since the same ordinal can occur as $\mu_i \oplus \nu_j$ for several pairs (i, j), there is some freedom in the choice of f. It will be convenient to specify f so that $f(i, j) < f(k, l)$ if and only if one of the following three conditions is satisfied.

(1) $\mu_i \oplus \nu_j > \mu_k \oplus \nu_l$.

(2) $\mu_i \oplus \nu_j = \mu_k \oplus \nu_l$ and $i < k$.

(3) $\mu_i \oplus \nu_j = \mu_k \oplus \nu_l$ and $i = k$ and $j < l$.

There is a unique such function because the three clauses define a linear ordering of the pairs (i, j). The first condition suffices to ensure that the sequence $\langle \pi_{f(i,j)} \rangle$ is non-increasing, as required above. We record the following claim about f for future reference.

Claim 41. *If $f(i, j) < f(k, l)$ then $i < k$ or $j < l$ (or both).*

Proof. The claim is obvious if the inequality $f(i, j) < f(k, l)$ holds by virtue of clause (2) or (3). If the inequality holds by virtue of clause (1), we argue by contradiction. If we had both $i \geq k$ and $j \geq l$, then, since the μ and ν sequences are non-increasing, we would have $\mu_i \leq \mu_k$ and $\nu_j \leq \nu_l$. But then $\mu_i \oplus \nu_j \leq \mu_k \oplus \nu_l$, contrary to clause (1). □

Now partition α into consecutive segments $A_1, \ldots, A_m$ of lengths $\omega^{\mu_1}, \ldots, \omega^{\mu_m}$ respectively, and partition β into consecutive segments $B_1, \ldots, B_n$ of lengths $\omega^{\nu_1}, \ldots, \omega^{\nu_n}$ respectively. Each $A_i \times B_j$ can be viewed as a partially ordered set where the order is componentwise. For each pair (i, j), fix a linearization $C_{f(i,j)}$ of height $\omega^{\mu_i \oplus \nu_j}$. Such linearizations exist by the induction hypothesis, since every $\omega^{\mu_i} < \alpha$ and every $\omega^{\nu_j} \neq 0$. By the definition of $\alpha \otimes \beta$, the concatenation

$$C = C_1^\frown \cdots ^\frown C_{mn}$$

is of length $\alpha \otimes \beta$. It remains to show that C extends the partially ordered set $\alpha \times \beta$. That will complete Case 1 of the proof of the proposition.

Within each block $A_i \times B_j$, there is no problem, since $C_{f(i,j)}$ extends $A_i \times B_j$ by the definition of $C_{f(i,j)}$. The only possible problem arises between elements of different blocks. Suppose, toward a contradiction, that something goes wrong, i.e., we have

$$\begin{aligned} & f(i,j) < f(k,l), \\ \text{and} \quad & (\gamma, \delta) \in C_{f(i,j)},\ (\varepsilon, \zeta) \in C_{f(k,l)}, \\ \text{but} \quad & (\varepsilon, \zeta) \le (\gamma, \delta) \text{ in } \alpha \times \beta. \end{aligned}$$

By Claim 41, either $i < k$ or $j < l$. If $i < k$, then the segment A_i of α, which contains γ, precedes the segment A_j, which contains ε; hence $\gamma < \varepsilon$. Similarly, $j < l$ implies $\delta < \zeta$. In either case, this contradicts that $(\varepsilon, \zeta) \le (\gamma, \delta)$ in $\alpha \times \beta$.

Case 2: Both α and β are powers of ω, say $\alpha = \omega^\mu$ and $\beta = \omega^\nu$. We need to prove that there is a linearization of the poset $\omega^\mu \times \omega^\nu$ of length $\omega^{\mu \oplus \nu}$.

Recall that every ω^γ is the length of a linearly ordered set B_γ such that $\mathrm{Dom}(B_\gamma)$ is the set of finite-support functions $f : \gamma \to \omega$ and the order of B_γ is reverse lexicographic [§ 2.4]. Since the posets $\omega^\mu \times \omega^\nu$ and $B_\mu \times B_\nu$ are isomorphic, it suffices to prove that there is a linearization of $B_\mu \times B_\nu$ of the length of $B_{\mu\oplus\nu}$. To this end, it suffices to produce a bijection C from $B_\mu \times B_\nu$ onto $B_{\mu\oplus\nu}$ that is monotone in both arguments. Indeed such a map gives a linearization

$$(f, g) < (f', g') \iff C(f, g) < C(f', g')$$

of $B_\mu \times B_\nu$ of the length of $B_{\mu\oplus\nu}$.

