Special Families of Sets and Baer-Specker Groups

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Abstract

We prove that the Baer-Specker group $\Pi = \mathbb{Z}^{\aleph_0}$ contains a pure subgroup isomorphic to the direct sum of $2^{\aleph_0}$ copies of itself. We produce $2^{2^{\aleph_0}}$ non-isomorphic subgroups of Π, each isomorphic to its dual group. Finally, we show that the isomorphism type of a generalized product of $\mathbb{Z}$'s, the set of functions $I \to \mathbb{Z}$ with support of size at most α, uniquely determines both the cardinality of I and α (as long as there are no measurable cardinals $\leq \alpha$). All three of these results are obtained using set-theoretic existence theorems, namely the existence of large independent families, large almost disjoint families, and Δ-systems.

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1 Introduction

We establish in this paper three results that share two common features. First, they all involve the Baer-Specker group $\mathbb{Z}^{\aleph_0}$ or closely related groups. Second, their proofs all involve classical results from combinatorial set theory that assert the existence of some special families of sets.

Specifically, we shall use the existence of large independent families to show that the direct sum of $2^{\aleph_0}$ copies of the Baer-Specker group can be embedded as a pure subgroup of a single copy. We shall use the existence of large almost disjoint families to show that the Baer-Specker group has many subgroups that are isomorphic to their dual groups. Finally, we shall use the existence of Δ-systems to show that the isomorphism type of a group of the form

$$\{x \in \mathbb{Z}^\kappa : |\{\xi \in \kappa : x(\xi) \neq 0\}| \leq \alpha\}$$

determines the cardinals $\alpha \leq \kappa$ uniquely, at least as long as there are no measurable cardinals below α.

Apart from this introduction, the paper consists of a section presenting the necessary background, both from abelian group theory and from set theory, followed by three sections, each devoted to the proof of one of the three results just stated. Those three sections can be read independently of each other.

2 Background

2.1 Abelian Group Theory

All groups considered in this paper are abelian, so "free" means "free abelian." We write $\mathbb{Z}$ for the additive group of integers and $\mathbb{Z}^I$, where I is an arbitrary index set, for the direct product, i.e., the group of functions from I to $\mathbb{Z}$ with the operation of componentwise addition. Usually, I will be an infinite cardinal number κ, which we identify, as is customary in set theory, with the initial ordinal number of cardinality κ. In particular, the Baer-Specker group $\mathbb{Z}^{\aleph_0}$ is the group of all ordinary (i.e., indexed by the set $\mathbb{N}$ of natural numbers) infinite sequences of integers. Since we shall need to refer to it so often, we introduce a short notation for it.

Definition 1 $\Pi = \mathbb{Z}^{\aleph_0}$.

We use the notation e_i ($i \in I$) for the "standard unit vectors" in $\mathbb{Z}^I$, namely the functions defined by $e_i(j) = 1$ if $j = i$ and 0 otherwise.

We shall need the following theorem of Specker [8] (for $\kappa = \aleph_0$) and Łoś (for larger κ); see [4, Theorem 94.4].

Theorem 2 *If $h : \mathbb{Z}^I \to \mathbb{Z}$ is a homomorphism, then $h(e_i) = 0$ for all but finitely many $i \in I$. If, in addition, there are no measurable cardinals $\leq |I|$, then h is completely determined by the values of the $h(e_i)$'s.*

For readers unfamiliar with measurable cardinals, we remark that they are extremely large. Thus, the theorem applies to all the cardinals that one would ordinarily meet, including $\aleph_0$ and the cardinal $2^{\aleph_0}$ of the continuum. It is consistent with the usual axioms of set theory (the Zermelo-Fraenkel axioms including the axiom of choice) that there are no measurable cardinals. We shall also need the fact that no successor cardinals are measurable.

The unit vectors e_i are obviously linearly independent, so they generate a free subgroup $\bigoplus^I \mathbb{Z}$ of $\mathbb{Z}^I$. In contrast, it follows from Theorem 2 that $\mathbb{Z}^I$ is not free unless I is finite. (When I is countably infinite, the theorem implies that $\mathbb{Z}^I$ has too few homomorphisms to $\mathbb{Z}$ to be free; for larger I, possibly even larger than a measurable cardinal, the product $\mathbb{Z}^I$ cannot be free because it includes a copy of $\mathbb{Z}^{\aleph_0}$.)

Definition 3 The *support* of an element $x \in \mathbb{Z}^I$ is the set

$$\mathrm{supp}(x) = \{i \in I : x(i) \neq 0\}.$$

If $\alpha \leq \kappa$ are infinite cardinals, then we define

$$\Pi(\kappa, \alpha) = \{x \in \mathbb{Z}^\kappa : |\mathrm{supp}(x)| \leq \alpha\}$$

Thus, for example, $\Pi(\kappa, \kappa) = \mathbb{Z}^\kappa$, and $\bigoplus^I \mathbb{Z}$ consists of the members of $\mathbb{Z}^I$ whose supports are finite.

