SIMPLE CARDINAL CHARACTERISTICS OF THE CONTINUUM

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Abstract. We classify many cardinal characteristics of the continuum according to the complexity, in the sense of descriptive set theory, of their definitions. The simplest characteristics ($\mathbf{\Sigma}^0_2$ and, under suitable restrictions, $\mathbf{\Pi}^0_2$) are shown to have pleasant properties, related to Baire category. We construct models of set theory where (unrestricted) $\mathbf{\Pi}^0_2$-characteristics behave quite chaotically and no new characteristics appear at higher complexity levels. We also discuss some characteristics associated with partition theorems and we present, in an appendix, a simplified proof of Shelah's theorem that the dominating number is less than or equal to the independence number.

1. Introduction

Cardinal characteristics of the continuum are cardinal numbers, usually between $\aleph_1$ and $\mathbf{c} = 2^{\aleph_0}$ inclusive, that give information about the real line $\mathbb{R}$ or the closely related sets $\mathcal{P}(\omega)$ (the power set of the set ω of natural numbers), $[\omega]^\omega$ (the set of infinite subsets of ω), and ${}^\omega\omega$ (the set of functions from ω to ω). We give a few examples here (others will be given later) and refer to [3, 23] for more examples and an extensive discussion.

The most obvious characteristic is $\mathbf{c}$, the cardinality of $\mathbb{R}$ and of the other sets, $\mathcal{P}(\omega)$, etc., mentioned above.

Baire category gives rise to several characteristics, of which we mention here the covering number

$$\mathbf{cov}(B) = \text{ minimum number of meager sets needed to cover } \mathbb{R}$$

and the uniformity number

$$\mathbf{unif}(B) = \text{ minimum cardinality of a non-meager set of reals.}$$

Lebesgue measure gives rise to analogous characteristics, $\mathbf{cov}(L)$ and $\mathbf{unif}(L)$, defined by putting "measure zero" in place of "meager" in the previous definitions.

For $X, Y \in [\omega]^\omega$, we say that X *splits* Y if both $X \cap Y$ and $Y - X$ are infinite. Then the *splitting number* $\mathbf{s}$ is the minimum cardinality of a splitting family, i.e., a

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family $\mathcal{S} \subseteq [\omega]^\omega$ such that every $Y \in [\omega]^\omega$ is split by some $X \in \mathcal{S}$. The *unsplitting number* $\mathbf{r}$ (sometimes called the reaping number or the refining number) is the minimum cardinality of an unsplittable family, i.e., a family $\mathcal{R} \subseteq [\omega]^\omega$ such that no single $X \in [\omega]^\omega$ splits all the sets $Y \in \mathcal{R}$.

For $f, g \in {}^\omega\omega$, we say that g *eventually majorizes* f, written $f <^* g$, if $f(n) < g(n)$ for all but finitely many n. The *dominating number* $\mathbf{d}$ is the minimum cardinality of a dominating family, i.e., a family $\mathcal{D} \subseteq {}^\omega\omega$ such that every $f \in {}^\omega\omega$ is eventually majorized by some $g \in \mathcal{D}$. The *bounding number* $\mathbf{b}$ is the minimum cardinality of an unbounded family, i.e., a family $\mathcal{B} \subseteq {}^\omega\omega$ such that no single $g \in {}^\omega\omega$ eventually majorizes all the functions $f \in \mathcal{B}$.

For $X, Y \in [\omega]^\omega$, we say that X is *almost included* in Y, written $X \subseteq^* Y$, if $X - Y$ is finite. The tower number $\mathbf{t}$ is the smallest ordinal (necessarily a regular cardinal) such that there is a $\mathbf{t}$-sequence $(X_\alpha)_{\alpha<\mathbf{t}}$ from $[\omega]^\omega$ that is almost decreasing (i.e., $X_\alpha \subseteq^* X_\beta$ for $\beta < \alpha$) and cannot be extended, i.e., no $Y \in [\omega]^\omega$ satisfies $Y \subseteq^* X_\alpha$ for all $\alpha < \mathbf{t}$. Such a sequence, not necessarily of minimum length, is called a *tower*.

Many cardinal characteristics have definitions of the form "the minimum cardinality of a set $\mathcal{X}$ of reals such that, for every real y, there is at least one $x \in \mathcal{X}$ such that $R(x, y)$", where R is some binary relation on reals. If, as is customary in set theory, we apply the name reals not only to members of $\mathbb{R}$ but also to members of $\mathcal{P}(\omega)$ and ${}^\omega\omega$, then the definitions of $\mathbf{s}$, $\mathbf{r}$, $\mathbf{d}$, and $\mathbf{b}$ are clearly of this form. The definitions of the covering and uniformity numbers for category and measure can be put into this form by coding meager or measure zero Borel sets by reals. The definition of $\mathbf{t}$, however, seems not to be of this form, since the family $\mathcal{X} = \{X_\alpha \mid \alpha < \mathbf{t}\}$ is subject to the additional requirement of being well-ordered by $\supseteq^*$. Other characteristics, particularly $\mathbf{h}$ and $\mathbf{g}$ (see [23]) are even further from the simple form we are discussing.

The primary purpose of this paper is to discuss the possibility of classifying those characteristics definable in this simple form by applying, to the relations R involved in these definitions, the familiar hierarchical classifications of descriptive set theory. Specifically, we define for any pointclass Γ (e.g. for any level of the Borel hierarchy) a uniform Γ-characteristic to be an uncountable cardinal definable in the simple form described above, with $R \in \Gamma$. We also define a somewhat broader notion of (non-uniform) Γ-characteristic, which encompasses $\mathbf{t}$ and several other familiar characteristics. Our goal is to infer combinatorial information about a characteristic κ from the descriptive set-theoretic information that κ is a Γ-characteristic or a uniform Γ-characteristic for a reasonably small Γ.

For example, when $\Gamma = \mathbf{\Sigma}^0_2$, we can show that Martin's axiom implies that all $\mathbf{\Sigma}^0_2$-characteristics equal $\mathbf{c}$. In fact, in ZFC alone, all $\mathbf{\Sigma}^0_2$-characteristics are $\geq \mathbf{cov}(B)$, which equals $\mathbf{c}$ under MA and which is itself a uniform $\mathbf{\Pi}^0_1$-characteristic.

For $\mathbf{\Pi}^0_2$, the situation is not so pleasant. $\aleph_1$ is a uniform $\mathbf{\Pi}^0_2$-characteristic, and there is a great deal of arbitrariness as to which cardinals are $\mathbf{\Pi}^0_2$-characteristics and which are not. Nevertheless, for certain well-behaved $\mathbf{\Pi}^0_2$-characteristics, we can obtain results dual to those for $\mathbf{\Sigma}^0_2$.

This paper is organized as follows. Section 2 contains definitions, examples, and elementary properties of Γ-characteristics and uniform Γ-characteristics. Sections

3 through 5 contain the results cited above, $\mathbf{\Sigma}_2^0$ being treated in Section 3 and $\mathbf{\Pi}_2^0$ in Sections 4 and 5. In Section 6, we record some information about characteristics related to Ramsey's theorem and other partition theorems. These characteristics first attracted my attention as interesting (and possibly new) examples of $\mathbf{\Sigma}_2^0$ and $\mathbf{\Pi}_2^0$ characteristics, but they have since acquired considerable combinatorial interest in their own right. Finally, following the example of Vaughan [23], we present in an appendix a proof of Shelah's theorem that the independence number $\mathbf{i}$ is no smaller than $\mathbf{d}$. This proof is a reformulation of Shelah's original proof [23] avoiding a few unnecessary complications. Although it contains no new (vis à vis [23]) ideas, its relative simplicity seems to justify recording it in print.

2. Definitions, Basic Properties, and Examples

Throughout this paper, Γ denotes a pointclass in the sense of descriptive set theory. The specific Γ's that we consider will usually be low levels of the Borel hierarchy, particularly $\mathbf{\Pi}_1^0$ and $\mathbf{\Pi}_2^0$, their lightface analogs Π_1^0 and Π_2^0, and the class $OD\mathbb{R}$ of relations ordinal-definable from reals. The relevance of $OD\mathbb{R}$ is that it includes all "reasonable" pointclasses, so, by proving that a cardinal is not an $OD\mathbb{R}$-characteristic, we establish that it is not a Γ-characteristic for "any" Γ.

Definition. An uncountable cardinal κ is a Γ-*characteristic* if there is a family of κ sets, each in Γ, such that ${}^\omega\omega$ is covered by the family but not by any subfamily of cardinality $< \kappa$. An uncountable cardinal κ is a *uniform* Γ-*characteristic* if there is a binary relation R on ${}^\omega\omega$ such that $R \in \Gamma$ and such that κ is the minimum cardinality of a family $\mathcal{X} \subseteq {}^\omega\omega$ such that for every $y \in {}^\omega\omega$ there exists $x \in \mathcal{X}$ with $R(x, y)$.

The remainder of this section is devoted to examples and elementary properties of Γ-characteristics and uniform Γ-characteristics. The following proposition is obvious.

Proposition 1. *(a) If $\Gamma \subseteq \Delta$ then every (uniform) Γ-characteristic is a (uniform) Δ-characteristic.*
(b) Every Γ-characteristic lies between $\aleph_1$ and $\mathbf{c}$, inclusive.
(c) If Γ is closed under pre-images by continuous functions, then every uniform Γ-characteristic is a Γ-characteristic. □

In connection with (c), we note that the only continuous functions needed are $y \mapsto (x, y)$ for arbitrary fixed x. The hypothesis of (c) means in practice that Γ is a boldface pointclass; for lightface pointclasses Γ there are no Γ-characteristics but there are uniform Γ-characteristics.

In defining (uniform) Γ-characteristics, we used the descriptive set-theorist's usual version ${}^\omega\omega$ of the "reals". Had we used the actual reals $\mathbb{R}$ or $\mathcal{P}(\omega)$ or $[\omega]^\omega$ instead, any (uniform) Γ-characteristic in the modified sense would also be a (uniform) Γ-characteristic in the original sense, provided Γ is closed under pre-images by continuous functions with recursive codes. This follows immediately from the well-known fact [18, p. 12] that ${}^\omega\omega$ can be mapped onto each of $\mathbb{R}$, $\mathcal{P}(\omega)$, and $[\omega]^\omega$ by continuous functions with recursive codes. The specific Γ's that we deal with will have this closure property, so, when verifying that some cardinal is a (uniform) Γ-characteristic, we may use $\mathbb{R}$ or $\mathcal{P}(\omega)$ or $[\omega]^\omega$ or similar spaces instead of ${}^\omega\omega$.

The next proposition lists some examples of Γ-characteristics, mostly uniform ones. The proof in each case consists of exhibiting the family of sets or the binary relation required by the definition and then verifying membership in Γ. We shall see later that $\mathbf{\Sigma}^0_2$ can always be improved to $\mathbf{\Pi}^0_1$, so part (b) of the proposition is not optimal. In fact, it is easy to prove directly that $\mathbf{d}$, $\mathbf{r}$, and $\mathbf{cov}(B)$ are uniform $\mathbf{\Pi}^0_1$-characteristics. But for the time being, we give the complexity bounds that follow directly from the definitions.

Proposition 2. *(a)* $\mathbf{c}$ *is a uniform* Π^0_1*-characteristic.*
(b) $\mathbf{d}$, $\mathbf{r}$, $\mathbf{cov}(B)$, *and* $\mathbf{unif}(L)$ *are uniform* Σ^0_2*-characteristics.*
(c) $\mathbf{b}$, $\mathbf{s}$, $\mathbf{unif}(B)$, $\mathbf{cov}(L)$, *and* $\aleph_1$ *are uniform* Π^0_2*-characteristics.*
(d) $\mathbf{t}$ *is a* $\mathbf{\Pi}^0_2$*-characteristic.*

Proof. (a) Take $R(x,y)$ to be $x=y$, a Π^0_1 relation.