By the definition of natural sum [§ 2.4], $\mu \oplus \nu$ can be partitioned into two subsets, M of length μ and N of length ν. Let B_M (resp. B_N) be the poset of finite-support functions f from M (resp. N) to ω ordered in the reverse lexicographic way. Since posets $B_M \times B_N$ and $B_\mu \times B_\nu$ are isomorphic, it suffices to prove that there is a bijection D from $B_M \times B_N$ onto $B_{\mu\oplus\nu}$ that is monotone in both arguments.

The desired map D sends (f, g) to the union $f \cup g$ (where we view a function as a set of ordered pairs). Then D is clearly a one-to-one map from $B_M \times B_N$ to $B_{M \cup N} = B_{\mu\oplus\nu}$. D is monotone; that is, if $f \le f'$ and $g \le g'$ then $D(f, g) \le D(f', g')$. This is because the last place (in $\mu \oplus \nu$) where $D(f, g)$ and $D(f', g')$ differ either lies in M and is the last place where f and f' differ or lies in N and is the last place where g and g' differ.

It remains to check that every finite-support function $h : \omega^\mu \oplus \omega^\nu \to \omega$ has the form $f \cup g$ where $f : M \to \omega$ and $g : N \to \omega$. The desired f, g are the restrictions of h to M, N respectively. □

Proposition 42. *For every natural number n and all ordinals $\alpha_1, \dots, \alpha_n$,*

1. *There is a linearization of $\alpha_1 \times \cdots \times \alpha_n$ of length $\alpha_1 \otimes \cdots \otimes \alpha_n$.*
2. *$\|\alpha_1 \times \cdots \times \alpha_n\| \geq \alpha_1 \otimes \cdots \otimes \alpha_n$.*

Proof. Again, in virtue of Proposition 31, the first claim implies the second. So it suffices to prove claim 1. We do that by induction on n. Cases $n < 2$ are trivial, and case $n = 2$ is Lemma 40. We suppose that $n \geq 2$ and that claim 1 holds for all natural numbers $\leq n$, and we prove the case $n+1$ of claim 1. By the induction hypothesis, there is a linearization A of $\alpha_1 \times \cdots \times \alpha_n$ of length $\alpha_1 \otimes \cdots \otimes \alpha_n$. So the (componentwise) partial order of $\alpha_1 \times \cdots \times \alpha_n \times \alpha_{n+1}$ can be extended to the componentwise partial order $A \times \alpha_{n+1}$ isomorphic to $(\alpha_1 \otimes \cdots \otimes \alpha_n) \times \alpha_{n+1}$. By Lemma 40, this can in turn be extended to a linear order of length $\alpha_1 \otimes \cdots \otimes \alpha_n \otimes \alpha_{n+1}$. □

6.2 The Natural Product is Large Enough

In this section, we prove that the stature of $\alpha_1 \times \cdots \times \alpha_n$ is at most as large as the natural product of the n ordinals.

Proposition 43. *For every natural number n and all ordinals $\alpha_1, \dots, \alpha_n$, we have $\|\alpha_1 \times \cdots \times \alpha_n\| \leq \alpha_1 \otimes \cdots \otimes \alpha_n$.*

Proof. Without loss of generality $n \geq 2$. We prove the proposition by induction on $\alpha_1 \otimes \cdots \otimes \alpha_n$. The base case $\alpha_1 \otimes \cdots \otimes \alpha_n = 0$ (so that one of the ordinals α_i is 0) is trivial. The induction step splits into two cases.

Case 1: At least one of ordinals α_i is not of the form ω^μ. Without loss of generality, the Cantor normal form of α_1 has at least two terms. Notice that any ordinal is the *natural* sum (as well as the ordinary sum) of the terms in its Cantor normal form. So we have $\alpha_1 = \alpha' \oplus \alpha'' = \alpha' + \alpha''$ with both summands $< \alpha$. (To be specific, take α'' to be the last term in the Cantor normal form and α' to be the sum of all the earlier terms.) So α is the disjoint union of its initial segment α' and a final segment F of length α''. Therefore

$$\alpha \times \alpha_2 \times \cdots \times \alpha_n = (\alpha' \times \alpha_2 \times \cdots \times \alpha_n) \cup (F \times \alpha_2 \times \cdots \times \alpha_n).$$