We shall use the "inner product" notation

$$\langle x, y \rangle = \sum_{i \in I} x(i)y(i)$$

for $x, y \in \mathbb{Z}^I$, but of course the sum makes sense only if all but finitely many of the summands are zero. That is, $\langle x, y \rangle$ is defined if and only if $\mathrm{supp}(x) \cap \mathrm{supp}(y)$ is finite. In particular, the inner product always makes sense if one of the factors is in $\bigoplus^I \mathbb{Z}$.

Definition 4 The *dual* of an abelian group G is the group $G^* = \mathrm{Hom}(G, \mathbb{Z})$ of homomorphisms from G to $\mathbb{Z}$. The group operation in G^* is pointwise addition.

Because $\bigoplus^I \mathbb{Z}$ is free on the generators e_i, it is easy to see that its dual can be identified with $\mathbb{Z}^I$ via the inner product pairing. That is, each element $x \in \mathbb{Z}^I$ defines a homomorphism $\bigoplus^I \mathbb{Z} \to \mathbb{Z} : y \mapsto \langle x, y\rangle$, and every member of $\left(\bigoplus^I \mathbb{Z}\right)^*$ arises in this way from a unique x.

The Specker-Łoś Theorem 2 implies that, as long as there are no measurable cardinals $\leq |I|$, the dual of $\mathbb{Z}^I$ can be identified with $\bigoplus^I \mathbb{Z}$ via the same inner product pairing.

So far, we have used the notation $\bigoplus$ only in connection with $\mathbb{Z}$, but we shall also use it more generally, writing $\bigoplus^I G$ for the direct sum of copies of G indexed by I, that is, the set of functions $I \to G$ with finite supports.

2.2 Set Theory

Since we often have to refer to the cardinality $2^{\aleph_0}$ of the continuum, we use the (fairly standard) notation $\mathfrak{c}$ for it. We also use the standard notation κ^+ for the next infinite cardinal after κ.

We shall need three well-known existence theorems from set theory. They can be found in the textbook [6, Lemmas 24.8, 23.9, and 22.6]. (Theorem 10 is stated there only for countable κ, but the proof generalizes to arbitrary infinite κ.)

Definition 5 Let A and B any sets. A family $\mathcal{F}$ of functions from A to B is *independent* if, whenever a finite list of elements $f_1, \ldots, f_n$ of $\mathcal{F}$ and an equally long list of elements $b_1, \ldots, b_n$ of B are given, there is at least one $a \in A$ that simultaneously satisfies the equations $f_1(a) = b_1, \ldots, f_n(a) = b_n$.

The following result was first proved in [3]; it was extended to larger cardinals in [5] but we shall not need the extension here.

Theorem 6 *If A and B are countably infinite then there is an independent family of $\mathfrak{c}$ functions $A \to B$.*

Definition 7 Two sets are said to be *almost disjoint* if their intersection is finite. An *almost disjoint family* is a family of infinite sets every two of whose members are almost disjoint.

Theorem 8 *For any countably infinite set A, there is an almost disjoint family of $\mathfrak{c}$ subsets of A.*

This theorem is folklore, but its proof is so short that we cannot resist giving it. It obviously doesn't matter which countably infinite set A we consider, so take $A = \mathbb{Q}$, the set of rational numbers. Associate to each real number r some sequence of distinct rational numbers converging to r. These sequences, considered simply as sets of rational numbers, are obviously pairwise almost disjoint.

Definition 9 A family $\mathcal{D}$ of sets is a Δ-*system* if there is a single set R, called the *root* of the Δ-system, such that every two distinct members of $\mathcal{D}$ have intersection exactly R.

Theorem 10 *Let κ be an infinite cardinal number, and let $\mathcal{S}$ be a family of κ^+ finite sets. Then $\mathcal{S}$ has a subfamily $\mathcal{D}$, also of cardinality κ^+, that is a Δ-system.*

3 Embedding a Sum of Π's in One Π

Theorem 11 *There is a pure subgroup of Π that is isomorphic to $\bigoplus^{\mathfrak{c}} \Pi$.*

Proof Apply Theorem 6 with $A = \mathbb{N}$ and $B = \Pi^*$. Remember that B can be identified with $\bigoplus^{\aleph_0} \mathbb{Z}$, so it is countable and the theorem applies. Let $\mathcal{F}$ be an independent family of $\mathfrak{c}$ functions $\mathbb{N} \to \Pi^*$. We shall produce an embedding

$$j : \bigoplus^{\mathcal{F}} \Pi \to \Pi,$$

and we shall show that its range is a pure subgroup of Π.