(b) For $\mathbf{d}$, take

$$R(x,y) \iff y \le^* x \iff \exists k\,(\forall n \ge k)\, y(n) \le x(n).$$

For $\mathbf{r}$, work in $[\omega]^\omega$ and take

$$\begin{aligned} R(x,y) &\iff y \text{ does not split } x \\ &\iff \exists k\,[(\forall n \ge k)\,(n \in x \implies n \in y) \\ &\qquad\qquad \text{or } (\forall n \ge k)\,(n \in x \implies n \notin y)]. \end{aligned}$$

For $\mathbf{cov}(B)$, we need a coding of the meager sets, or rather of the countable unions of nowhere dense closed sets, by "reals". Any nowhere dense closed subset F of ${}^\omega\omega$ is of the form

$$\{x \in {}^\omega\omega \mid \text{ No finite initial segment of } x \text{ is in } D\}$$

where $D \subseteq {}^{<\omega}\omega$ (the set of finite sequences from ω) is dense in the sense that every $s \in {}^{<\omega}\omega$ has an extension $s^\frown t \in D$ (where $^\frown$ means concatenation). If $f : {}^{<\omega}\omega \to {}^{<\omega}\omega$, then $\{s^\frown f(s) \mid s \in {}^{<\omega}\omega\}$ is such a D, and every dense D includes one of this form. Thus,

$$\begin{aligned} &\mathrm{NWD}(f) = \\ &\qquad \{x \in {}^\omega\omega \mid (\forall s \in {}^{<\omega}\omega)\, s^\frown f(s) \text{ is not an initial segment of } x\} \end{aligned}$$

is a nowhere dense closed set, and every nowhere dense set is included in one of this form. Therefore, if $g : \omega \times {}^{<\omega}\omega \to {}^{<\omega}\omega$, then

$$\begin{aligned} &\mathrm{Meager}(g) = \\ &\qquad \{x \in {}^\omega\omega \mid \exists n\,(\forall s \in {}^{<\omega}\omega) s^\frown g(n,s) \text{ is not an initial segment of } x\} \end{aligned}$$

is a meager F_σ set, and every meager set is included in one of this form. It follows that $\mathbf{cov}(B)$ is the minimum size of a family $\mathcal{X}$ of g's (in ${}^{\omega\times{}^{<\omega}\omega}({}^{<\omega}\omega)$ which is recursively homeomorphic to ${}^\omega\omega$) such that for every $y \in {}^\omega\omega$ there is $g \in \mathcal{X}$ with $y \in \mathrm{Meager}(g)$. By inspection of the definition of $\mathrm{Meager}(g)$, we see that the relation $y \in \mathrm{Meager}(g)$ is a Σ^0_2 relation $R(g, y)$. Therefore $\mathbf{cov}(B)$ is a uniform Σ^0_2-characteristic.

(Remark: By regarding $\mathbf{cov}(B)$ as the minimum number of nowhere dense (rather than meager) sets needed to cover the reals, we could work with $\mathrm{NWD}(f)$ rather than $\mathrm{Meager}(g)$ and get, with less work, the better result that $\mathbf{cov}(B)$ is a uniform Π^0_1-characteristic. But we shall need $\mathrm{Meager}(g)$ later, and the improvement from Σ^0_2 to Π^0_1 will be automatic when we establish Proposition 3(a) below.)

For $\mathbf{unif}(L)$, we also need a coding, this time of the measure zero G_δ sets, by reals. The idea is to code a measure zero G_δ set N by coding a sequence of open sets U_n, with measures $\mu(U_n) \le 2^{-n}$, and with intersection N; an open set U_n in turn is coded by listing codes for basic open sets whose union is U_n. For convenience, we work in ${}^\omega 2$ (or equivalently $\mathcal{P}(\omega)$), where every finite sequence $s \in {}^{<\omega}2$ determines a basic open set

$$B_s = \{x \in {}^\omega 2 \mid s \text{ is an initial segment of } x\}$$

of measure $2^{-\mathrm{length}(s)}$. We intend to use functions $g : \omega \times \omega \to {}^{<\omega}2$ to code G_δ sets $\bigcap_{n\in\omega} \bigcup_{k\in\omega} B_{g(n,k)}$, but we wish to ensure that only G_δ sets of measure zero are obtained. Therefore, we set

$$\begin{aligned}\mathrm{Null}(g) =&\{x \in {}^\omega\omega \mid [\forall n\, \exists k\, g(n,k) \text{ is an initial segment of } x] \text{ and} \\ &[\forall n\, \forall k \sum_{j=0}^{k} 2^{-\mathrm{length}(g(n,j))} \le 2^{-n}]\}.\end{aligned}$$

The second clause here means that $\mu(\bigcup_{j\in\omega} B_{g(n,j)}) \le 2^{-n}$ and so the coded G_δ set must have measure zero. If this clause, which is independent of x, is not satisfied by g, then $\mathrm{Null}(g) = \emptyset$. Thus, $\mathrm{Null}(g)$ is always a G_δ set of measure zero, and every set of measure zero is included in one of this form. Therefore, $\mathbf{unif}(L)$ is the minimum cardinality of a family $\mathcal{X}$ of reals (in ${}^\omega 2$) such that, for every g (in ${}^{\omega\times\omega}({}^{<\omega}2)$ which is homeomorphic to ${}^\omega\omega$), there is $x \in \mathcal{X}$ with $x \notin \mathrm{Null}(g)$. The relation $x \notin \mathrm{Null}(g)$ is, by inspection of the definition of $\mathrm{Null}(g)$, Σ^0_2, and so $\mathbf{unif}(L)$ is a uniform Σ^0_2-characteristic.

(c) For $\mathbf{b}$, take,

$$R(x, y) \iff x \not\le^* y \iff \forall k\, (\exists n \ge k)\, y(n) < x(n).$$

For $\mathbf{s}$, take

$$\begin{aligned}R(x, y) &\iff x \text{ splits } y \\ &\iff \forall k\, [(\exists n \ge k)\, (n \in x \wedge n \in y) \text{ and} \\ &\qquad (\exists n \ge k)\, (n \notin x \wedge n \in y)].\end{aligned}$$

For **unif**(B), take

$$R(x,g) \iff x \notin \mathrm{Meager}(g).$$

For **cov**(L), take

$$R(g,y) \iff y \in \mathrm{Null}(g).$$

Finally, for $\aleph_1$, take, for $x \in {}^\omega\omega$ and $y \in {}^{\omega\times\omega}\omega$,

$$\begin{aligned} R(x,y) &\iff x \text{ differs from all the functions } y(n,-) \text{ for } n \in \omega \\ &\iff (\forall n)\,(\exists k)\,x(k) \neq y(n,k). \end{aligned}$$

All these R's are Π^0_2 by inspection.

(d) Let $(X_\alpha)_{\alpha<\mathbf{t}}$ be as in the definition of $\mathbf{t}$. For each $\alpha < \mathbf{t}$, let

$$Q_\alpha = \{Y \in [\omega]^\omega \mid Y - X_\alpha \text{ is infinite}\}.$$

Then Q_α is $\boldsymbol{\Pi}^0_2$ for each α, and the family $\{Q_\alpha \mid \alpha < \mathbf{t}\}$ covers $[\omega]^\omega$ because the sequence $(X_\alpha)_{\alpha<\mathbf{t}}$ cannot be extended. A subfamily of cardinality $< \mathbf{t}$ has the form $\{Q_\alpha \mid \alpha \in S\}$ where S is a subset of $\mathbf{t}$ of cardinality $< \mathbf{t}$ and is therefore not cofinal in $\mathbf{t}$. So let $\beta < \mathbf{t}$ be larger than every $\alpha \in S$. Then, for $\alpha \in S$, we have $X_\beta \subseteq^* X_\alpha$. Therefore $X_\beta \notin \bigcup_{\alpha\in S} Q_\alpha$, and the subfamily fails to cover $[\omega]^\omega$. □

We remark that the proof of (d) shows that, if a regular cardinal κ is the length of a tower, then κ is a $\boldsymbol{\Pi}^0_2$-characteristic.

Proposition 3. *(a) Every uniform Σ^0_{n+1}-characteristic is a uniform Π^0_n-characteristic.*
(b) Every (uniform) $\boldsymbol{\Sigma}^0_{n+1}$-characteristic is a (uniform) $\boldsymbol{\Pi}^0_n$-characteristic.
(c) Every $\boldsymbol{\Sigma}^1_2$-characteristic is a $\boldsymbol{\Delta}^1_1$-characteristic.

Proof. (a) Let R be a Σ^0_{n+1} binary relation on ${}^\omega\omega$, given by

$$R(x,y) \iff (\exists n \in \omega)\, Q(n,x,y)$$

where Q is Π^0_n. If $\mathcal{X}$ is a subset of ${}^\omega\omega$ such that $\forall y\,(\exists x \in \mathcal{X})\,R(x,y)$, then $\omega \times \mathcal{X}$ is a subset $\mathcal{Z}$ of $\omega \times {}^\omega\omega$ such that $\forall y\,(\exists (n,x) \in \mathcal{Z})\,Q(n,x,y)$. Conversely, for any such $\mathcal{Z} \subseteq \omega \times {}^\omega\omega$, we obtain such an $\mathcal{X} \subseteq {}^\omega\omega$ by taking $\mathcal{X} = \{x \mid (\exists n)\,(n,x) \in \mathcal{Z}\}$. In each case $\mathcal{X}$ and $\mathcal{Z}$ have the same cardinality provided they are infinite. So the uniform Σ^0_{n+1}-characteristic defined by R equals the uniform Π^0_n-characteristic defined by $Q \subseteq (\omega \times {}^\omega\omega) \times {}^\omega\omega$.

(b) The uniform case is proved exactly like (a); just make all the Σ's and Π's boldface. The non-uniform case is (at least) equally easy; in the given cover of ${}^\omega\omega$ by $\boldsymbol{\Sigma}^0_{n+1}$ sets, replace each of these sets A with countably many $\boldsymbol{\Pi}^0_n$ sets whose union is A.

(c) Every $\boldsymbol{\Sigma}^1_2$ set is a union of $\aleph_1$ Borel (i.e., $\boldsymbol{\Delta}^1_1$) sets [18,p.96]. So if κ is a $\boldsymbol{\Sigma}^1_2$-characterstic and $\mathcal{X}$ is a covering of ${}^\omega\omega$ by κ $\boldsymbol{\Sigma}^1_2$ sets such that no subfamily of size $< \kappa$ covers ${}^\omega\omega$, we can replace each of the $\boldsymbol{\Sigma}^1_2$ sets in $\mathcal{X}$ by its $\aleph_1$ Borel constituents to obtain a new covering $\mathcal{X}'$ of ${}^\omega\omega$ by κ (since $\kappa \geq \aleph_1$) $\mathbf{\Delta}^1_1$ sets such that no subfamily of size $< \kappa$ covers ${}^\omega\omega$. $\square$

Neither Proposition 3 nor the list of examples in Proposition 2 mentioned $\mathbf{\Sigma}^0_1$-characteristics, and for a good reason. There are none. If ${}^\omega\omega$ is covered by a family of $\mathbf{\Sigma}^0_1$ (i.e., open) subsets, then it is covered by a countable subfamily (i.e., the space ${}^\omega\omega$ is Lindelöf), but we required characteristics to be uncountable.

Corollary 4. **d**, **r**, **cov**(B), *and* **unif**(L) *are uniform* Π^0_1*-characteristics.*

Proof. Combine Propositions 2(b) and 3(a). $\square$

In view of Proposition 3, the Σ levels of the arithmetical and finite Borel hierarchies are, in the context of characteristics, equivalent to the immediately preceding Π levels. ("Finite" is inessential here. Proposition 3(b) remains true, with the same proof, if n is allowed to be transfinite.) For the rest of the paper, we shall use the Π rather than the Σ class.

To avoid leaving an obvious and unnecessary gap in our list of examples, we comment on the additivity and cofinality characteristics of measure and category. These characteristics are defined by

$$\begin{aligned}\mathbf{add}(B) &= \text{minimum number of meager sets whose union is not meager,}\\ \mathbf{cof}(B) &= \text{minimum number of meager sets in a family}\\ &\qquad \text{such that every meager set has a superset in the family,}\end{aligned}$$

and analogous definitons for **add**(L) and **cof**(L). In these definitions, we can restrict attention to meager F_σ sets and measure zero G_δ sets, i.e., to sets easily coded by reals. The coding shows that these are characteristics in the sense of our definitions and in fact that both additivities are uniform Σ^1_1-characteristics and both cofinalities are uniform Π^1_1-characteristics. One can do considerably better by using combinatorial characterizations of these cardinals. Specifically, by writing out the characterizations of **add**(L) and **cof**(L) implicit in Theorems 0.9 and 0.10 of [1] (and using Lemma 0.5 to eliminate a quantifier), we find that **add**(L) is a uniform Π^0_2-characteristic and **cof**(L) is a uniform Σ^0_2- (hence Π^0_1-)characteristic. (Actually, Theorems 0.9 and 0.10 are concerned with the equations **add**$(L) =$ **c** and **cof**$(L) =$ **c**, but the generalizations we need can be proved exactly the same way. The same remark applies to other results cited below.) For the Baire category characteristics, we have that **add**(B) is either **cov**(B) or **b**, whichever is smaller [15], and **cof**(B) is either **unif**(B) or **d**, whichever is larger [16]. So both **add**(B) and **cof**(B) are uniform Π^0_2-characteristics. We remark that Bartoszyński's elegant characterization [1, Thm. 1.7] of **cov**(B) as the smallest cardinality of any $\mathcal{X} \subseteq {}^\omega\omega$ such that

$$(\forall y \in {}^\omega\omega)\,(\exists x \in \mathcal{X})\,(\exists n)\,(\forall k \geq n) x(k) \neq y(k)$$

gives an alternative proof that **cov**(B) is a uniform Σ^0_2- (hence Π^0_1-)characteristic.