By Lemma 36, we have

$$\begin{aligned}\|\alpha \times \alpha_2 \times \cdots \times \alpha_n\| &\le \|\alpha' \times \alpha_2 \times \cdots \times \alpha_n\| \oplus \|F \times \alpha_2 \times \cdots \times \alpha_n\| \\ &\le (\alpha' \otimes \alpha_2 \otimes \cdots \otimes \alpha_n) \oplus (\alpha'' \otimes \alpha_2 \otimes \cdots \otimes \alpha_n) \\ &= \alpha \otimes \alpha_2 \otimes \cdots \otimes \alpha_n,\end{aligned}$$

where the second inequality comes from the induction hypothesis and the final equality is the distributivity of $\otimes$ over $\oplus$.

Case 2: $\alpha_i = \omega^{\mu_i}$ for $i = 1, \ldots, n$. Let $P = \alpha_1 \times \cdots \times \alpha_n$. By Lemma 35, $\|P\|$ is the smallest ordinal strictly above $\|\mathrm{Ideal}_P(\xi_1, \ldots, \xi_n)\|$ for any $\xi_i \in \alpha_i$. So it suffices to show that, for all such $\xi_1, \ldots, \xi_n$,

$$\|\mathrm{Ideal}_P(\xi_1, \ldots, \xi_n)\| < \alpha_1 \otimes \cdots \otimes \alpha_n.$$

Fix an arbitrary tuple $(\xi_1, \ldots, \xi_n) \in P$ and let $I = \mathrm{Ideal}_P(\xi_1, \ldots, \xi_n)$. We show that $\|I\| < \alpha_1 \otimes \cdots \otimes \alpha_n$.

The set I is covered by the subsets $P_1, \ldots, P_n$ where P_i is obtained from P by replacing the i^{th} factor α_i with ξ_i. By Lemma 36, $\|I\| \le \|P_1\| \oplus \cdots \oplus \|P_n\|$. By the induction hypothesis, each $\|P_i\|$ is bounded by the natural product of ξ_i and $n-1$ ordinals α_j with $j \ne i$. By the strict monotonicity of $\oplus$, we have

$$\|P_i\| < \alpha_1 \otimes \cdots \otimes \alpha_n = \omega^{\mu_1 \oplus \cdots \oplus \mu_n}.$$

for each i. By Corollary 34, the ordinal $\omega^{\mu_1 \oplus \cdots \oplus \mu_n}$, is strictly larger than the natural sum of any finite number of smaller ordinals. It follows that

$$\|I\| \le \|P_1\| \oplus \cdots \oplus \|P_n\| < \omega^{\mu_1 \oplus \cdots \oplus \mu_n} = \alpha_1 \otimes \cdots \otimes \alpha_n.$$

□

Propositions 42 and 43 imply Theorem 6.

7 Stature is Maximal Linearization Length

In this section, we prove Theorem 10: The stature of a wpo set P is the largest among the lengths of linearizations of P. In § 4.1, we showed that every linearization of P is well-founded and has length $\le \|P\|$ (Proposition 31). It remains to prove that the supremum of linearization lengths of P is attainable and equal to $\|P\|$. This is easy if P is linearly ordered.

Lemma 44. *If P is a well-ordered set then the supremum of linearization lengths of P is attainable and equal to $\|P\|$.*

Proof. The first claim is trivial as there is only one linearization and so the supremum is $|P|$. To prove the second claim, recall that, by Proposition 30, $\|P\| = |\mathcal{P}(P)|$. But $\mathcal{P}(P)$ is isomorphic to P via by the map $(D, d) \mapsto d$. □

In the rest of this section, we prove that, for any wpo set P, the supremum of linearization heights of P is attainable and equal to $\|P\|$.

7.1 Long Consistent Sequence Suffices

Definition 45. Two posets P and Q are *consistent* if there is no pair $\{x, y\}$ such that $x <_P y <_Q x$.

A sequence $s = \langle x_\beta : \beta < \alpha \rangle$ of distinct elements of a poset P can be viewed as a linearly ordered set where $x_\beta \leq_s x_\gamma$ if and only if $\beta \leq \gamma$.

Definition 46. A sequence $s = \langle x_\beta : \beta < \alpha \rangle$ of distinct elements of P is *consistent* with P if the posets P and s are consistent.