Recall that an element x of $\bigoplus^{\mathcal{F}} \Pi$ is an $\mathcal{F}$-indexed family (equivalently, a function on $\mathcal{F}$) $x = (x_f)_{f \in \mathcal{F}}$ such that $x_f \in \Pi$ for all f and $x_f = 0$ for all but finitely many f. We define

$$j(x) = \left(\sum_{f \in \mathcal{F}} f(n)(x_f) \right)_{n \in \mathbb{N}} .$$

To decipher this, notice first that, since each $f \in \mathcal{F}$ is a function $\mathbb{N} \to \Pi^*$, each $f(n)$ is an element of Π^*, i.e., a homomorphism from Π to $\mathbb{Z}$. So $f(n)$

can be applied to the element x_f of Π to produce an element $f(n)(x_f) \in \mathbb{Z}$. So the sum in our formula for $j(x)$ is a sum of integers. Although there are $\mathfrak{c}$ summands, one for each $f \in \mathcal{F}$, only finitely many are non-zero, because $x_f = 0$ for all but finitely many $f \in \mathcal{F}$. So the sum is a well-defined integer. Forming this sum for each $n \in \mathbb{N}$, we get a sequence of integers, i.e., an element of Π. This is the element that is defined as $j(x)$.

It is clear that j is a homomorphism, because all the $f(n)$ are homomorphisms. We must check that j is one-to-one and that its range is a pure subgroup of Π. Both of these facts are immediate consequences of the following claim: If $x = (x_f)_{f\in\mathcal{F}} \in \bigoplus^{\mathcal{F}} \Pi$ and if d is the greatest common divisor of all the components $x_{f,n}$ (in $\mathbb{Z}$) of all the components x_f (in Π) of x, then d is one of the components of $j(x)$.

Indeed, if the claim is true, then for any non-zero x we shall have a non-zero d and thus a non-zero $j(x)$, so j will be an embedding. Furthermore, if $j(x)$ is divisible in Π by some integer m, then in particular d is divisible by m, and so all components $x_{f,n}$ are divisible by m. But then x is divisible by m in $\bigoplus^{\mathcal{F}} \Pi$. So the range of j is pure in Π.

It remains to prove the claim, so let x and d be as there. List as $f_1, \dots, f_r$ the finitely many $f \in \mathcal{F}$ for which $x_f \neq 0$. By elementary number theory, d is a linear combination, with integer coefficients, of finitely many components of these x_{f_i}'s. So we can write d as the sum of the inner products of these x_{f_i}'s and certain elements of $\bigoplus^{\mathbb{N}} \mathbb{Z}$. Equivalently, we can write d as

$$d = \sum_{i=1}^{r} s_i(x_{f_i})$$

for certain $s_i \in \Pi^*$. By the independence of the family $\mathcal{F}$, there is some $n \in \mathbb{N}$ such that $f_i(n) = s_i$ for each $i = 1, \dots, r$. Then, using the definition of j and remembering that $x_f = 0$ for all f except $f_1, \dots, f_r$, we find that the n^{th} component of $j(x)$ is

$$\sum_{f\in\mathcal{F}} f(n)(x_f) = \sum_{i=1}^{r} f_i(n)(x_{f_i}) = \sum_{i=1}^{r} s_i(x_{f_i}) = d,$$

as required. This completes the proof of the claim and therefore also the proof of the theorem. $\square$

4 Groups Associated to Ideals of Sets

Since $\Pi = \mathbb{Z}^{\aleph_0}$ and $\bigoplus^{\aleph_0} \mathbb{Z}$ are each isomorphic to the other's dual, and since dualization preserves direct sums, $\Pi \oplus \bigoplus^{\aleph_0} \mathbb{Z}$ is isomorphic to its own dual. Furthermore, it is isomorphic to a subgroup of Π, for example the subgroup of those $x \in \Pi$ such that only finitely many even-numbered components x_{2n} are non-zero.

Some time ago, the second author asked whether there are any essentially different examples of (pure) subgroups of Π that are isomorphic to their own duals. The primary purpose of this section is to give a strong affirmative answer: There are $2^{\mathfrak{c}}$ such subgroups. Along the way, we shall obtain additional results about a fairly natural class of subgroups of Π, namely those associated to ideals of subsets of $\mathbb{N}$.

We have been informed that Greg Schlitt also found, at approximately the same time, examples of subgroups of Π isomorphic to their own duals, but we have not seen his construction, and we do not know whether his examples are the same as ours.

Definition 12 A *non-trivial ideal of subsets of* $\mathbb{N}$, or simply an *ideal* (since no other sorts of ideals will occur in this paper), is a family $\mathcal{I}$ of subsets of $\mathbb{N}$ with the following properties:

- Every finite subset of $\mathbb{N}$ is a member of $\mathcal{I}$.
- If $X \in \mathcal{I}$ and $Y \subseteq X$ then $Y \in \mathcal{I}$.
- If $X, Y \in \mathcal{I}$ then $X \cup Y \in \mathcal{I}$.

As an immediate consequence of the definition, if two subsets of $\mathbb{N}$ differ by only finitely many elements, then both are in $\mathcal{I}$ or neither is in $\mathcal{I}$.

We shall need two constructions starting with an ideal. One constructs a group and the other constructs another ideal.