3. Lower Bound for $\mathbf{\Pi}^0_1$-Characteristics

Theorem 5. *If κ is a $\mathbf{\Pi}^0_1$-characteristic, then $\kappa \geq \mathbf{cov}(B)$.*

Proof. Let $\mathcal{X}$ be a covering of ${}^\omega\omega$ by κ closed (i.e., $\mathbf{\Pi}^0_1$) sets. We shall show that either $\kappa \geq \mathbf{cov}(B)$ or a countable subfamily of $\mathcal{X}$ covers ${}^\omega\omega$. In particular, if $\mathcal{X}$ witnesses that κ is a $\mathbf{\Pi}^0_1$-characteristic, then the second alternative is impossible, so the theorem will follow.

Fix a countable base for the topology of ${}^\omega\omega$, and let U be the union of all the basic open sets B such that some countable subfamily of $\mathcal{X}$ covers B. (We do not claim that any such B's exist; U might be empty.) As the base is countable, U itself can be covered by a countable subfamily of $\mathcal{X}$. So if $U = {}^\omega\omega$, we are done.

Henceforth, we assume that $U \neq {}^\omega\omega$, and we let $C = {}^\omega\omega - U$. So C is a nonempty closed subset of ${}^\omega\omega$. We claim that C is perfect. Indeed, if C had an isolated point x, then some basic neighborhood B of x would be included in $U \cup \{x\}$, which can be covered by countably many sets from $\mathcal{X}$ — countably many to cover U and one more to cover x. But then $B \subseteq U$, contrary to $x \in B \cap C = B - U$.

So C is a perfect subset of ${}^\omega\omega$, and clearly C is covered by the closed sets $X \cap C$ for $X \in \mathcal{X}$. We claim that each of these closed sets is nowhere dense in C. Indeed, if this claim were false, there would be $X \in \mathcal{X}$ and a basic open set B such that $B \cap C$ is nonempty and $\subseteq X \cap C$. But then $B \subseteq U \cup X$, so B can be covered by a countable subfamily of $\mathcal{X}$ — countably many to cover U, plus X. But then $B \subseteq U$, contrary to $B \cap C \neq \emptyset$.

Thus, C is covered by κ nowhere dense (in C) sets $X \cap C$. But it is well-known that the Baire covering number for any perfect subset C of ${}^\omega\omega$ is the same as for ${}^\omega\omega$. So we must have $\kappa \geq \mathbf{cov}(B)$. □

Corollary 6. *All of* $\mathbf{d}$, $\mathbf{r}$, *and* $\mathbf{unif}(L)$ *are* $\geq \mathbf{cov}(B)$. □

This corollary is, of course, well-known, but the usual proof of $\mathbf{unif}(L) \geq \mathbf{cov}(B)$, due to Rothberger [20], uses more specific information about Lebesgue measure. Our proof, by contrast, uses only that $\mathbf{unif}(L)$ is a $\mathbf{\Sigma}^0_2$- (and therefore $\mathbf{\Pi}^0_1$-)characteristic. Specifically, if we replace the ideal L of measure zero sets by any ideal that, like L, has a cofinal subfamily indexed by reals, say $\{S_x \mid x \in {}^\omega\omega\}$, such that the relation $y \in S_x$ is $\mathbf{\Pi}^0_2$ (or even $\mathbf{\Pi}^0_2$ in x for each fixed y) then $\mathbf{unif}$ for this ideal is $\geq \mathbf{cov}(B)$. Rothberger's proof, on the other hand, is symmetric between category and measure; it shows that $\mathbf{unif}(B) \geq \mathbf{cov}(L)$ for the same reason as $\mathbf{unif}(L) \geq \mathbf{cov}(B)$. Our proof lacks this symmetry because $\mathbf{unif}(B)$ and $\mathbf{cov}(L)$ are not $\mathbf{\Pi}^0_1$- but $\mathbf{\Pi}^0_2$-characteristics. We shall discuss this matter further in Section 5.

Corollary 7. *Martin's axiom implies that every $\mathbf{\Pi}^0_1$-characteristic is equal to* $\mathbf{c}$.

Proof. Martin's axiom, even when weakened to apply only to countable (rather than ccc) partial orders, implies $\mathbf{cov}(B) = \mathbf{c}$. [5,14]. □

4. Consistency Results for $\mathbf{\Pi}^0_2$ and Higher Characteristics

All the familiar cardinal characteristics of the continuum — those defined in Section 1 above as well as numerous others [3,23] — are equal to $\mathbf{c}$ under Martin's axiom, and Corollary 7 may be viewed as a partial explanation of this fact. Unfortunately, this explanation does not extend beyond the $\mathbf{\Pi}^0_1$ level, since $\aleph_1$ is a uniform Π^0_2-characteristic. This leaves open several possibilities. For example, $\aleph_1$ might be the only exception to a very general situation. That is, there might be a large pointclass Γ (perhaps even $\Gamma = \mathrm{OD}\mathbb{R}$) such that, under MA, all Γ-characteristics greater than $\aleph_1$ are equal to $\mathbf{c}$. On the other hand, one might think that, as one goes up the Borel hierarchy, more and more small cardinals become characteristics; since $\aleph_1$ is a $\mathbf{\Pi}^0_2$-characteristic, perhaps $\aleph_2$ is a $\mathbf{\Pi}^0_3$-characteristic and so on.

In this section, we present consistency results showing that none of these possibilities is provable. We shall see that, from $\mathbf{\Pi}^0_2$ upward, there is considerable arbitrariness in the characteristics. For example, as the following theorem shows, it is consistent that, for all $n \in \omega$,

$$\begin{aligned} \aleph_n \text{ is a } \mathbf{\Pi}^0_2\text{-characteristic} &\iff \aleph_n \text{ is an } \mathrm{OD}\mathbb{R}\text{-characteristic} \\ &\iff n \text{ is a power of 17.} \end{aligned}$$

Theorem 8. *Assume GCH, and let A be a subset of ω containing 1 but not 0. Then there is a forcing extension of the universe in which $\mathbf{c} = \aleph_{\omega+1}$; the $\mathbf{\Pi}^0_2$-characteristics are $\aleph_n$ for $n \in A$, $\aleph_\omega$, and $\aleph_{\omega+1}$; and these are the only OD$\mathbb{R}$-characteristics.*

We shall prove a somewhat more general result, but it seems worthwhile to point out first that the presence of $\aleph_\omega$ among the $\mathbf{\Pi}^0_2$-characteristics is unavoidable if A is infinite. Indeed, if Γ is any non-trivial pointclass closed under pre-images by recursively coded continuous functions, then the supremum of any countably many Γ-characteristics is also a Γ-characteristic. Indeed, if $\mathcal{X}_n$ witnesses that κ_n is a Γ-characteristic, then $\{\{(n)^\frown x \mid x \in X\} \mid n \in \omega, X \in \mathcal{X}_n\}$ witnesses that $\sup_n \kappa_n$ is a Γ-characteristic.

Theorem 9. *Assume GCH, and let C be a closed set of uncountable cardinals containing $\aleph_1$, containing all uncountable cardinals $\leq |C|$ and containing the immediate successors of all its members of cofinality ω. Then there is a notion of forcing satisfying the countable chain condition and forcing that $\mathbf{c} = \max(C)$ and that both the set of $\mathbf{\Pi}^0_2$-characteristics and the set of OD$\mathbb{R}$-characteristics are equal to C.*

We remark that, because C is closed, it has a largest element, so the equation $\mathbf{c} = \max(C)$ makes sense.

Clearly, Theorem 8 is a special case of Theorem 9 with $C = \{\aleph_n \mid n \in A\} \cup \{\aleph_\omega, \aleph_{\omega+1}\}$. In contrast to the situation with Theorem 8, we do not know that all the hypotheses in Theorem 9 are really needed in their full strength. We certainly need that C be closed under ω-limits (see the remark preceding the theorem), that it have a largest element, that this element not have cofinality ω (so that $\mathbf{c} = \max(C)$ is consistent), and that it contain $\aleph_1$ (see Proposition 2(a, c)). But the remaining hypotheses might be mere artifacts of the proof technique.

Proof. The proof consists of first ensuring that every $\kappa \in C$ is a $\mathbf{\Pi}^0_2$-characteristic by forcing a maximal almost disjoint family of κ subsets of ω and second showing that no cardinals $\lambda \notin C$ are ODℝ-characteristics in the forcing extension. The first part of the proof uses Hechler's technique [9] for forcing maximal almost disjoint families of different sizes. The second is related to a theorem of Miller [17] that, when many independent Cohen reals are added to a model of GCH, no cardinal strictly between $\aleph_1$ and $\mathbf{c}$ is a Borel-characteristic; we strengthen the conclusion from "Borel" to "ODℝ", and we work with a Hechler-type model rather than the Cohen model.

Lemma 10. *If $\mathcal{X}$ is a maximal almost disjoint family of infinite subsets of ω, then $|\mathcal{X}|$ is a $\mathbf{\Pi}^0_2$-characteristic.*

Proof. We work in $[\omega]^\omega$ and define, for each $X \in \mathcal{X}$,

$$M(X) = \{Y \in [\omega]^\omega \mid X \cap Y \text{ is infinite}\}.$$

The family $\{M(X) \mid X \in \mathcal{X}\}$ covers $[\omega]^\omega$, by maximality of $\mathcal{X}$. But no proper subfamily $\{M(X) \mid X \in \mathcal{X}'\}$ with $\mathcal{X}' \subset \mathcal{X}$ covers $[\omega]^\omega$, because if $Y \in \mathcal{X} - \mathcal{X}'$ then $Y \notin M(X)$ for any $X \in \mathcal{X}'$. By counting quantifiers, we see that each $M(X)$ is a $\mathbf{\Pi}^0_2$ set, so $|\mathcal{X}|$ is a $\mathbf{\Pi}^0_2$-characteristic. □

Returning to the proof of the theorem, we recall that we wish to force, for each $\kappa \in C$, a maximal almost disjoint family of cardinality κ; by Lemma 10, this will ensure that every $\kappa \in C$ is a $\mathbf{\Pi}^0_2$-characteristic in the extension (provided cardinals are preserved, which they will be). For each $\kappa \in C$, let $I_\kappa = \{(\kappa, \xi) \mid \xi < \kappa\}$, and let $I = \bigcup_{\kappa \in C} I_\kappa$. The maximal almost disjoint family of size κ that we adjoin will be indexed by I_κ, so altogether we shall adjoin an I-indexed family of subsets of ω. Since we want the forcing to satisfy the countable chain condition, we use finite conditions; we build the desired almost-disjointness into the forcing, and genericity will yield the desired maximality.

A forcing condition p is a function into 2 whose domain is of the form $F \times n$ where F is a finite subset of I and $n \in \omega$. (We make the customary identification of n with $\{0, 1, \dots, n-1\}$.) An extension of $p : F \times n \to 2$ is a condition $p' : F' \times n' \to 2$ such that $p' \supseteq p$ (and therefore $F' \supseteq F$ and $n' \geq n$) and, whenever (κ, ξ) and (κ, η) are distinct elements of I_κ (for the same κ) and $n \leq k < n'$, then $p'(\kappa, \xi, k)$ and $p'(\kappa, \eta, k)$ are not both 1. (Intuitively, we regard $p : F \times n \to 2$ as giving the following information about the generic sets $a_{\kappa,\xi} \subseteq \omega$ being adjoined. First, if $p(\kappa, \xi, k) = 1$ (resp. 0), then $k \in a_{\kappa,\xi}$ (resp. $k \notin a_{\kappa,\xi}$), and second, if (κ, ξ) and (κ, η) are distinct elements of $F \cap I_\kappa$, then $a_{\kappa,\xi} \cap a_{\kappa,\eta} \subseteq n$. Then the definition of extension corresponds to giving more information.) We call this notion of forcing P.