In this subsection, we prove that a wpo set P has a linearization of length $\|P\|$ if it has a consistent sequence of elements of length $\|P\|$. We start with an auxiliary result.

Lemma 47. *Let P be a poset (not necessarily wpo), and let A be a linearly ordered set with $Dom(A) \subseteq Dom(P)$. If A and P are consistent then there is a linearization of P that extends A.*

Proof. Let R be the binary relation $\leq_P \cup \leq_A$. It suffices to prove that the digraph $G = (\mathrm{Dom}(P), R)$ is acyclic. Indeed if G is acyclic then the transitive closure R^* of R is a partial order. Extend R^* to a linear order (by Zorn's lemma, any poset can be extended to a linearly ordered set) and get the desired linearization of P.

So suppose, toward a contradiction, that G has a cycle C. Since $\leq_P$ and $\leq_A$ are transitive, we can combine consecutive "steps" in the same ordering. Thus, without loss of generality, C has the form $x_0, x_1, \ldots, x_{n-1}$ where $x_i <_P x_{i+1}$ for even i and $x_i <_A x_{i+1}$ for odd i. Here we take the subscripts modulo n, so that when $i = n-1$ we interpret $i+1$ as 0. Also, n is even. Note that each x_i is in A because it is related by $<_A$ to either x_{i-1} or x_{i+1}. Since $\leq_A$

linearly orders A, let j be the index for which x_j is largest, with respect to $\leq_A$.

In particular, $x_j >_A x_{j+1}$. But if j is odd then $x_j <_A x_{j+1}$, a contradiction. So j is even and $x_j <_P x_{j+1}$. But then A and P are inconsistent contrary to the hypothesis of the lemma. □

Remark 48. The linearity of A is essential for the proof of the lemma. It is not true that, if two posets P, Q with the same domain are consistent, then there is a partial order extending both of them. For a counterexample, take the common domain to be a four-element set $\{a, b, c, d\}$, take $<_P = \{(a, b), (c, d)\}$, and take $<_Q = \{(b, c), (d, a)\}$. Then P and Q are consistent, yet the union of the order relations contains a cycle. □

Lemma 49. *Let P be a wpo set, and suppose that there is a sequence $s = \langle x_\alpha : \alpha < \|P\| \rangle$ of elements of P consistent with P. Then there is a linearization of P of length $\|P\|$.*

Proof. By Lemma 47, there is a linearization A of P that extends s. By Proposition 31, A is well-founded and $|A| \leq \|P\|$. Since A extends s, we have $|A| \geq |s|$; one easy way to see that $|A| \geq |s|$ is to use a game as in § 3. But $|s| = \|P\|$. So $|A| = \|P\|$. □

7.2 Producing a Long Consistent Sequence

Proposition 50. *For every wpo set P, there is a linearization of P of length $\|P\|$.*

Proof. Fix a wpo set P. According to Lemma 49, it suffices to prove that there is a sequence $s = \langle x_\alpha : \alpha < \|P\| \rangle$ of elements of P consistent with P. We do that by induction on $\|P\|$.

Case 1: $\|P\| = 0$. Trivial.

Case 2: $\|P\|$ is a successor ordinal $\alpha + 1$. By Lemma 35, there is an element $x \in P$ such that $\|\text{Ideal}_P(x)\| = \alpha$. Let I be $\text{Ideal}_P(x)$ viewed as a poset. By induction hypothesis, we have an α-sequence s of elements of I consistent with I. Appending x to s, we get the desired $\|P\|$-sequence of elements of P consistent with P.

Case 3: $\|P\|$ is a limit ordinal but not of the form ω^α. So the Cantor normal form of $\|P\|$ has at least two summands; let ω^ε be the last summand, and let δ be the sum of all the other terms in the Cantor normal form. So

$$\|P\| = \delta + \omega^\varepsilon = \delta \oplus \omega^\varepsilon.$$

By Lemma 35, there is $x \in P$ such that $\delta < \|\mathrm{Ideal}_P(x)\| < \|P\|$. Applying Lemma 36 with the posets $\mathrm{Ideal}_P(x)$ and $\mathrm{Filter}_P(x)$ in the roles of Q_1 and Q_2, we obtain

$$\|P\| \leq \|\mathrm{Ideal}_P(x)\| \oplus \|\mathrm{Filter}_P(x)\|.$$

It follows, by Lemma 32, that we cannot have $\|\mathrm{Filter}_P(x)\| < \omega^\varepsilon$.