Definition 13 Let $\mathcal{I}$ be an ideal. Then the associated subgroup of Π is defined to be

$$P(\mathcal{I}) = \{x \in \Pi : \mathrm{supp}(x) \in \mathcal{I}\}.$$

Since $\mathrm{supp}(x - y) \subseteq \mathrm{supp}(x) \cup \mathrm{supp}(y)$, the closure properties of $\mathcal{I}$ immediately imply that $P(\mathcal{I})$ is a subgroup of Π. In fact, it is clearly a pure subgroup of Π. Furthermore, it includes the subgroup $\bigoplus^{\aleph_0}\mathbb{Z}$ because $\mathcal{I}$ contains all finite sets.

Distinct ideals have distinct associated groups. Indeed, if a set $X \subseteq \mathbb{N}$ is in one of the ideals but not in the other, then any $x \in \Pi$ with support exactly X is in one of the associated groups but not in the other.

The smallest ideal consists of just the finite subsets of $\mathbb{N}$; its associated group is $\bigoplus^{\aleph_0}\mathbb{Z}$. The largest ideal consists of all subsets of $\mathbb{N}$; its associated group is Π.

Definition 14 Let $\mathcal{A}$ be any family of subsets of $\mathbb{N}$. The *orthogonal* or *dual ideal* is defined to be

$$\mathcal{A}^{\perp} = \{Y \subseteq \mathbb{N} : Y \cap X \text{ is finite for all } X \in \mathcal{A}\}.$$

It is trivial to check that $\mathcal{A}^{\perp}$ is always an ideal, whether or not $\mathcal{A}$ is one.

We shall have to deal with double dualization; the following lemma gives an alternative formulation of that notion.

Lemma 15 *For any family $\mathcal{A}$ of subsets of $\mathbb{N}$,*

$$\mathcal{A}^{\perp\perp} = \{Z \subseteq \mathbb{N} : (\forall \textit{ infinite } Y \subseteq Z)(\exists X \in \mathcal{A})\, X \cap Y \textit{ is infinite}\}.$$

Proof This is essentially just a matter of chasing through the definitions. A set Z is *not* in $\mathcal{A}^{\perp\perp}$ if and only if it has infinite intersection with some set in $\mathcal{A}^{\perp}$. Equivalently, since $\mathcal{A}^{\perp}$ is closed under subsets, Z has an infinite subset $Y \in \mathcal{A}^{\perp}$. In other words, Z has an infinite subset Y whose intersection with every $X \in \mathcal{A}$ is finite. But this is precisely the negation of the criterion offered in the lemma. □

Definition 16 $\mathrm{Idl}(\mathcal{A})$ is the ideal generated by $\mathcal{A}$, i.e., the smallest ideal that includes $\mathcal{A}$. It consists of all sets that are covered by finitely many sets from $\mathcal{A}$ and a finite set.

The following lemma collects some elementary properties of the notions just introduced.

Lemma 17 *For any families $\mathcal{A}$ and $\mathcal{B}$ of subsets of $\mathbb{N}$, we have:*

- $\mathcal{A} \subseteq \mathcal{B}^{\perp}$ *if and only if* $\mathcal{B} \subseteq \mathcal{A}^{\perp}$.

- *If* $\mathcal{A} \subseteq \mathcal{B}$ *then* $\mathcal{B}^\perp \subseteq \mathcal{A}^\perp$.
- $\mathcal{A} \subseteq \mathcal{A}^{\perp\perp}$.
- $\mathcal{A}^{\perp\perp\perp} = \mathcal{A}^\perp$.
- $\mathcal{A}$ *is of the form* $\mathcal{B}^\perp$ *for some* $\mathcal{B}$ *if and only if* $\mathcal{A}^{\perp\perp} = \mathcal{A}$.
- $\mathit{Idl}(\mathcal{A}) \subseteq \mathcal{A}^{\perp\perp}$.
- $(\mathit{Idl}(\mathcal{A}))^\perp = \mathcal{A}^\perp$.

Proof The first statement is clear beause both sides of the alleged equivalence say that each set in $\mathcal{A}$ is almost disjoint from each set in $\mathcal{B}$. The next four items follow from the first by general facts about Galois correspondences, but they are also easy to prove directly. The sixth item follows from the third, since $\mathrm{Idl}(\mathcal{A})$ is the smallest ideal that includes $\mathcal{A}$. Finally, the $\subseteq$ half of the last item follows from the second, while the $\supseteq$ half follows by combining the second, fourth, and sixth. $\square$

The next result tells us what orthogonality of ideals has to do with the associated groups.