Before proceeding, we should point out that this notion of forcing is essentially a part of the forcing defined by Hechler in [9, Theorem 3.2]. Hechler adjoins maximal almost disjoint families of all cardinalities from $\aleph_1$ to $\mathbf{c}$ and towers of all lengths $\leq \mathbf{c}$ of uncountable cofinality, and he works with a ground model where $\mathbf{c}$ is (in the interesting cases) already large so that his forcing does not alter cardinal exponentiation. If one ignores the parts of Hechler's forcing that refer to towers and one replaces the interval $[\aleph_1, \mathbf{c}]$ of cardinals (destined to become the sizes of maximal almost disjoint families) by C, then one obtains a notion of forcing having a dense subset isomorphic to our P. The next two lemmas are transcriptions for P of corresponding arguments in Hechler's proof; we include their proofs for the reader's convenience.

Lemma 11. *P satisfies the countable chain condition.*

Proof. Let $\aleph_1$ elements $p_\alpha : F_\alpha \times n_\alpha \to 2$ of P be given ($\alpha < \aleph_1$). By passing to a subfamily of size $\aleph_1$, we can assume that all the n_α are the same n and that the F_α constitute a Δ-system [11, p.225], i.e., $F_\alpha \cap F_\beta$ is the same set K for all $\alpha < \beta < \aleph_1$. Again, by passing to a subfamily of size $\aleph_1$, we can assume that the restrictions $p_\alpha | K \times n$ are all equal. But then, for any $\alpha < \beta$, $p_\alpha \cup p_\beta$ is a common extension of p_α and p_β. So the $\aleph_1$ given conditions do not form an antichain. $\square$

(This proof shows that in any family of $\aleph_1$ conditions, there is a subfamily of $\aleph_1$ conditions every finitely many of which have a common extension, i.e., P has precaliber $\aleph_1$.)

Let G be a P-generic filter over the universe V. (Formally, we are passing to a Boolean-valued extension of V.) For $(\kappa, \xi) \in I$ let

$$a_{\kappa,\xi} = \{k \mid (\exists p \in G)\, p(\kappa, \xi, k) = 1\},$$

and for $\kappa \in C$ let

$$\mathcal{A}_\kappa = \{a_{\kappa,\xi} \mid \xi < \kappa\}.$$

Lemma 12. *For each $\kappa \in C$, $\mathcal{A}_\kappa$ is a maximal almost disjoint family of subsets of ω in $V[G]$.*

Proof. Fix $\kappa \in C$. If $\xi < \eta < \kappa$, then G contains some $p : F \times n \to 2$ with both (κ, ξ) and (κ, η) in F, because the set (in V) of all such p's is clearly dense. Fix such a $p \in G$ and consider an arbitrary $q \in G$. As G is a filter, p and q have a common extension r. By definition of extension, we cannot have $r(\kappa, \xi, k) = r(\kappa, \eta, k) = 1$ for any $k \geq n$, and thus we cannot have $q(\kappa, \xi, k) = q(\kappa, \eta, k) = 1$ for any $k \geq n$. As q was arbitrary in G, it follows that $a_{\kappa,\xi} \cap a_{\kappa,\eta} \subseteq n$. So $\mathcal{A}_\kappa$ is an almost disjoint family of subsets of ω.

To prove maximality, suppose x were (in $V[G]$) an infinite subset of ω almost disjoint from $a_{\kappa,\xi}$ for all $\xi < \kappa$. Because of the countable chain condition (Lemma 11), x has name $\dot{x} \in V$ that involves only countably many conditions. Fix a countable set $J \subseteq I$ such that all the conditions involved in $\dot{x}$ have domains $\subseteq J \times \omega$. Also fix a condition $p : F \times n \to 2$ forcing "$\dot{x}$ is an infinite subset of ω almost disjoint from $\dot{a}_{\kappa,\xi}$ for all $\xi < \kappa$" (where $\dot{a}$ is the standard name for the function a that sends (κ, ξ) to $a_{\kappa,\xi}$, and where we have written κ instead of its canonical name $\check{\kappa}$.) Enlarging J if necessary, we assume $F \subseteq J$. Since $\kappa \in C$, κ is uncountable, so fix $\xi < \kappa$ with $(\kappa, \xi) \notin J$. Since p forces "$\dot{x} \cap \dot{a}_{\kappa,\xi}$ is finite", it has an extension $p' : F' \times n' \to 2$ forcing "$\dot{x} \cap \dot{a}_{\kappa,\xi} \subseteq m$" for some $m \in \omega$. Extending p' further, we can assume $n' \geq m$, and then, as m can be increased trivially, we can assume $m = n'$, so p' forces "$\dot{x} \cap \dot{a}_{\kappa,\xi} \subseteq n'$." Now, as p' extends p and therefore forces "$\dot{x}$

is infinite and almost disjoint from all $\dot{a}_{\kappa,\eta}$", it also forces (since F' is finite) "there is $k \geq n'$ such that

$$k \in \dot{x} \text{ and } (\forall \eta \in X)\, k \notin \dot{a}_{\kappa,\eta}, \text{''} \tag{1}$$

where $X = \{\eta \mid (\kappa,\eta) \in F' \cap J\}$.

Extend p' to a condition $q : H \times n'' \to 2$ forcing (1) for a specific $k \geq n'$. Let $F'' = H \cap J$ and let $p'' = q \mid (F'' \times n'')$.

Claim. *p'' forces (1) for the same value of k.*

To prove this claim, suppose it failed, and extend p'' to a condition r forcing the negation of (1). As p'' and all the conditions involved in (1) (i.e., in $\dot{x}$ and in $\dot{a}_{\kappa,\eta}$ for $\eta \in X$) have domains $\subseteq J \times \omega$, we can take r to also have domain $\subseteq J \times \omega$. Then the function $q \cup r$ can be extended by zeros to a condition that extends both q and r. This is absurd, as q forces (1) and r forces its negation. So the claim is proved.

Notice that p'' agrees with p' on the common part of their domains, $(F' \cap J) \times n'$, because q extends them both. Extending p'' if necessary, we assume $n'' > k$; then,

$$(\forall \eta \in X)\, p''(\kappa,\eta,k) = 0 \tag{2}$$

because p'' forces (1).

Define a function $s : F' \times n'' \to 2$ by making s agree with p' on $F' \times n'$, making s agree with p'' on $(F' \cap J) \times n''$, setting $s(\kappa,\xi,k) = 1$, and setting all remaining values of s equal to zero. It is obvious that s extends p' as a function; we claim it extends p' as a condition. We must check that, for any $(\lambda,\alpha) \neq (\lambda,\beta)$ both in $F' \cap I_\lambda$ and any j with $n' \leq j < n''$, the values of s at the two locations (λ,α,j) and (λ,β,j) are not both 1. If they were, then neither of these values could be given by the first clause in the definition of s, because $j \geq n'$ and the first clause gives values on $F' \times n'$. Neither of these values could be given by the last clause, since the last clause gives values of zero. One of the values could be given by the second clause, but not both, for the values given by the second clause agree with the values of q, an extension of p'. So one of the two values was given by the third clause and the other by the second. That is, $\lambda = \kappa$, $j = k$, and one of α and β, say α, is ξ. Then

$$1 = s(\lambda,\beta,j) = s(\kappa,\beta,k)$$

is given by the second clause, i.e., $(\kappa,\beta) \in F' \cap J$. But then

$$s(\kappa,\beta,k) = p''(\kappa,\beta,k) = 0,$$

by (2). This contradiction shows that s extends p' as a condition.

Therefore, s, like p', forces "$k \in \dot{x}$" and "$\dot{x} \cap \dot{a}_{\kappa,\xi} \subseteq n'$." As $k \geq n'$, s must also force "$k \notin \dot{a}_{\kappa,\xi}$." But this is absurd, as $s(\kappa,\xi,k) = 1$. This contradiction completes the proof of maximality. $\square$

By the lemmas proved so far, every $\kappa \in C$ is a $\mathbf{\Pi}^0_2$-characteristic in $V[G]$. It remains to prove that in $V[G]$ no cardinal $\lambda \notin C$ is an ODℝ-characteristic and that $\mathbf{c} = \max(C)$. The latter actually follows from the former, since $\mathbf{c}$ is the largest $\mathbf{\Pi}^0_2$-characteristic, but it is also easy to see directly, since the notion of forcing P has cardinality $\max(C)$, whose cofinality is uncountable, and P satisfies the countable chain condition, and GCH holds in the ground model V.

To complete the proof, consider any uncountable $\lambda \notin C$, and suppose we have, in $V[G]$, a λ-sequence of ODℝ sets $X_\alpha (\alpha < \lambda)$ that cover ${}^\omega\omega$. Fix a sequence of reals u_α and a sequence of ordinals θ_α such that X_α is ordinal-definable with real parameter u_α and in fact is the θ_αth set ordinal-definable from u_α (in some standard well-ordering of the $\mathrm{OD}(u_\alpha)$ sets). Choose in V names $\dot{X}$, $\dot{u}$, $\dot{\theta}$ for the sequences (X_α), (u_α), (θ_α) such that P forces "$\dot{u}$ is a λ-sequence of reals, $\dot{\theta}$ is a λ-sequence of ordinals, and, for each $\alpha < \lambda$, $\dot{X}_\alpha$ is the $\dot{\theta}_\alpha$th element (in the standard order) of $\mathrm{OD}(\dot{u}_\alpha)$."

Let μ be the largest element of C below λ. The hypotheses of the theorem imply that μ exists and has uncountable cofinality. (Here and below, we tacitly use Lemma 11 to avoid having to say whether cardinals and cofinalities refer to V or to $V[G]$; the countable chain condition makes these concepts absolute.) It follows that, in V where GCH holds, $\mu^{\aleph_0} = \mu$.

We intend to find a set $M \subseteq \lambda$ of cardinality μ, such that the sets X_α for $\alpha \in M$ cover ${}^\omega\omega$. M will be obtained in the ground model V as the union of an increasing $\aleph_1$-sequence of approximations M_σ of cardinality $\leq \mu$ for $\sigma < \aleph_1$. Recall that $\mu \geq \aleph_1$, so the union M also has cardinality $\leq \mu$. For limit ordinals σ, M_σ will be the union of the M_τ's for $\tau < \sigma$. As M_0 we take the empty set. The non-trivial part of the construction is the successor step, and for this we need some preliminary work.

Until further notice, we work in V. We let $\dot{x}$ range over names for reals, and we identify two names if P forces that they are equal. With this convention, the countable chain condition (Lemma 11) allows us to assume that each name $\dot{x}$ involves only countably many conditions. So there is a set $J \subseteq I$ such that

(a) all the conditions involved in $\dot{x}$ have domains $F \times n$ with $F \subseteq J$, and

(b) $|J| = \aleph_0$.

In particular, we can choose such a J_α for each of the names $\dot{u}_\alpha$ $(\alpha < \lambda)$ that we chose earlier for the real parameters in ordinal definitions of $\dot{X}_\alpha$. Enlarging J_α, but keeping it countable, we can similarly arrange that all conditions involved in $\dot{\theta}_\alpha$ have domains $F \times n$ with $F \subseteq J_\alpha$. Let S be the union of these λ countable sets J_α and the sets I_κ for $\kappa \leq \mu$ in C. So $|S| = \lambda$.

Until further notice, consider a fixed but arbitrary set $K \subseteq S$ of cardinality μ such that $I_\kappa \subseteq K$ for all $\kappa \leq \mu$ in C. Notice that, for $\kappa > \mu$ in C, we have $\kappa > \lambda$ (by choice of μ and because $\lambda \notin C$), so $I_\kappa - S$ and $I_\kappa - K$ have cardinality κ. We shall call $J \subseteq I$ a *support* for a name $\dot{x}$ (of a real) if it satisfies (a) and (b) above and also

(c) for each $\kappa \in C$, if $J \cap I_\kappa - K$ is nonempty, then it is infinite.

Notice that the new clause (c) is easy to satisfy by enlarging J.

Let $\mathcal{G}$ be the group of those permutations of I that map each I_κ into itself and that fix all members of K. Clearly, $\mathcal{G}$ acts as a group of automorphisms of the notion of forcing P, by

$$g(p)(g(\kappa,\xi),k)=p(\kappa,\xi,k),$$

and it is well known that such automorphisms also act on the class of P-forcing names (i.e., on the associated Boolean-valued model) and preserve the forcing relation. It is easy to check that, if J supports $\dot{x}$, then $g(J)$ supports $g(\dot{x})$; if, in addition, g fixes all members of J, then it also fixes $\dot{x}$.