By induction hypothesis, $\mathrm{Ideal}_P(x)$ contains a consistent sequence s of length $\|\mathrm{Ideal}_P(x)\| > \delta$. And $\mathrm{Filter}_P(x)$ contains a consistent sequence t of length $\geq \omega^\varepsilon$. Indeed, let Q be the poset $\mathrm{Filter}_P(x)$. If $\|Q\| < \|P\|$ then the induction hypothesis gives t of length $\|\mathrm{Filter}_P(x)\| \geq \omega^\varepsilon$. Otherwise $\|Q\| = \|P\|$. Use Lemma 35 to find $w \in Q$ such that $\omega^\varepsilon \leq \|\mathrm{Ideal}_Q(w)\| < \|P\|$. By applying the induction hypothesis to $\mathrm{Ideal}_Q(w)$, we again get a consistent sequence t of length $\geq \omega^\varepsilon$.

The concatenation $s^\frown t$ has length at least $\delta + \omega^\varepsilon = \|P\|$. It is consistent with P because s and t are consistent with P and because all elements of t and no elements of s are $\geq_P x$.

Case 4: $\|P\| = \omega^\alpha$ for some non-zero ordinal α. Let κ be the cofinality of ω^α, and let $\langle \beta_\xi : \ \xi < \kappa\rangle$ be a strictly increasing sequence of ordinals cofinal with ω^α. Recall that κ, being the cofinality of something, must be a regular cardinal.

Lemma 51. *There is an increasing sequence $\langle x_\xi : \ \xi \in \kappa\rangle$ of elements of P such that $\|Ideal_P(x_\xi)\| > \beta_\xi$ for all $\xi \in \kappa$.*

Proof. For each $\xi < \kappa$, use Lemma 35 to obtain some $y_\xi \in P$ with $\|\mathrm{Ideal}_P(y_\xi)\| > \beta_\xi$. Although there may be repetitions in the sequence $\langle y_\xi\rangle$, no single element y can be y_ξ for κ different ordinals ξ. The reason is that, if there were such a y, then $\|\mathrm{Ideal}_P(y)\|$ would be greater than the corresponding ordinals β_ξ. As any κ of these ordinals have supremum ω^α, we would have $\|\mathrm{Ideal}_P(y_\xi)\| \geq \omega^\alpha = \|P\|$, which is absurd.

Because no element occurs κ times in the sequence $\langle y_\xi\rangle$ and because κ is regular, the set $S = \{\xi : \ y_\xi \neq y_\eta \text{ for all } \eta < \xi\}$ contains κ ordinals. We can therefore extract a subsequence of $\langle y_\xi\rangle$ in which there are no repetitions. Specifically, let $f(\xi)$ be the ξ^{th} ordinal in S, and let $y'_\xi = y_{f(\xi)}$. Then all the y'_ξ are distinct and, since $f(\xi) \geq \xi$, we have $\|\mathrm{Ideal}_P(y'_\xi)\| > \beta_{f(\xi)} \geq \beta_\xi$. (That

$f(\xi) \geq \xi$ is probably intuitively evident; for a proof see [17, Theorem 5.1.1].) From now on, we work with the y'_ξ and we omit the primes.

Invoking again the regularity of κ, we can apply the Dushnik-Miller theorem, Theorem 21, to the partition where $S_1 = \{\{\xi < \eta\} : y_\xi \leq_P y_\eta\}$ and $S_2 = [\kappa]^2 - S_1$. If an infinite subset T of κ had $[T]^2 \subseteq S_2$, then the first ω elements of T would constitute an infinite bad sequence, contrary to the assumption that P is wpo. So, by the Dushnik-Miller theorem, there must be a κ-element subset $T \subseteq \kappa$ such that $[T]^2 \subseteq S_1$. Letting $g(\xi)$ denote the ξ^{th} ordinal in T and letting $x_\xi = y_{g(\xi)}$, we obtain the conclusion of the lemma. Indeed, the homogeneity of T ensures that the sequence $\langle x_\xi : \xi \in \kappa \rangle$ is increasing, and because $g(\xi) \geq \xi$ we have $\|\text{Ideal}_P(x_\xi)\| = \|\text{Ideal}_P(y_{g(\xi)})\| > \beta_{g(\xi)} \geq \beta_\xi$. □