Proposition 18 *Let* $\mathcal{I}$ *be an ideal. Then the dual* $P(\mathcal{I})^*$ *of its associated group and the group* $P(\mathcal{I}^\perp)$ *associated to its dual ideal are isomorphic, via the inner product pairing defined in Section 2.*

Proof First observe that, if $x \in P(\mathcal{I}^\perp)$ and $y \in P(\mathcal{I})$, then their supports, being in $\mathcal{I}^\perp$ and $\mathcal{I}$, respectively, have finite intersection, so the inner product $\langle x, y\rangle$ is defined. It is obviously linear in both x and y, so each $x \in P(\mathcal{I}^\perp)$ gives a homomorphism $\langle x, -\rangle : P(\mathcal{I}) \to \mathbb{Z}$, i.e., an element of $P(\mathcal{I})^*$, and this assignment $x \mapsto \langle x, -\rangle$ is a homomorphism from $P(\mathcal{I}^\perp)$ to $P(\mathcal{I})^*$. What remains to be shown is that this homomorphism is an isomorphism, i.e., that every homomorphism $h : P(\mathcal{I}) \to \mathbb{Z}$ is of the form $\langle x, -\rangle$ for a unique $x \in P(\mathcal{I}^\perp)$.

Uniqueness is easy, for if x works then its n^{th} component must be $x_n = \langle x, e_n\rangle = h(e_n)$. (Note that $e_n \in P(\mathcal{I})$ as $\mathcal{I}$ contains all finite sets.) So we know what x should be. We must check that $x \in P(\mathcal{I}^\perp)$ and that $\langle x, -\rangle$ agrees with h on all of $P(\mathcal{I})$, not just on the e_n's.

For the first of these checks, we must consider an arbitrary set $A \in \mathcal{I}$ and show that $A \cap \mathrm{supp}(x)$ is finite, i.e., that $h(e_n) = 0$ for all but finitely many

$n \in A$. This is trivial if A is finite, so assume A is infinite. Since $A \in \mathcal{I}$, $P(\mathcal{I})$ includes all the functions $y \in \Pi$ whose support is included in A. These functions form a subgroup G of Π that is isomorphic to Π itself. Indeed, an isomorphism is induced by any bijection between $\mathbb{N}$ and A, and under such an isomorphism the unit vectors e_n ($n \in \mathbb{N}$) of Π correspond to the unit vectors e_n for $n \in A$. Applying Specker's Theorem 2 to the restriction of h to G, we find that h maps all but finitely many of the e_n ($n \in A$) to 0, as required.

Finally, we must check that h and $\langle x, - \rangle$, which are known to agree on all the e_n ($n \in \mathbb{N}$), actually agree on all of $P(\mathcal{I})$. But $P(\mathcal{I})$ is, by definition, the union of subgroups of the form considered in the preceding paragraph, namely

$$G = \{x \in \Pi : \mathrm{supp}(x) \subseteq A\}$$

for $A \in \mathcal{I}$. So it suffices to show that h and $\langle x, - \rangle$ agree on each such G. But this follows from Specker's Theorem, since G is isomorphic to Π and h agrees with $\langle x, - \rangle$ on the images, under the isomorphism, of the standard unit vectors. □

Corollary 19 *Let $\mathcal{I}$ be an ideal. The associated group $P(\mathcal{I})$ is reflexive if and only if $\mathcal{I} = \mathcal{I}^{\perp\perp}$.*

Proof Applying the proposition twice, we find that the double dual of $P(\mathcal{I})$ can be identified with $P(\mathcal{I}^{\perp\perp})$. Furthermore, because of the simple form of the pairings, the canonical homomorphism of $P(\mathcal{I})$ into its double dual corresponds, under this identification, with the inclusion map of $P(\mathcal{I})$ into $P(\mathcal{I}^{\perp\perp})$. So this homomorphism is an isomorphism, i.e., $P(\mathcal{I})$ is reflexive, if and only if this inclusion is an equality. But we have seen earlier that this happens if and only if the ideals $\mathcal{I}$ and $\mathcal{I}^{\perp\perp}$ are equal. □

A convenient way to get dual ideals, and thus dual groups, is to split a maximal almost disjoint family, as follows. We know, by Theorem 8, that there is an almost disjoint family of $\mathfrak{c}$ subsets of $\mathbb{N}$. By Zorn's Lemma, we can extend this to a maximal almost disjoint family $\mathcal{M}$. (Later, we shall want an additional condition on $\mathcal{M}$, but for now any infinite maximal almost disjoint family will do.) Partition $\mathcal{M}$ into two subfamilies $\mathcal{A}$ and $\mathcal{B}$.

Lemma 20 *In the situation just described, $\mathcal{A}^{\perp} = \mathcal{B}^{\perp\perp}$ and $\mathcal{B}^{\perp} = \mathcal{A}^{\perp\perp}$.*

Proof Since $\mathcal{M}$ is an almost disjoint family, every member of $\mathcal{A}$ is in $\mathcal{B}^\perp$. By Lemma 17, $\mathcal{A}^\perp \supseteq \mathcal{B}^{\perp\perp}$.