If J is a support then, thanks to clause (c), its $\mathcal{G}$-orbit (i.e., its equivalence class under the action of $\mathcal{G}$) is determined by $J\cap K$ and

$$\bar{J}=\{\kappa\in C\mid J\cap I_\kappa-K\neq\emptyset\}.$$

That is, if J' is another support with $J'\cap K=J\cap K$ and $\overline{J'}=\bar{J}$, then there is $g\in\mathcal{G}$ with $g(J)=J'$. Since $J\cap K$ is countable and $|K|=\mu=\mu^{\aleph_0}$ there are, as J varies over all supports, only μ possibilities for $J\cap K$. Also, since $\bar{J}$ is a countable subset of C and $|C|\leq\mu$ (because all uncountable cardinals $\leq|C|$ are in C), the number of possibilities for $\bar{J}$ is $\leq\mu^{\aleph_0}=\mu$. Therefore, there are only μ $\mathcal{G}$-orbits of supports.

For each $\mathcal{G}$-orbit of supports, choose one member J such that $J\cap S=J\cap K$, i.e., such that J is disjoint from $S-K$. Such a J is easy to find, starting with an arbitrary J' in the orbit. For each $\kappa\in C$ such that J' meets $I_\kappa\cap S-K$, we have $\kappa>\lambda$ (for otherwise $I_\kappa\subseteq K$) and then $|I_\kappa-S|>\lambda$ so there are permutations of I_κ fixing $I_\kappa\cap K$ pointwise and mapping the (countable) rest of J' out of S. Combine such permutations for all relevant κ to get $g\in\mathcal{G}$ for which $J=g(J')$ is as desired. Call the μ orbit-representatives just chosen the *standard* supports.

For any fixed support J, any name $\dot{x}$ supported by J can be specified by giving, for each $n\in\omega$, a maximal antichain of conditions that are supported by J and that decide $\dot{x}(n)$ and giving those decisions. It follows, by CH, that there are only $\aleph_1$ such names for each J.

Thus, there are only μ names $\dot{x}$ that have standard supports. For each of these, fix a countable set $A=A(\dot{x})\subseteq\lambda$ such that P forces "$(\exists\alpha\in\check{A})\,\dot{x}\in\dot{X}_\alpha$." The existence of such an A follows from the countable chain condition and the fact that "$(\exists\alpha<\lambda)\,\dot{x}\in\dot{X}_\alpha$" is forced. Let B be the union of these sets $A(\dot{x})$ for all $\dot{x}$ with standard support. As the union of μ countable sets, B has cardinality $\leq\mu$.

Now un-fix K. The preceding discussion produces, for each $K\subseteq S$ of size μ with $I_\kappa\subseteq K$ for $\kappa\leq\mu$, a subset B of λ of cardinality $\leq\mu$.

At last, we are in a position to complete the definition of the sequence $(M_\sigma)_{\sigma<\aleph_1}$ by carrying out the successor step. Recall that $\dot{u}_\alpha$ and $\dot{\theta}_\alpha$ are such that P forces "$\dot{X}_\alpha$ is the $\dot{\theta}_\alpha$th set ordinal-definable from $\dot{u}_\alpha$" and that J_α was chosen so that all conditions involved in $\dot{u}_\alpha$ and $\dot{\theta}_\alpha$ have domains $F\times n$ with $F\subseteq J$. Now, given M_σ, to define $M_{\sigma+1}$, apply the preceding construction of B from K with

$$K=K_\sigma=\bigcup_{\alpha\in M_\sigma}J_\alpha\cup\bigcup_{\kappa\leq\mu\text{ in }C}I_\kappa.$$

As each J_α is countable and $|M_\sigma| \leq \mu$ we have $|K| = \mu$ so the construction of B makes sense. We set $M_{\sigma+1} = B$ and note that $|M_{\sigma+1}| \leq \mu$ as desired. We also note that the K_σ form a continuous monotone sequence because the M_σ do.

Having defined M_σ for all $\sigma < \aleph_1$ and thus also their union M, we complete the proof of the theorem by showing that, for every name $\dot{x}$ of a real, P forces "$(\exists\alpha \in M)\, \dot{x} \in \dot{X}_\alpha$". Let $\dot{x}$ be given and let J satisfy (a) and (b) in the definition of support for $\dot{x}$. Let $K_\infty = \bigcup_{\tau<\aleph_1} K_\tau$, so $|K_\infty| = \mu$. As J is countable, we can fix $\sigma < \aleph_1$ such that

$$J \cap K_\infty \subseteq K_\sigma. \tag{1}$$

Henceforth, we use notation as in the construction of $M_{\sigma+1}$ from M_σ. In particular, K means K_σ and the notion of support uses this K in clause (c). J need not be a support, i.e., clause (c) may fail, but we can enlarge J to a support by adding elements of $I_\kappa - K_\infty$ for all necessary $\kappa > \mu$. (This is possible as, for such κ, $|I_\kappa| > \lambda > \mu = |K_\infty|$. We needn't worry about $\kappa \leq \mu$ as $I_\kappa \subseteq K_\sigma$ for such κ). This enlargement preserves (1), so we assume from now on that J is a support.

Recall that we chose a standard support in every $\mathcal{G}$-orbit of supports. So fix $g \in \mathcal{G}$ such that $g(J)$ is standard. Neither J nor $g(J)$ meets $K_{\sigma+1} - K$. In the case of J this follows from (1), while in the case of $g(J)$ it follows from the fact that standard supports don't meet $S - K$ (and clearly $K_\tau \subseteq S$ for all τ). Thus, there is $h \in \mathcal{G}$ such that h agrees with g on J and with the identity map on $K_{\sigma+1} - K$. In particular, $h(J) = g(J)$ is standard, and h leaves $K_{\sigma+1}$ pointwise fixed (because all elements of $\mathcal{G}$ fix K and h fixes $K_{\sigma+1} - K$.)

Since $h(\dot{x})$ has standard support $h(J)$, it is one of the μ names for which we chose a set $A = A(h(\dot{x}))$ to include in B. By the defining property of A, P forces "$(\exists\alpha \in \check{A})\, h(\dot{x}) \in \dot{X}_\alpha$ " and thus also

"$(\exists\alpha \in \check{A})\, h(\dot{x})$ is in the $\dot{\theta}_\alpha$th set ordinal-definable from $\dot{u}_\alpha$."

For any $\alpha \in A$, we have $\alpha \in B \subseteq M_{\sigma+1}$ by definition of B and $M_{\sigma+1}$. We also have $J_\alpha \subseteq K_{\sigma+1}$, and so h fixes J_α pointwise. It follows, by definition of J_α, that h fixes the names $\dot{u}_\alpha$ and $\dot{\theta}_\alpha$. So, by (2), P forces

"$(\exists\alpha \in \check{A})\, h(\dot{x})$ is in the $h(\dot{\theta}_\alpha)$th set ordinal-definable from $h(\dot{u}_\alpha)$,"

and, since h is an automorphism,

"$(\exists\alpha \in \check{A})\, \dot{x}$ is in the $\dot{\theta}_\alpha$th set ordinal-definable from $\dot{u}_\alpha$."

Since $A \subseteq B \subseteq M_{\sigma+1} \subseteq M$, we have that P forces "$(\exists\alpha \in \check{M})\, \dot{x} \in \dot{X}_\alpha$," as required to complete the proof. $\square$

Remark. At the January, 1991, Bar-Ilan conference on the set theory of the reals, I described many of the results in this paper and made some conjectures, one of which was that there might be very few uniform Π^0_1-characteristics and that one might be able to classify them all. Shelah promptly informed me that, by a countable-support product of forcing notions from [21], he can produce models with infinitely many uniform Π^0_1-characteristics, all of the form "the smallest number of g-branching subtrees of ${}^{<\omega}\omega$ needed to cover all the paths through an f branching subtree of ${}^{<\omega}\omega$." (Here f and g are suitable recursive functions on ω, and an f-branching tree is one in which each node of level n has exactly $f(n)$ immediate successors.) These characteristics can be prescribed rather freely, and one can get uncountably many of them if one allows boldface $\mathbf{\Pi}^0_1$, i.e., non-recursive f and g. I do not know to what extent Shelah's models satisfy the additional property, enjoyed by the models in Theorem 9, that cardinals not explicitly made to be characteristics are not even OD$\mathbb{R}$-characteristics. This work of Shelah will appear (with some modifications) in [7].

It is clear from Theorem 9 (or even from Theorem 8) that we cannot expect restrictive results about $\mathbf{\Pi}^0_2$-characteristics in ZFC. But these theorems leave open the possibility of restrictive results in stronger theories, perhaps ZFC + MA. The following theorem, a corollary of a result of Harrington [8], shows that even Martin's axiom gives no restrictions on the $\mathbf{\Delta}^1_1$-characteristics.

Theorem 13. *It is consistent with Martin's axiom that* $\mathbf{c}$ *be arbitrarily large and that every uncountable cardinal* $\leq \mathbf{c}$ *be a* $\mathbf{\Delta}^1_1$*-characteristic and a uniform* $\mathbf{\Sigma}^1_2$*-characteristic.*

Proof. Harrington [8, Theorem B] obtained models of Martin's axiom in which $\mathbf{c}$ is arbitrarily large and there is a $\mathbf{\Pi}^1_2$ well-ordering $\leq$ of a set of reals, having length $\mathbf{c}$. For any uncountable cardinal $\kappa < \mathbf{c}$, let a be the κth element in this well-ordering. Then the binary relation

$$R(x,y) \iff y \not\leq a \text{ or } y = x$$

is $\mathbf{\Sigma}^1_2$, and a set $\mathcal{X}$ of reals satisfies $\forall y\,(\exists x \in \mathcal{X})\, R(x,y)$ if and only if $\mathcal{X}$ contains all the predecessors of a, of which there are κ. This proves the $\mathbf{\Sigma}^1_2$ part of the theorem for $\kappa < \mathbf{c}$. The case of $\kappa = \mathbf{c}$ is trivial as $\mathbf{c}$ is a uniform $\mathbf{\Pi}^0_1$-characteristic. The $\mathbf{\Delta}^1_1$ part follows by Proposition 3(c). □

Notice that in this proof the well-ordering was used only to produce $\mathbf{\Pi}^1_2$ sets of arbitrary uncountable cardinality $\kappa < \mathbf{c}$.

5. Duality

There is an intuition that some of the familiar cardinal characteristics of the continuum occur in dual pairs. For example, in the abstract of [16], Miller refers to dualizing the proof of $\mathbf{add}(L) \leq \mathbf{b}$ to obtain $\mathbf{d} \leq \mathbf{cof}(L)$. In this section, we make some remarks about this sort of duality, and we attempt to relate it to our theory of Γ-characteristics.

At first sight, duality seems quite easy to describe. Indeed, the Σ^0_2 relations used in the proof of Proposition 2(b) for $\mathbf{d}$, $\mathbf{r}$, $\mathbf{cov}(B)$, and $\mathbf{unif}(L)$ are precisely the negations of the converses of the Π^0_2 relations used in the proof of Proposition 2(c) for the dual characteristics **b**, **s**, **unif**(B), and **cov**(L), respectively. Thus, if we define the dual of a binary relation by

$$\tilde{R}(x,y) \iff \neg R(y,x),$$

then the uniform characteristic determined by $\tilde{R}$ seems to be, in the intuitive sense, dual to the uniform characteristic determined by R.

Some caution is needed, however, with this notion of duality. For example, the Σ^0_2-characteristics in Proposition 2(b) are in fact uniform Π^0_1-characteristics by Proposition 3(a), and in fact the first three of them have quite natural Π^0_1 descriptions. (In the definition of **d**, one can replace $\leq^*$ with "everywhere $\leq$", a similar deletion of "mod finite" in the notion of splitting works for **r**, and for **cov**(B) one can work with nowhere dense closed sets instead of meager sets.) Dualizing those definitions leads to a Σ^0_1 form, for which the cardinal is $\aleph_0$ (and thus not a characteristic, by our definition). It would appear that, before dualizing, one must be careful to put R into the proper form. But what is the proper form?

A hint can be obtained by comparing the "good", i.e., nicely dualizable definitions of **d**, **r**, **b**, and **s** in the proof of Proposition 2 with the "bad" versions obtained by deleting "mod finite". The most evident difference, apart from the complexity difference which makes the bad Π^0_1 definitions look better than the good Π^0_2 ones, is that the good R's are unchanged by finite modifications of their arguments. That is, if x and x' differ only finitely and if y and y' differ only finitely, then for the good R's, but not for the bad ones, $R(x,y)$ implies $R(x',y')$. We call such R's invariant.

What about **cov**(B), **unif**(B), and their measure analogs? The relations used in the proof of Proposition 2 are not invariant, but they can be replaced with ones that are invariant, have the same complexity, and lead to the same duals.