Lemma 52. *There is an increasing sequence $\langle x_\xi : \xi \in \kappa \rangle$ of elements of P such that*

$$\|\textit{Ideal}(x_{\xi+1})\| > \|\textit{Ideal}(x_\xi)\| \oplus \beta_\xi \textit{ for all } \xi < \kappa.$$

Proof. By Lemma 51, there is an increasing sequence $s = \langle y_\xi : \xi \in \kappa \rangle$ such that every $\|\text{Ideal}_P(y_\xi)\| > \beta_\xi$. The desired $\langle x_\xi : \xi \in \kappa \rangle$ is a subsequence of s built by recursion. Start with $x_0 = y_0$ and, at limit stages of the recursion, simply take the next y_η after all those previously taken. The nontrivial case is the successor step, where we already have x_ξ and must find an appropriate $x_{\xi+1}$. Since the statures of the sets $\text{Ideal}(y_\eta)$ approach $\|P\| = \omega^\alpha$, it suffices to check that $\|\text{Ideal}(x_\xi)\| \oplus \beta_\xi < \omega^\alpha$. Fortunately, this follows immediately from Corollary 33. This completes the proof of the lemma. □

Let $\langle x_\xi : \xi \in \kappa \rangle$ be as in Lemma 52. Temporarily fix some $\xi < \kappa$. Since $\text{Ideal}(x_{\xi+1})$ is obviously the union of $\text{Ideal}(x_\xi)$ and $\text{Ideal}(x_{\xi+1}) \cap \text{Filter}(x_\xi)$, Lemma 36 gives us that

$$\|\text{Ideal}(x_{\xi+1})\| \leq \|\text{Ideal}(x_\xi)\| \oplus \|\text{Ideal}(x_{\xi+1}) \cap \text{Filter}(x_\xi)\|.$$

Comparing this with Lemma 52, we find that

$$\|\text{Ideal}(x_{\xi+1}) \cap \text{Filter}(x_\xi)\| \geq \beta_\xi.$$

Applying the induction hypothesis to $\text{Ideal}(x_{\xi+1}) \cap \text{Filter}(x_\xi)$ (which has lower stature than P because it's a subset of $\text{Ideal}(x_{\xi+1})$), we obtain, in $\text{Ideal}(x_{\xi+1}) \cap \text{Filter}(x_\xi)$, a sequence s_ξ of length at least β_ξ which is consistent with $\text{Ideal}(x_{\xi+1}) \cap \text{Filter}(x_\xi)$ and therefore is consistent with P.

Now un-fix ξ. Let t be the concatenation of all the sequences s_ξ, in order of increasing ξ. The length of t is, for each ξ, at least β_ξ, since s_ξ is a segment of t. So the length of t is at least the supremum of the β_ξ's, which is ω^α.

To complete the proof, it remains only to check that t is consistent with P. Since each s_ξ has this property, the only thing that can go wrong is that there are $\xi < \eta$ with some y in s_η being $\leq_P$ some $x \in s_\xi$. To see that this cannot happen, suppose it did, and recall where these sequences s_ξ and s_η came from. The former was chosen from $\text{Ideal}(x_{\xi+1}) \cap \text{Filter}(x_\xi)$, so $x \not\geq x_{\xi+1}$, while the latter was chosen from $\text{Ideal}(x_{\eta+1}) \cap \text{Filter}(x_\eta)$, so $y \geq x_\eta$. Since the sequence $\langle x_\xi : \ \xi \in \kappa \rangle$ is increasing, and since $\xi < \eta$, we have

$$x \geq y \geq x_\eta \geq x_{\xi+1},$$

a contradiction. Proposition 50 is proved. □

Propositions 31 and 50 imply Theorem 10.

8 Related Work

We describe in this section earlier work on two concepts central to this paper, namely well partially ordered sets and natural products of ordinals.

8.1 Natural Products

Natural sums and natural products of ordinals are defined in Hausdorff's book [9, pages 68–70]. Hausdorff credits these concepts to Hessenberg, citing [10, § 75], but the cited section contains only natural sums, not products, nor have we found natural products elsewhere in [10].