For the converse inclusion, consider any $Z \in \mathcal{A}^\perp$. To show that $Z \in \mathcal{B}^{\perp\perp}$, we use Lemma 15. So consider any infinite $Y \subseteq Z$. If $Y \notin \mathcal{M}$ then, by maximality of $\mathcal{M}$ among almost disjoint families, Y must have infinte intersection with some $X \in \mathcal{M}$. If, on the other hand, $Y \in \mathcal{M}$, then the same conclusion holds with $X = Y$. So, in either case, Y has infinite intersection with some set $X \in \mathcal{M}$. This X cannot be in $\mathcal{A}$, because $Y \subseteq Z \in \mathcal{A}^\perp$. So $X \in \mathcal{B}$, and the proof of the converse inclusion is complete.

This establishes the first equation in the lemma; the second holds by symmetry. □

Corollary 21 *In the situation described above, each of the groups* $P(\mathcal{A}^\perp) = P(\mathcal{B}^{\perp\perp})$ *and* $P(\mathcal{B}^\perp) = P(\mathcal{A}^{\perp\perp})$ *is isomorphic to the dual of the other, via the inner product pairing.*

Proof Combine the lemma with Proposition 18. □

Finally, we are ready to construct the groups that answer the question at the beginning of this section.

Theorem 22 *There are* $2^\mathfrak{c}$ *pairwise non-isomorphic, pure subgroups of* Π*, each of which is isomorphic to its dual.*

Proof We shall construct the required groups as $P(\mathcal{I})$ for suitably chosen ideals $\mathcal{I}$. To construct these ideals, first partition $\mathbb{N}$ into two infinite sets P and Q, and let π be a permutation of $\mathbb{N}$ that interchanges P and Q and satisfies $\pi^2 = \mathrm{id}$. So π consists of a bijection from P to Q together with its inverse. For example, take P to consist of the even numbers and Q of the odd ones, and let $\pi(n)$ be $n+1$ for even n and $n-1$ for odd n.

Next, use Theorem 8 to obtain a family of $\mathfrak{c}$ almost disjoint subsets of P, and use Zorn's Lemma to enlarge it to a maximal family $\mathcal{P}$ of almost disjoint subsets of P. Let $\mathcal{Q}$ be the image of $\mathcal{P}$ under π; it is of course a maximal family of $\mathfrak{c}$ almost disjoint subsets of Q. It follows immediately that $\mathcal{P} \cup \mathcal{Q}$ is a maximal almost disjoint family of subsets of $\mathbb{N}$.

Arbitrarily partition $\mathcal{P}$ into two subfamilies $\mathcal{X}$ and $\mathcal{Y}$; then of course $\mathcal{Q}$ is correspondingly partitioned into $\pi(\mathcal{X})$ and $\pi(\mathcal{Y})$. Note for future reference that, as $|\mathcal{P}| = \mathfrak{c}$, there are $2^\mathfrak{c}$ ways to do this partitioning; for the time being we concentrate on one fixed partition.

Let $\mathcal{A} = \mathcal{X} \cup \pi(\mathcal{Y})$ and $\mathcal{B} = \pi(\mathcal{X}) \cup \mathcal{Y}$. Notice that $\mathcal{A}$ and $\mathcal{B}$ constitute a partition of the maximal almost disjoint family $\mathcal{P} \cup \mathcal{Q}$, so the preceding lemma and its corollary apply. That is, each of $P(\mathcal{A}^\perp)$ and $P(\mathcal{B}^\perp)$ is isomorphic, via the inner product pairing, to the dual of the other. But $P(\mathcal{A}^\perp)$ and $P(\mathcal{B}^\perp)$ are also isomorphic to each other, because the permutation π of $\mathbb{N}$ interchanges $\mathcal{A}$ and $\mathcal{B}$. Therefore, $P(\mathcal{A}^\perp)$ is isomorphic to its own dual. (The pairing here is $\langle x, y^\pi \rangle$, where y^π is defined by $(y^\pi)_n = y_{\pi(n)}$.)

Thus, we have obtained a self-dual, pure subgroup $P(\mathcal{A}^\perp)$ of Π. Notice that, for any set $S \in \mathcal{P}$, an element of Π with support S will be in $P(\mathcal{A}^\perp)$ if and only if $S \in \mathcal{Y}$. Thus, the partition $\{\mathcal{X}, \mathcal{Y}\}$ can be recovered from the resulting group $P(\mathcal{A}^\perp)$. That is, distinct partitions produce distinct groups. Since there are $2^{\mathfrak{c}}$ partitions of $\mathcal{P}$ into two pieces, we have $2^{\mathfrak{c}}$ distinct self-dual, pure subgroups of Π.

But the theorem claims $2^{\mathfrak{c}}$ subgroups that are not merely distinct but non-isomorphic. Some of our groups $P(\mathcal{A}^\perp)$ may be isomorphic, so a little more work is needed.