For the category situation, we define a new coding of meager sets by

$$x \in \mathrm{Meager}'(y) \iff (\exists s,t \in {}^{<\omega}\omega)\, x * s \in \mathrm{Meager}(y * t),$$

where Meager is as in the proof of Proposition 2(b) and where $x * s$ is x but with s in place of the initial segment of the same length in x, i.e.,

$$(x * s)(n) = \begin{cases} s(n) & \text{if } n \in \mathrm{domain}(s) \\ x(n) & \text{otherwise.} \end{cases}$$

Thus, the relation "$x \in \mathrm{Meager}'(y)$" is "$x \in \mathrm{Meager}(y)$" enlarged just enough to be invariant. Notice that $\mathrm{Meager}'(y)$ is meager (since there are only countably many possibilities for s and t) and every meager set has a superset of the form $\mathrm{Meager}'(y)$. Furthermore, "$x \in \mathrm{Meager}'(y)$" is a Σ^0_2 relation, so Meager$'$ can replace Meager in the proof of Proposition 2.

For the measure situation, things are a bit more complicated. Defining Null$'$ in exact analogy with Meager$'$ would make "$x \in \mathrm{Null}'(y)$" Σ^0_3 because "$x \in \mathrm{Null}(y)$" is Π^0_2, so the complexity would increase. Changing $(\exists s,t)$ to $(\forall s,t)$ would make $\mathrm{Null}'(y)$ empty, since there is always a t for which $\mathrm{Null}(y*t)$ is empty. We observe however, that we can safely use $\forall s$; that is, if we define

$$x \in \mathrm{Null}^*(y) \iff (\forall s \in {}^{<\omega}\omega)\, x * s \in \mathrm{Null}(y),$$

then this relation is Π^0_2, invariant with respect to x, and still enjoys the crucial property that every set A of measure zero is included in one of the form $\mathrm{Null}^*(y)$. (For the proof, simply observe that $\{x * s \mid x \in A, s \in {}^{<\omega}\omega\}$ has measure zero and is therefore included in some $\mathrm{Null}(y)$.) To obtain invariance with respect to y, we use a different coding. Every set A of measure zero can be covered by a sequence of sets A_n each of which is a union of finitely many basic open sets B_s (as in the proof of Proposition 2) and has measure below some prescribed positive bound ε_n. Dovetailing infinitely many such constructions with suitable ε's, we can find a sequence of sets A_n such that A_n is a finite union of basic open sets, A_n has measure $\leq 2^{-n}$, and every element of A is in infinitely many A_n's. (This construction is in [2, Lemma 1.1].) Conversely, if each A_n has measure $\leq 2^{-n}$, then $\{x \mid x \in A_n \text{ for infinitely many } n\}$ has measure zero. Let $f : \omega \times \omega \to [{}^{<\omega}\omega]^{<\omega}$ be a recursive function such that, if n is fixed and k varies over ω, $f(n, k)$ enumerates all finite sets $F \subseteq {}^{<\omega}\omega$ such that $\bigcup_{s \in F} B_s$ has measure $\leq 2^{-n}$. Then

$$x \in \mathrm{Null}^\dagger(y) \iff (\forall m)\,(\exists n \geq m)\, x \text{ has a initial segment in } f(n, y(n))$$

defines a Π^0_2 relation, invariant with respect to y, such that all sets of the form $\mathrm{Null}^\dagger(y)$ have measure zero and all sets of measure zero have supersets of this form. Finally, we obtain the desired Null', invariant in both variables, by putting $\mathrm{Null}^\dagger$ in place of Null in the defintion of Null^*:

$$x \in \mathrm{Null}'(y) \iff (\forall s \in {}^{<\omega}\omega)\, x * s \in \mathrm{Null}^\dagger(y),$$

The preceding discussion suggests that a suitable context for duality is uniform characteristics given by invariant relations. We leave it to the reader to verify that the relation $\mathrm{Meager}'(y) \subseteq \mathrm{Meager}'(x)$ and its dual $\mathrm{Meager}'(x) \not\subseteq \mathrm{Meager}'(y)$ determine $\mathbf{cof}(B)$ and $\mathbf{add}(B)$, respectively, so these are dual characteristics determined by invariant relations. The analog for measure also holds, as does the improvement, described in Section 2, from Σ^1_1 to Π^0_2 for $\mathbf{add}(L)$ and from Π^1_1 to Σ^0_2 for $\mathbf{cof}(L)$, because the relations obtained from Theorems 0.9 and 0.10 of [1] are invariant.

We close this section with a very easy result, dual to Theorem 5, as propaganda for this notion of duality.

Proposition 14. *Let κ be the uniform $\mathbf{\Pi}^0_2$-characteristic determined by an invariant $\mathbf{\Pi}^0_2$ relation R. Then $\kappa \leq \mathbf{unif}(B)$.*

Proof. Since R determines a characteristic, there must be, for each $y \in {}^\omega\omega$, at least one $x \in {}^\omega\omega$ such that $R(x, y)$. But then, being invariant under finite modifications,

$$R_y = \{x \in {}^\omega\omega \mid R(x, y)\}$$

is dense. It is also $\mathbf{\Pi}^0_2$, i.e., a G_δ set, so it is comeager. Thus, letting $\mathcal{X}$ be a non-meager set of the smallest possible size $\mathbf{unif}(B)$, we have that $\mathcal{X}$ meets R_y for each y. Thus, by definition of κ, we have $\kappa \leq |\mathcal{X}| = \mathbf{unif}(B)$. □

In particular, we have the following analog of Corollary 6, including the other half of Rothberger's theorem [20] along with some easier known results.

Corollary 15. *All of* $\mathbf{b}$, $\mathbf{s}$, *and* $\mathbf{cov}(L)$ *are* $\leq \mathbf{unif}(B)$. □

We remark that, in the proof of Proposition 14, invariance of R was used only with respect to the variable x.

6. Partition Characteristics

This section is devoted to some characteristics connected with partition theorems. Some of these characteristics first attracted my attention as possible new examples of uniform Π^0_1-characteristics. (This was before Shelah showed how to get a plentiful supply of uniform Π^0_1-characteristics [7].) Others arose as duals. They seem to have some intrinsic interest, so we present here what is known about them.

We begin with Ramsey's theorem in the simple form: If $[\omega]^2$ is partitioned into two pieces, then there is an infinite $H \subseteq \omega$ such that $[H]^2$ is included in one piece. As usual, $[X]^n$ is the set of n-element subsets of X, and an H as in the theorem is said to be *homogeneous* for the partition. We call H *almost homogeneous* if there is a finite $F \subseteq H$ such that $H - F$ is homogeneous.

Let

$$\begin{aligned}\mathbf{par} = &\text{minimum number of partitions } \Pi : [\omega]^2 \to 2 \text{ such that} \\ &\text{no single } H \in [\omega]^\omega \text{ is almost homogeneous for them all}\end{aligned}$$

and

$$\begin{aligned}\mathbf{hom} = &\text{minimum number of infinite subsets } H \text{ of } \omega \\ &\text{such that every partition } \Pi : [\omega]^2 \to 2 \\ &\text{has an almost homogeneous set among these } H\text{'s.}\end{aligned}$$

Both of these are easily seen to be uncountable. Since "H is almost homogeneous for Π" is an invariant Σ^0_2 relation, **par** is a uniform Π^0_2-characteristic, **hom** is a uniform Σ^0_2- (hence Π^0_1-)characteristic, and their definitions are dual to each other. (That **hom** is a uniform Π^0_1-characteristic can also be seen by observing that it is unchanged if "almost" is removed from the definition; the same change would turn **par** into $\aleph_0$.)

Our first result is that **par** is nothing new.

Theorem 16. **par** *is the smaller of* **b** *and* **s**.

Proof. First, we consider any $\kappa < \mathbf{par}$, i.e., any κ such that every κ partitions $[\omega]^2 \to 2$ have a common, infinite, almost homogeneous set.

We claim that $\kappa < \mathbf{b}$. To prove this, let a family $\mathcal{F}$ of κ non-decreasing functions $f : \omega \to \omega$ be given; we seek a single g eventually majorizing them. Each $f \in \mathcal{F}$ induces a partition $\Pi_f : [\omega]^2 \to 2$, namely

$$\Pi_f\{a < b\} = 0 \iff f(a) < b.$$

A homogeneous set of color 1 for Π_f must be finite, being bounded by f of its first element. So the common, infinite, almost homogeneous set H for all the Π_f, $f \in \mathcal{F}$, must be almost homogeneous of color 0. That is, for each $f \in \mathcal{F}$, we have $f(a) < b$

for all sufficiently large $a < b$ in H. It follows that the function sending each $n \in \omega$ to the second element of A after n eventually majorizes each $f \in \mathcal{F}$.

We claim further that $\kappa < \mathbf{s}$. Let a family $\mathcal{S}$ of κ infinite subsets S of ω be given; we seek an infinite set not split by any of them. Each $S \in \mathcal{S}$ induces a partition $\Pi_S : [\omega]^2 \to 2$, namely

$$\Pi_S\{a < b\} = 0 \iff a \in S.$$

Clearly, a set almost homogeneous for Π_S is not split by S, so the hypothesis on κ provides the desired unsplit set.

The preceding two claims establish that $\mathbf{par} \leq \min\{\mathbf{b}, \mathbf{s}\}$.

To prove the converse, consider any $\kappa < \min\{\mathbf{b}, \mathbf{s}\}$, and let a family of κ partitions $\Pi_\alpha : [\omega]^2 \to 2$, for $\alpha < \kappa$, be given. We seek a set almost homogeneous for all the Π_α's. For each $\alpha < \kappa$ and each $a \in \omega$, let

$$S_{\alpha,a} = \{b \in \omega - \{a\} \mid \Pi_\alpha\{a, b\} = 0\}.$$

Since there are only $\kappa \cdot \aleph_0 < \mathbf{s}$ sets $S_{\alpha,a}$, they do not form a splitting family. So let A be an infinite set not split by any of them. This means that, for each α and a, the value of $\Pi_\alpha\{a, b\}$ is the same, say $v_\alpha(a)$, for all sufficiently large $b \in A$, say all such that $b > f_\alpha(a)$.

The same argument, applied within A to the κ sets $\{a \mid v_\alpha(a) = 0\}$, provides an infinite $B \subseteq A$ such that $v_\alpha(a)$ has the same value, say i_α, for all sufficiently large $a \in B$, say all such $a \geq u_\alpha$. Since $\kappa < \mathbf{b}$, the κ functions f_α are all eventually majorized by a single function g. Increasing u_α if necessary, we can arrange that $g(a) \geq f_\alpha(a)$ for all $a \geq u_\alpha$. Finally, we construct the desired almost homogeneous infinite set $H \subseteq B$ by choosing its members inductively from B so that, if $a < b$ are in H, then $g(a) < b$. To see that H is almost homogeneous for each Π_α, suppose a and b are in H and $u_\alpha \leq a < b$. Then $f_\alpha(a) \leq g(a) < b$ and so $\Pi_\alpha\{a, b\} = v_\alpha(a) = i_\alpha$. $\square$

Attempting to dualize the preceding argument, to obtain $\mathbf{hom} = \max\{\mathbf{d}, \mathbf{r}\}$, we succeed only partially. To state the result that we obtain, we need the following variant of $\mathbf{r}$ introduced and studied by Vojtáš [24,25].

$$\begin{aligned} \mathbf{r}_\sigma = {} & \text{smallest cardinality of any } \mathcal{X} \subseteq [\omega]^\omega \\ & \text{such that, for any countably many sets } Y_n \in [\omega]^\omega,\ n \in \omega, \\ & \text{there is } X \in \mathcal{X} \text{ not split by any } Y_n. \end{aligned}$$

It is clear that $\mathbf{r}_\sigma$ is a uniform Π^0_3-characteristic and that $\mathbf{r}_\sigma \geq \mathbf{r}$. It is an open problem whether $\mathbf{r}_\sigma = \mathbf{r}$ (provably in ZFC). $\mathbf{r}_\sigma$ arises naturally in analysis as the characteristic associated to the Bolzano-Weierstrass theorem; it is the smallest cardinality of any $\mathcal{X} \subseteq [\omega]^\omega$ such that every bounded sequence $(x_n)_{n \in \omega}$ of real numbers has a convergent subsequence of the form $(x_n)_{n \in X}$ with $X \in \mathcal{X}$ [25].

Observe that the Π^0_3 relation determining $\mathbf{r}_\sigma$, namely "no term of the sequence coded by y splits x," is invariant (for reasonable coding), so it makes sense to consider the dual relation, defining the uniform Σ^0_3- (hence Π^0_2-)characteristic "the minimum number of ω-sequences of sets such that every infinite set is split by some term of one of these ω-sequences." But this is simply $\mathbf{s}$. Thus, $\mathbf{s}$ is dual to both $\mathbf{r}$ and $\mathbf{r}_\sigma$, a circumstance that helps to explain why the following dual of Theorem 16 looks weaker than one might expect.