Carruth [2] proved that every linearization of the componentwise partial order on $\alpha \times \beta$ has length at most $\alpha \otimes \beta$. In our presentation, this fact is a consequence of Propositions 31 and 43. Carruth's argument is fairly complex, using neither any notion of stature nor indeed any notion of well partially ordered set. Carruth's motivation came from the theory of ordered abelian groups; he shows how to bound, in terms of the length of a well-ordered set X of positive elements in such a group, the length of the (necessarily also well-ordered) subsemigroup generated by X.

8.2 Well Partially Ordered Sets

Well partially ordered sets were introduced by Higman [11]. He called them partially ordered sets with the finite basis property. This terminology refers to the characterization given by item 3 in our Lemma 17, which Higman used as the definition. He proved several equivalent characterizations, including the main points of Lemma 17 and the well-foundedness of $\mathcal{I}$. His main result is that when P is wpo then so is the set of finite sequences from P, ordered by "componentwise majorized by a subsequence of".

The second author [8] independently discovered the notion of wpo, introduced the terminology "tight partial order", and proved some cases of Higman's result that he needed for investigations about decidability in predicate logic. The word "tight" was meant to refer to a boot, where one cannot move downward or sideways but only upward.

Kruskal [13] developed the theory of wpo sets further, proving a celebrated result about certain posets of trees being wpo. He seems to be the first to use the terminology "well-quasi-ordering". ("Quasi" in place of "partial" means that $\leq$ is not required to be antisymmetric. Many authors write "preorder" instead of "quasi-order", but "prewellorder" means something different from well-quasi-order. A prewellorder is a preorder whose partially ordered quotient, obtained by identifying x and y whenever $x \leq y \leq x$, is a well-order.) Kruskal mentions that previous authors have used the terms "well-partial-ordering" and "partial well-ordering. Even at this early stage of the development of wpo theory, the terminology had become so chaotic that Kruskal gives, at the end of [13], a glossary for matching his terminology with Higman's.

In [14], Kruskal describes much of the early history of the wpo concept (though he was unaware of [8]). He mentions yet another name for the concept, "fairly well-ordered", used by Michael [15].

De Jongh and Parikh [5] give several equivalent characterizations of wpo, adding to Higman's list the property that all linearizations are well-ordered. Furthermore, they show that among the ordinal lengths of these linear orderings there is a largest one. In the case of a Cartesian product $\alpha \times \beta$ of ordinals, they show that this largest length of a linearization is $\alpha \otimes \beta$. Recall that, by Theorem 10, the stature of a wpo set P equals the largest length of a linearization, which de Jongh and Parikh call $o(P)$. In this sense, [5] can be regarded as introducing the notion of stature, though without a name and without other equivalent descriptions (such as our definition in terms of the forest of nonempty bad sequences).

The definition of stature that we use, the height of the forest of nonempty bad sequences, was studied by Kříž and Thomas [12], who called it the type of P and used the notation $c(P)$ for it. They assert (in their Theorem 4.7) that this equals the largest length of a linearization, but there seems to be a problem with the proof. Their Theorem 4.6 uses in an essential way that the disjoint union of two posets was defined with the two parts incomparable, but then this theorem is applied in a situation where the incomparability requirement is violated. Nevertheless, their Lemma 4.5 motivated our use of the Dushnik-Miller theorem in the proof of Lemma 51; in fact, their Lemma 4.5 essentially re-proves the relevant case of the Dushnik-Miller theorem.

Remark 53. As already indicated, the notion of wpo set has acquired many names as a result of being discovered many times. (Yet another name, "Noetherian", is used in [1, page 33]; other authors, however, use "Noetherian" to mean that the reverse ordering is well-founded.) If we could choose between the many names, we would prefer "tight", and not just because one of us introduced it. It's short and (with the boot metaphor) descriptive, and it doesn't use "well" as an adjective (as in "well partial order"). A second choice would probably be "finite basis property". Although longer, it summarizes nicely one of the equivalent characterizations of the notion. It also has the advantage of being the name used by Higman, who introduced the concept.

Unfortunately, the terminology "well partially ordered" and its close relative "well quasi-ordered" are used so commonly, and the alternatives so rarely, that it seems hopeless to advocate a change of terminology now. We have therefore resigned ourselves to wpo.

An imaginative interpretation of "well partial order" is to invoke the other meaning of "well", namely a source of water from underground. Like a boot, a well (at least an old-fashioned one) is closed at the bottom and sides but open at the top. Think of a well partial order as a partial order where, as in a well, the only direction for unrestricted motion is upward.

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[12]. We thank Alfons Geser for reference [7].

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