Concentrate for the moment on one group of the form $P(\mathcal{I})$ for some ideal $\mathcal{I}$ (for example a group of the form $P(\mathcal{A}^\perp)$). We know that its dual group is isomorphic to a subgroup $P(\mathcal{I}^\perp)$ of Π, so this dual group has cardinality at most $|\Pi| = \mathfrak{c}$. That is, $P(\mathcal{I})$ admits at most $\mathfrak{c}$ homomorphisms to $\mathbb{Z}$. Now a homomorphism h to Π is determined by countably many homomorphisms to $\mathbb{Z}$, namely the compositions of h with the coordinate projections $\Pi \to \mathbb{Z}$. So the number of homomorphisms from $P(\mathcal{I})$ to Π is at most $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$. In particular, there are at most $\mathfrak{c}$ subgroups of Π that are isomorphic to $P(\mathcal{I})$.

Therefore, if we take the collection of $2^{\mathfrak{c}}$ groups $P(\mathcal{A}^\perp)$ constructed above and if we split it into isomorphism classes, then each isomorphism class has cardinality at most $\mathfrak{c}$. Since $2^{\mathfrak{c}} > \mathfrak{c}$, it follows that there are at least $2^{\mathfrak{c}}$ isomorphism classes. $\square$

5 Embeddings and Isomorphisms of Partial Products

It was pointed out in [1] that $\mathbb{Z}^\lambda$ cannot be embedded as a subgroup of $\mathbb{Z}^\kappa$ for $\kappa < \lambda$, even if, as can happen in some models of set theory, these groups have the same cardinality. Here we extend this result to certain subgroups obtained by restricting the cardinality of supports, at least as long as no measurable cardinals interfere. (The situation in the presence of measurable cardinals will be treated in [2].) Recall the groups $\Pi(\kappa, \alpha) = \{x \in \mathbb{Z}^\kappa : |\mathrm{supp}(x)| \leq \alpha\}$ defined in Section 2.

Theorem 23 *Let four infinite cardinals $\alpha \leq \kappa$ and $\beta \leq \lambda$ be given, and assume that $\Pi(\lambda, \beta)$ is isomorphic to a subgroup of $\Pi(\kappa, \alpha)$. Then $\lambda \leq \kappa$ and, if there are no measurable cardinals $\leq \alpha$, then $\beta \leq \alpha$.*

Proof Assume all the hypotheses of the theorem, and fix a homomorphic embedding $j : \Pi(\lambda, \beta) \to \Pi(\kappa, \alpha)$. To prove that $\lambda \leq \kappa$, we use the following analog and consequence of Specker's Theorem 2.

Lemma 24 *If $h : \Pi(\lambda, \beta) \to \mathbb{Z}$ is a homomorphism, then $h(e_i) = 0$ for all but finitely many $i \in \lambda$.*

Proof Suppose not, so there are infinitely many indices $i \in \lambda$ for which $h(e_i) \neq 0$. Let C be a *countably* infinite set of such indices. Then

$$G = \{x \in \mathbb{Z}^\lambda : \mathrm{supp}(x) \subseteq C\}$$

is a subgroup of $\Pi(\lambda, \beta)$ isomorphic to Π via a bijection between C and $\mathbb{N}$. Composing $h \restriction G$ with that isomorphism, we get a homomorphism $\Pi \to \mathbb{Z}$ which, by our choice of C, maps all the standard unit vectors of Π to non-zero integers. This contradicts Specker's Theorem. □

We apply the lemma to the κ homomorphisms $h_i = \pi_i \circ j : \Pi(\lambda, \beta) \to \mathbb{Z}$ obtained by composing the embedding j with the projections $\pi_i : \Pi(\kappa, \alpha) \to \mathbb{Z}$ for $i \in \kappa$. According to the lemma, each of the sets

$$F_i = \{m \in \lambda : h_i(e_m) \neq 0\}$$

is a finite subset of λ. On the other hand, each $m \in \lambda$ belongs to at least one F_i. Indeed, as j is an embedding, $j(e_m)$ is non-zero, so at least one of its components $\pi_i(j(e_m)) = h_i(e_m)$ must be non-zero. So we have λ covered by the κ finite sets F_i. Therefore, $\lambda \leq \aleph_0 \cdot \kappa = \kappa$. This proves the first of the two conclusions of the theorem.

To prove the second conclusion, assume, toward a contradiction, that $\alpha < \beta$ and there are no measurable cardinals $\leq \alpha$. Since the finite sets F_i cover λ, there must be at least λ of them that are distinct. But we have $\lambda \geq \beta > \alpha$, so we can find α^+ distinct sets among the F_i's. By the Δ-system

Theorem 10, we can find a subfamily of α^+ of the F_i's that form a Δ-system, say with root R. Thus, all these F_i's include R, and at most one of them equals R, so we have a set $I \subseteq \kappa$ of size $|I| = \alpha^+$ such that the sets $F_i - R$ for $i \in I$ are all nonempty and pairwise disjoint. Select one element from each of these sets $F_i - R$, and let x be any member of $\mathbb{Z}^\lambda$ whose support consists of exactly the selected elements. This x has two key properties:

1. $\mathrm{supp}(x)$ has cardinality α^+, and

2. $\mathrm{supp}(x) \cap F_i$ has exactly one element for each $i \in I$.