Theorem 17. $\max\{\mathbf{r}, \mathbf{d}\} \leq \mathbf{hom} \leq \max\{\mathbf{r}_\sigma, \mathbf{d}\}$.

Proof. For the first inequality, fix a family $\mathcal{X}$ of $\mathbf{hom}$ infinite subsets H of ω, containing an almost homogeneous set for every $\Pi : [\omega]^\omega \to 2$. For any $S \subseteq \omega$, define Π_S as in the proof of Theorem 16, and observe that an H almost homogeneous for Π_S is not split by S. Thus, $\mathcal{X}$ is unsplittable and so $\mathbf{r} \leq \mathbf{hom}$. Similarly, given a non-decreasing $f \in {}^\omega\omega$, define Π_f as in the proof of Theorem 16, and observe that, if H is homogeneous for Π_f then, as in that proof,

$$g_H(n) = \text{the second element of } H \text{ after } n$$

defines a g_H eventually majorizing f. Thus $\{g_H \mid H \in \mathcal{X}\}$ is a dominating family and so $\mathbf{d} \leq \mathbf{hom}$.

For the second inequality, let $\kappa = \max\{\mathbf{r}_\sigma, \mathbf{d}\}$. Let $\mathcal{X}$ be a family of $\mathbf{r}_\sigma$ sets as in the definition of $\mathbf{r}_\sigma$. Inside each $X \in \mathcal{X}$, let $\mathcal{Y}(X)$ be an unsplittable family of $\mathbf{r}$ sets. Let $\mathcal{Y} = \bigcup_{X \in \mathcal{X}} \mathcal{Y}(X)$, and let $\mathcal{D} \subseteq {}^\omega\omega$ be a dominating family of cardinality $\mathbf{d}$. For each $Y \in \mathcal{Y}$ and $f \in \mathcal{D}$, let $Z = Z(Y, f)$ be an infinite subset of Y such that, if $a < b$ are in Z then $f(a) < b$. Since all of $\mathbf{r}$, $\mathbf{r}_\sigma$, and $\mathbf{d}$ are $\leq \kappa$, there are at most κ sets $Z(Y, f)$. We shall complete the proof by showing that every partition $\Pi : [\omega]^\omega \to 2$ has an almost homogeneous set among the $Z(Y, f)$'s.

So let Π be given, and consider the countably many sets

$$S_a = \{b \in \omega - \{a\} \mid \Pi\{a, b\} = 0\}$$

for $a \in \omega$. By choice of $\mathcal{X}$, find $X \in \mathcal{X}$ not split by any S_a. Thus, for each a, there are $v(a) \in \{0, 1\}$ and $g(a) \in \omega$ such that, whenever $b \in X$ and $b \geq g(a)$, then $\Pi\{a, b\} = v(a)$. By choice of $\mathcal{Y}(X)$, find $Y \in \mathcal{Y}(X) \subseteq \mathcal{Y}$ not split by $\{a \mid v(a) = 0\}$. Thus, there are $i \in \{0, 1\}$ and $u \in \omega$ such that, if $a \in Y$ and $a \geq u$, then $v(a) = i$. By choice of $\mathcal{D}$, find $f \in \mathcal{D}$ eventually majorizing g; increasing u if necessary, we may assume that $f(a) > g(a)$ for all $a \geq u$. Now if a and b are in $Z(Y, f)$ and $u < a < b$, then, $g(a) < f(a) < b$ (by definition of $Z(Y, f)$) and therefore $\Pi\{a, b\} = v(a) = i$. So $Z(Y, f)$ is almost homogeneous for Π, as required. $\square$

We generalize $\mathbf{par}$ and $\mathbf{hom}$ by considering partitions of $[\omega]^k$ instead of $[\omega]^2$. (One could also consider partitions into a larger (finite) number of pieces, but it is easy to check that this would not affect either characteristic.) Let $\mathbf{par}_k$ and $\mathbf{hom}_k$ be defined exactly like $\mathbf{par}$ and $\mathbf{hom}$ except that $[\omega]^2$ is replaced by $[\omega]^k$. Notice that $\mathbf{par}_1 = \mathbf{s}$ and $\mathbf{hom}_1 = \mathbf{r}$. Henceforth, we consider only $k \geq 2$. The proofs of Theorems 16 and 17 generalize easily to these higher values of k, but in fact one can say slightly more, as was pointed out to me by Laflamme who attributed the observation to Shelah.

Proposition 18. $\mathbf{par}_k = \min\{\mathbf{b}, \mathbf{s}\}$ *and* $\mathbf{hom}_k = \max\{\mathbf{r}_\sigma, \mathbf{d}\}$ *for* $k \geq 3$.

Proof. In view of the preceding remarks and the obvious fact that $\mathbf{hom}_k \leq \mathbf{hom}_l$ for $k \leq l$, all we need to prove is that $\mathbf{r}_\sigma \leq \mathbf{hom}_3$. Let $\mathcal{X}$ be a family of $\mathbf{hom}_3$ infinite subsets of ω, containing almost homogeneous sets for all partitions $\Pi : [\omega]^3 \to 2$. We claim that $\mathcal{X}$ is as required in the definition of $\mathbf{r}_\sigma$. Let countably many sets Y_n be given. Define $\Pi : [\omega]^3 \to 2$ by

$$\Pi\{a < b < c\} = 0 \iff (\forall n \leq a)\,(b \in Y_n \iff c \in Y_n),$$

and let $H \in \mathcal{X}$ be almost homogeneous for Π; deleting finitely many elements from H we get a homogeneous set H', and we complete the proof by showing that H' (and therefore also H) is not split by any Y_n. Let a be the smallest element of H'.

If Π maps $[H']^3$ to 1, then infinitely many sets

$$\{n \leq a \mid b \in Y_n\}$$

are all distinct, as b varies over $H' - \{a\}$, but they are all subsets of the finite set $\{0, 1, ..., a\}$, so this is absurd. Therefore Π maps $[H']^3$ to 0. This means that, for any n, Y_n contains all or none of those $b \in H'$ that are greater than the next element of H' after n. So Y_n does not split H'. □

Going beyond Ramsey's theorem, we can define analogous characteristics associated with the partition theorems of Nash-Williams [19], Galvin and Prikry [6], and Silver [22]. Little is known about these characteristics, but we list for reference some elementary facts. As we go from weaker to stronger partition theorems, the **hom** characteristics weakly increase and in particular are all $\geq \max\{\mathbf{r}_\sigma, \mathbf{d}\}$, and the **par** characteristics weakly decrease and in particular are all $\leq \min\{\mathbf{b}, \mathbf{s}\}$. A lower bound for the **par** characteristics is the distributivity number $\mathbf{h}$ defined as follows. Call a family $\mathcal{D} \subseteq [\omega]^\omega$ *dense open* if (a) every $X \in [\omega]^\omega$ has a subset in $\mathcal{D}$ and (b) if $X \subseteq^* Y$ and $Y \in \mathcal{D}$ then $X \in \mathcal{D}$. Then $\mathbf{h}$ is the smallest cardinal κ such that some κ dense open families have empty intersection. That $\mathbf{h} \leq$ the analogs of **par** associated to various partition theorems follows easily from the fact that those theorems ensure that, for any partition of the appropriate sort, the almost homogeneous sets form a dense open family. (Duality provides an upper bound for the characteristics analogous to **hom**, namely the smallest cardinality of a family $\mathcal{X}$ of sets that meets every dense family. Unfortunately, this cardinal equals $\mathbf{c}$, a trivial upper bound.)

It is clear that one can similarly associate characteristics with other partition theorems, for example the canonical partition theorem of Erdős and Rado [4] or the finite sum theorem of Hindman [10]. We shall discuss only two more analogs of **par** and two analogs of **hom**, associated to very weak partition theorems.

The first of these theorems can be viewed as the "canonical partition theorem for singletons," so we denote the analogs of **par** and **hom** with the subscript $1c$, but it's really just the (infinitary) pigeonhole principle: If $f : \omega \to \omega$, then there is an infinite $H \subseteq \omega$ on which f is constant or one-to-one. We define

$$\begin{aligned}\mathbf{par}_{1c} =& \text{smallest cardinality of any } \mathcal{X} \subseteq {}^\omega\omega \text{ such that there is no} \\ & H \in [\omega]^\omega \text{ such that each } f \in \mathcal{X} \\ & \text{is constant or one-to-one on a cofinite subset of } H,\end{aligned}$$

and

$$\begin{aligned}\mathbf{hom}_{1c} =& \text{smallest cardinality of any } \mathcal{X} \subseteq [\omega]^\omega \text{ such that every} \\ & f \in {}^\omega\omega \text{ is constant or one-to-one on some } H \in \mathcal{X}.\end{aligned}$$

As usual, $\mathbf{hom}_{1c}$ would be unaffected if we included "mod finite", and then it is clearly dual to $\mathbf{par}_{1c}$.

Proposition 19. *(a)* $\mathbf{par}_{1c} = \min\{\mathbf{b}, \mathbf{s}\}$.
(b) $\max\{\mathbf{r}, \mathbf{d}\} \leq \mathbf{hom}_{1c} \leq \max\{\mathbf{r}_\sigma, \mathbf{d}\}$.

Proof. Notice that f is one-to-one or constant on H if and only if H is homogeneous for the partition of $[\omega]^2$ that sends $\{a, b\}$ to 0 if and only if $f(a) = f(b)$. This immediately implies $\mathbf{par}_{1c} \geq \mathbf{par}$ and $\mathbf{hom}_{1c} \leq \mathbf{hom}$. In view of Theorems 16 and 17, we have half of each of (a) and (b). It remains to prove $\mathbf{par}_{1c} \leq \mathbf{b}, \mathbf{s}$ and $\mathbf{hom}_{1c} \geq \mathbf{r}, \mathbf{d}$. The parts pertaining to $\mathbf{s}$ and $\mathbf{r}$ follow from the observation that an infinite set H is not split by X if and only if the characteristic function of X is constant or one-to-one on a cofinite subset of H. (It can't be one-to-one.)

To prove $\mathbf{par}_{1c} \leq \mathbf{b}$, consider an arbitrary $\kappa < \mathbf{par}_{1c}$, and let a family $\mathcal{F}$ of κ functions $f \in {}^\omega\omega$ be given; we must find a single $g \in {}^\omega\omega$ eventually majorizing them all. For each $f \in \mathcal{F}$, partition ω into finite intervals $[a_0, a_1)$, $[a_1, a_2)$, etc., where $0 = a_0 < a_1 < a_2 < \dots$ and $a_{n+1} > f(n)$. Define $\hat{f} \in {}^\omega\omega$ by letting $\hat{f}(k) = n$ for all $k \in [a_n, a_{n+1})$. As $\kappa < \mathbf{par}_{1c}$, find an infinite $H \subseteq \omega$ such that each $\hat{f}$ is constant or one-to-one on a cofinite subset of H. Define $g \in {}^\omega\omega$ by letting $g(n)$ be the 2^nth element of H; we shall show that g eventually majorizes every $f \in \mathcal{F}$. Fix any $f \in \mathcal{F}$. As $\hat{f}$ is not constant on any infinite set, the defining property of H ensures that only finitely many of the intervals $[a_n, a_{n+1})$ associated to f meet H more than once. It follows that, for sufficiently large n, $g(n)$ is in an interval later than the nth, so $g(n) \geq a_{n+1} > f(n)$. This completes the proof that $\kappa < \mathbf{b}$ and therefore $\mathbf{par}_{1c} \leq \mathbf{b}$.

The proof that $\mathbf{hom}_{1c} \geq \mathbf{d}$ is quite similar. Let $\mathcal{X}$ be as in the definition of $\mathbf{hom}_{1c}$, and associate to each $H \in \mathcal{X}$ the function g defined as above, sending n to the 2^nth element of H. To see that these $\mathbf{hom}_{1c}$ functions g form a dominating family, consider any $f \in {}^\omega\omega$, define $\hat{f}$ as above, find $H \in \mathcal{X}$ such that $\hat{f}$ is one-to-one or constant on a cofinite subset of H, and argue as above that g eventually majorizes f. □

The last pair of partition characteristics that we discuss is defined like the pair $\mathbf{par}_{1c}$ and $\mathbf{hom}_{1c}$ except that "one-to-one" is weakened to "finite-to-one". We call these $\mathbf{par}_{1cf}$ and $\mathbf{hom}_{1cf}$, where f stands for "finite". Clearly $\mathbf{par}_{1cf} \geq \mathbf{par}_{1c}$ and $\mathbf{hom}_{1cf} \leq \mathbf{hom}_{1c}$.

Proposition 20. *(a)* $\mathbf{par}_{1cf} = \mathbf{s}$.
(b) $\mathbf{r} \le \mathbf{hom}_{1cf} \le \mathbf{r}_\sigma$.

Proof. The proofs that $\mathbf{par}_{1cf} \le \mathbf{s}$ and $\mathbf{hom}_{1cf} \ge \mathbf{r}$ are the same as the corresponding proofs for $\mathbf{par}_{1c}$ and $\mathbf{hom}_{1c}$; just replace "one-to-one" with "finite-to-one".

To see that $\mathbf{hom}_{1cf} \le \mathbf{r}_\sigma$, we let $\mathcal{X}$ be as in the definition of $\mathbf{r}_\sigma$ and we show that $\mathcal{X}$ also has the property required in the definition of $\mathbf{hom}_{1cf}$. Let $f \in {}^\omega\omega$, and let $Y_n = f^{-1}\{n\}$ for each $n \in \omega$. By hypothesis, $\mathcal{X}$ contains an infinite set H not split by any Y_n. If, for some n, H is almost included in Y_n, then f is constant with value n on a cofinite part of H. Otherwise, H is almost disjoint from every Y_n, and this means that f is finite-to-one on H.

The proof that $\mathbf{par}_{1cf} \ge \mathbf{s}$ is quite similar. We consider any $\kappa < \mathbf{s}$ and show that $\kappa < \mathbf{par}_{1cf}$. Let κ functions $f \in {}^\omega\omega$ be given. The $\kappa \cdot \aleph_0 < \mathbf{s}$ sets $f^{-1}\{n\}$, for the given f's and all $n \in \omega$, do not form a splitting family, so let H be an infinite set not split by any of them. For each of the given f's, the argument in the preceding paragraph shows that f is finite-to-one or constant on a cofinite subset of H. □

7. Questions

1. Among the familiar cardinal characteristics of the continuum [23], the distributivity number $\mathbf{h}$ and the closely related groupwise density number $\mathbf{g}$ do not seem to fit our definition of Γ-characteristics, because their definitions involve counting (dense or groupwise dense) families of reals rather than counting reals. Can one give equivalent definitions of $\mathbf{h}$ and $\mathbf{g}$ showing that they are (at least) OD$\mathbb{R}$-characteristics? (Of course a smaller Γ than OD$\mathbb{R}$ would be preferable.)

2. Many more of the familiar characteristics are Γ-characteristics for a reasonable Γ but are not known to be uniform Γ-characteristics for any Γ. Examples include $\mathbf{p}$, $\mathbf{t}$, $\mathbf{a}$, $\mathbf{i}$, and $\mathbf{u}$. Are any of these provably uniform OD$\mathbb{R}$-characteristics?

3. We saw in Section 2 that $\mathbf{add}(B)$ and $\mathbf{cof}(B)$ are uniform Π^0_2-characteristics. Duality suggests that it should be possible to replace Π^0_2 with Σ^0_2 and therefore with Π^0_1 for one of the two. Theorem 5 requires that one to be $\mathbf{cof}(B)$, since $\mathbf{add}(B)$ can consistently be $< \mathbf{cov}(B)$. Therefore, we ask: Is $\mathbf{cof}(B)$ a uniform Π^0_1-characteristic?

4. To what extent are the hypotheses about C in Theorem 9 needed for the theorem and not just for our proof? We remarked before stating the theorem that C has to be closed under limits of cofinality ω. If, as in our proof, each $\kappa \in C$ is the cardinality of a maximal almost disjoint family, then, by a result of Hechler [9, Thm 1], C must be closed under singular limits. But there might be proofs that don't rely on maximal almost-disjoint families and allow non-closed sets C. The requirement that C contain the immediate successors of all its members of cofinality ω cannot be deleted entirely, as $\max(C)$, which is to be $\mathbf{c}$ in the extension, had better not have cofinality ω. But one might be able to significantly weaken it. And the requirement that C contain all uncountable cardinals $\le |C|$ is purely a technical requirement for our proof.

5. Is $\mathbf{hom} = \mathbf{hom}_3$? One way to settle this (affirmatively) would be to prove $\mathbf{r} = \mathbf{r}_\sigma$, but $\mathbf{hom} = \mathbf{hom}_3$ might be easier to prove. (A meta-question: Clarify the

connection between this question and the "2 versus 3" problem in the theory of initial segments of models of arithmetic [13, p. 226].) A related question, bringing the $\mathbf{r}$ versus $\mathbf{r}_\sigma$ question to the forefront without the extra complication of $\max\{-, \mathbf{d}\}$, is whether either of the inequalities in Proposition 20(b) is reversible.

The referee has pointed out a similar problem concerning the cardinal $\mathbf{cov}(L)$. Define $\mathbf{cov}_\sigma(L)$ to be the minimum cardinality for a family $\mathcal{X}$ of measure zero sets of reals such that every countable set of reals is a subset of some $X \in \mathcal{X}$. Is $\mathbf{cov}_\sigma(L)$ equal to $\mathbf{cov}(L)$?

Appendix. Shelah's Proof of $\mathbf{d} \leq \mathbf{i}$

An infinite family $\mathcal{I} \subseteq [\omega]^\omega$ is said to be independent if, whenever $\mathcal{X}$ and $\mathcal{Y}$ are disjoint, finite subfamilies of $\mathcal{I}$, then the intersection

$$W(\mathcal{X}, \mathcal{Y}) = (\bigcap_{X \in \mathcal{X}} X) \cap (\bigcap_{Y \in \mathcal{Y}} (\omega - Y))$$

is infinite. (Most authors only require that $W(\mathcal{X}, \mathcal{Y})$ be nonempty, but when $\mathcal{I}$ is infinite this definition is equivalent to ours, and we don't wish to consider finite independent families.) The characteristic $\mathbf{i}$ is defined to be the smallest cardinality of any maximal independent family.

Theorem 21. (Shelah, [23, appendix]) $\mathbf{d} \leq \mathbf{i}$.

The following proof is based on the one in [23], but it avoids a few of the complications in that proof. Claude Laflamme has informed me that Bill Weiss has simplified Shelah's proof in a very similar way. We begin with a lemma that is essentially due to Ketonen [12, Prop. 1.3].

Lemma 22. *Let C_n be a decreasing sequence of infinite subsets of ω, and let $\mathcal{A}$ be a family of fewer than $\mathbf{d}$ subsets of ω such that each $A \in \mathcal{A}$ has infinite intersection with each C_n. Then there is a subset B of ω such that $B \subseteq^* C_n$ for all n and $A \cap B$ is infinite for all $A \in \mathcal{A}$.*

Proof. For any $h : \omega \to \omega$, let

$$B_h = \bigcup_{n \in \omega} (C_n \cap h(n)).$$

As the C_n form a decreasing sequence, it is clear that $B_h \subseteq^* C_n$ for all n. Our goal is to choose h so that $A \cap B$ is infinite for all $A \in \mathcal{A}$. Define, for $A \in \mathcal{A}$ and $n \in \omega$,

$$f_A(n) = \text{the } n\text{th element of } A \cap C_n.$$

Notice that, if $h(n) > f_A(n)$ for a particular A and n, then $|A \cap B_h| \geq n$, as $A \cap B_h \supseteq A \cap C_n \cap (f_A(n) + 1)$. So it suffices to choose h so that, for each $A \in \mathcal{A}$, infinitely many n satisfy $h(n) \geq f_A(n)$, i.e., $h \not\leq^* f_A$. As $|\mathcal{A}| < \mathbf{d}$, the functions f_A for $A \in \mathcal{A}$ cannot constitute a dominating family, so such an h exists. $\square$

Proof of Theorem 21. Suppose $\mathcal{I}$ is an independent family of cardinality smaller than $\mathbf{d}$. We shall show that $\mathcal{I}$ is not a maximal independent family. That is, we shall find $Z \subseteq \omega$ such that, whenever $\mathcal{X}$ and $\mathcal{Y}$ are disjoint finite subfamilies of $\mathcal{I}$, then both $W(\mathcal{X},\mathcal{Y}) \cap Z$ and $W(\mathcal{X},\mathcal{Y}) - Z$ are infinite, so $\mathcal{I} \cup \{Z\}$ is independent and properly includes $\mathcal{I}$.

Partition $\mathcal{I}$ as $\mathcal{D} \cup \mathcal{E}$, where $\mathcal{D} = \{D_n \mid n \in \omega\}$ is countably infinite and $\mathcal{E}$ has (like $\mathcal{I}$) cardinality smaller than $\mathbf{d}$. Write D_n^0 and D_n^1 for D_n and $\omega - D_n$, respectively. For each $x \in {}^\omega 2$, we apply Lemma 22 with

$$C_n = \bigcap_{k<n} D_k^{x(k)}$$

and

$$\mathcal{A} = \{W(\mathcal{X},\mathcal{Y}) \mid \mathcal{X},\mathcal{Y} \text{ finite disjoint subfamilies of } \mathcal{E}\}.$$

The hypothesis of the lemma is satisfied because $\mathcal{I}$ is independent. So we obtain B_x such that

(1) $B_x \subseteq^* \bigcap_{k<n} D_k^{x(k)}$ for all n, and

(2) $B_x \cap W(\mathcal{X},\mathcal{Y})$ is infinite for every $W(\mathcal{X},\mathcal{Y}) \in \mathcal{A}$.

Notice that, by (1),

(3) $B_x \cap B_y$ is finite when $x \neq y$.

Fix two disjoint, countable, dense (in the usual product topology) subsets Q and Q' of ${}^\omega 2$. We can remove finitely many elements from B_x, for $x \in Q \cup Q'$, so that

(3*) $B_x \cap B_y = \emptyset$ for $x \neq y$ in $Q \cup Q'$.

(To see this, we use the countability of $Q \cup Q'$ to list the relevant B's in an ω-sequence, and remove from each one its (finite, by (3)) intersections with its (finitely many) predecessors in the list.) Notice that (1) and (2) remain true.

Now set

$$Z = \bigcup_{x \in Q} B_x \quad \text{and} \quad Z' = \bigcup_{x \in Q'} B_x,$$

so Z and Z' are disjoint, by (3*). We shall show that, for any finite disjoint $\mathcal{X},\mathcal{Y} \subseteq \mathcal{I}$, the intersection $W(\mathcal{X},\mathcal{Y}) \cap Z$ is infinite. The same reasoning with Q' in place of Q will yield that $W(\mathcal{X},\mathcal{Y}) \cap Z'$ is infinite, and therefore so is $W(\mathcal{X},\mathcal{Y}) - Z$, which will complete the proof.

Let finite disjoint $\mathcal{X},\mathcal{Y} \subseteq \mathcal{I}$ be given. As Q is dense in ${}^\omega 2$, it contains an x such that, if $D_k \in \mathcal{X}$ (resp. $\mathcal{Y}$), then $x(k) = 0$ (resp. 1), so $D_k^{x(k)} = D$ (resp. $\omega - D$). Thus,

$$\begin{aligned}
W(\mathcal{X},\mathcal{Y}) &= W(\mathcal{X}\cap\mathcal{E},\mathcal{Y}\cap\mathcal{E}) \cap W(\mathcal{X}\cap\mathcal{D},\mathcal{Y}\cap\mathcal{D}) \\
&= W(\mathcal{X}\cap\mathcal{E},\mathcal{Y}\cap\mathcal{E}) \cap \bigcap_{k:D_k\in\mathcal{X}\cup\mathcal{Y}} D_k^{x(k)} \\
&\supseteq W(\mathcal{X}\cap\mathcal{E},\mathcal{Y}\cap\mathcal{E}) \cap \bigcap_{k<n} D_k^{x(k)}, \\
&\qquad \text{for sufficiently large } n, \\
&\supseteq^* W(\mathcal{X}\cap\mathcal{E},\mathcal{Y}\cap\mathcal{E}) \cap B_x, \\
&\qquad \text{by (1).}
\end{aligned}$$

As $W(\mathcal{X} \cap \mathcal{E}, \mathcal{Y} \cap \mathcal{E}) \in \mathcal{A}$, we have by (2) that its intersection with B_x is infinite. We have just seen that this infinite set is almost included in $W(\mathcal{X}, \mathcal{Y})$, and it is also included in Z because $x \in Q$ implies $B_x \subseteq Z$. So $W(\mathcal{X}, \mathcal{Y}) \cap Z$ is infinite, as required. □

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