In view of property (1) and our assumption that $\alpha < \beta$, we have that $x \in \Pi(\lambda, \beta)$. So $j(x)$ is a well-defined element of $\Pi(\kappa, \alpha)$. In particular, since $|I| > \alpha$, there must be an index $i \in I$ such that the i^{th} component of $j(x)$ is 0. That is, $h_i(x) = \pi_i(j(x)) = 0$. Fix such an i. We shall obtain a contradiction by way of the following consequence of Łoś's part of Theorem 2.

Lemma 25 *Assume there is no measurable cardinal $\leq \alpha$ and that two homomorphisms from $\Pi(\lambda, \beta)$ to $\mathbb{Z}$ agree on all the standard unit vectors e_q. Then they agree on all elements of $\Pi(\lambda, \beta)$ whose support has size at most α^+.*

Proof Let x be an element of $\Pi(\lambda, \beta)$ whose support has size $\sigma \leq \alpha^+$ and let
$$G = \{y \in \Pi(\lambda, \beta) : \mathrm{supp}(y) \subseteq \mathrm{supp}(x)\}.$$
Using a bijection between σ and $\mathrm{supp}(x)$, we obtain an isomorphism between G and $\mathbb{Z}^\sigma$, taking standard unit vectors to standard unit vectors. Composing (the restrictions to G of) the two given homomorphisms with this isomorphism, we obtain two homomorphisms $\mathbb{Z}^\sigma \to \mathbb{Z}$ that agree on all the standard unit vectors.

We have assumed that there are no measurable cardinals $\leq \alpha$. Since successor cardinals like α^+ are never measurable, and since $\sigma \leq \alpha^+$, we can apply Łoś's part of Theorem 2 to conclude that the two homomorphisms we obtained from $\mathbb{Z}^\sigma$ to $\mathbb{Z}$ must coincide. But then the originally given homomorphisms coincide on G, which contains x. □

Corollary 26 *Assume there is no measurable cardinal $\leq \alpha$. Let $h : \Pi(\lambda, \beta) \to \mathbb{Z}$ be a homomorphism, and let $F = \{m \in \lambda : h(e_m) \neq 0\}$. Then, for any*

$x \in \Pi(\lambda, \beta)$ whose support has cardinality $\leq \alpha^+$,

$$h(x) = \sum_{m \in F} x_m h(e_m).$$

Proof The formula $\sum_{m \in F} x_m h(e_m)$ makes sense for any $x \in \Pi(\lambda, \beta)$, because F is finite by Lemma 24. It defines a homomorphism that clearly agrees with h on all the standard unit vectors. So, by Lemma 25, it agrees with h also at all elements x whose support has size at most α^+. $\square$

We apply the corollary to the element x and the homomorphism h_i obtained just before Lemma 25. Thus, the set called F in the corollary is what was previously called F_i. By the corollary, we have, since $|\mathrm{supp}(x)| = \alpha^+$,

$$h_i(x) = \sum_{m \in F_i} x_m h_i(e_m).$$

According to property (2) of x, there is exactly one index m, namely the unique element of $\mathrm{supp}(x) \cap F_i$, that contributes a non-zero term to this sum. Therefore the sum cannot vanish.

This contradicts the fact that we chose i so that $h_i(x) = 0$, and this contradiction completes the proof of the theorem. $\square$

Corollary 27 *If there are no measurable cardinals $\leq \alpha$, then both κ and α are determined by the isomorphism type of $\Pi(\kappa, \alpha)$.*

Remark 28 In analogy with the notation $\Pi(\kappa, \alpha)$, we can define $\Pi(\kappa, < \alpha)$ to be the subgroup of $\mathbb{Z}^\kappa$ consisting of elements whose support has cardinality strictly smaller than α. This new notation subsumes the old, since $\Pi(\kappa, \alpha) = \Pi(\kappa, < \alpha^+)$. Theorem 23 can easily be extended to the new context provided α and β are regular and uncountable. In more detail, if β is uncountable and $\Pi(\lambda, < \beta)$ embeds in $\Pi(\kappa, < \alpha)$ then $\lambda \leq \kappa$. If, in addition, α is uncountable, regular, and below all measurable cardinals, then $\beta \leq \alpha$. This extension is proved just like Theorem 23 but using the fact that the Δ-system Theorem 10 continues to hold when κ^+ is replaced by any uncountable regular cardinal.

The extension fails, however, if $\beta = \aleph_0$. In this case, $\Pi(\lambda, < \aleph_0)$ is the free group $\bigoplus^\lambda \mathbb{Z}$ on λ generators. A theorem of Nöbeling [7] says that the subgroup of bounded functions in $\mathbb{Z}^{\aleph_0}$ is free. So we have an embedding of $\Pi(\mathfrak{c}, < \aleph_0)$ as a pure subgroup of $\mathbb{Z}^{\aleph_0} = \Pi(\aleph_0, < \aleph_1)$.

Theorem 23 can also be extended to uncountable singular cardinals and to cardinals above a measurable one, but these extensions require some new set-theoretic information. They will be presented in [2].

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