ULTRAFILTERS: WHERE TOPOLOGICAL DYNAMICS = ALGEBRA = COMBINATORICS

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Abstract. We survey some connections between topological dynamics, semigroups of ultrafilters, and combinatorics. As an application, we give a proof, based on ideas of Bergelson and Hindman, of the Hales-Jewett partition theorem.

Furstenberg and his co-workers have shown [15, 16, 17] how to deduce combinatorial consequences from theorems about topological dynamics in compact metric spaces. Bergelson and Hindman [4] applied similar methods in non-metrizable spaces, particularly the Stone-Čech compactification $\beta\mathbb{N}$ of the discrete space of natural numbers. This approach and related ideas of Carlson [11] lead to particularly simple formulations since many of the basic concepts of dynamics, when applied to $\beta\mathbb{N}$, can be expressed in terms of a semigroup operation on $\beta\mathbb{N}$, the natural extension of addition on $\mathbb{N}$. The semigroup $\beta\mathbb{N}$ can also substitute, in many contexts, for the enveloping semigroups ([14]) traditionally used in topological dynamics. Further simplifications and applications of these ideas were developed in [3].

The purpose of this paper is to survey some of these developments. In contrast to most surveys, however, we include some detailed proofs, in order to emphasize their simplicity. In the first three sections, we develop the necessary theory of dynamics and the equivalent semigroup structure in $\beta\mathbb{N}$. In the fourth section, we apply the theory to present proofs of Hindman's partition theorem for finite sums and of the Hales-Jewett theorem about homogeneous combinatorial lines in cubes. A final section (omitted for lack of time in the talk on which this paper is based) compares the ultrafilters discussed in the earlier sections with other ultrafilters traditionally related to combinatorics, for example selective ultrafilters.

1. Ultrafilters

Throughout most of this paper we are concerned with ultrafilters on the set $\mathbb{N}$ of natural numbers. These are usually defined by some version of the following set-theoretic definition, in which we have included some redundant clauses for ease of future reference.

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Definition 1. An *ultrafilter* on $\mathbb{N}$ is a family $\mathcal{U}$ of subsets of $\mathbb{N}$ such that

(1) If $X \subseteq Y$ and $X \in \mathcal{U}$ then $Y \in \mathcal{U}$.
(2) If $X, Y \in \mathcal{U}$ then $X \cap Y \in \mathcal{U}$.
(3) $\emptyset \notin \mathcal{U}$.
(4) $\mathbb{N} \in \mathcal{U}$.
(5) For any $X \subseteq \mathbb{N}$, either $X \in \mathcal{U}$ or $\mathbb{N} - X \in \mathcal{U}$.
(6) If $X \cup Y \in \mathcal{U}$ then either $X \in \mathcal{U}$ or $Y \in \mathcal{U}$.

The first four of these clauses define *filters*.

The implications in (2) and (6) are reversible, by (1).

To each $a \in \mathbb{N}$ is associated a *principal* or *trivial* ultrafilter, namely $\hat{a} = \{X \subseteq \mathbb{N} \mid a \in X\}$. In many contexts, we identify $\hat{a}$ with a.

The following alternative definition expresses the usual way of viewing ultrafilters topologically.

Definition 2. An *ultrafilter* on $\mathbb{N}$ is a point in the Stone-Čech compactification $\beta\mathbb{N}$ of the discrete space $\mathbb{N}$.

The two definitions are equivalent in the following sense. For any point $p \in \beta\mathbb{N}$, the family of subsets of $\mathbb{N}$ whose closures in $\beta\mathbb{N}$ contain p satisfies Definition 1. Conversely, if $\mathcal{U}$ is as in Definition 1, then the closures in $\beta\mathbb{N}$ of the sets in $\mathcal{U}$ have exactly one point p in common. And the constructions described in the preceding two sentences are inverse to each other.

Although the two definitions of ultrafilters above are the most familiar ones, two other, equivalent definitions will be more useful for our purposes. The first of these uses the notion of a *quantifier* Q over $\mathbb{N}$. This is an operation which applies to a formula $\varphi(n)$ with a free variable ranging over $\mathbb{N}$ and produces a new formula $(Qn)\,\varphi(n)$ in which n is no longer free; it is required that replacing $\varphi(n)$ by an equivalent formula $\psi(n)$ yields an equivalent result $(Qn)\,\psi(n)$. Formally, a quantifier can be identified with the set of those $X \subseteq \mathbb{N}$ for which $(Qn)\, n \in X$ is true. Under this identification, the following definition amounts to Definition 1.

Definition 3. An *ultrafilter* on $\mathbb{N}$ is a quantifier $\mathcal{U}$ over $\mathbb{N}$ that respects the propositional connectives in the sense that the following equivalences hold for all formulas $\varphi(n)$ and $\psi(n)$

(1) $(\mathcal{U}n)\,\varphi(n) \wedge (\mathcal{U}n)\,\psi(n) \iff (\mathcal{U}n)\,(\varphi(n) \wedge \psi(n))$
(2) $(\mathcal{U}n)\,\varphi(n) \vee (\mathcal{U}n)\,\psi(n) \iff (\mathcal{U}n)\,(\varphi(n) \vee \psi(n))$
(3) $\neg(\mathcal{U}n)\,\varphi(n) \iff (\mathcal{U}n)\,\neg\varphi(n)$

If $\mathcal{U}$ is an ultrafilter in the sense of Definition 1, then the corresponding quantifier $(\mathcal{U}n)$, usually read "for $\mathcal{U}$-almost all n," is defined by

$$(\mathcal{U}n)\,\varphi(n) \iff \{n \in \mathbb{N} \mid \varphi(n)\} \in \mathcal{U};$$

conversely, from a quantifier as in Definition 3 we can define

$$\mathcal{U} = \{X \subseteq \mathbb{N} \mid (\mathcal{U}n)\, n \in X\}.$$

As all propositional connectives (of any number of arguments) can be expressed in terms of $\neg$ and $\wedge$, they are all respected by ultrafilter quantifiers.

Finally, we give another topological definition, which will provide a connection to dynamics.

Definition 4. An *ultrafilter* on $\mathbb{N}$ is a uniform operation on sequences in compact Hausdorff spaces. That is, it is an operator assigning to every sequence $(x_n)_{n\in\mathbb{N}}$ in every compact Hausdorff space X a point $\mathcal{U}\text{-}\lim_n x_n \in X$ subject to the requirement that, if $f : X \to Y$ is a continuous map to another compact Hausdorff space, then $f(\mathcal{U}\text{-}\lim_n x_n) = \mathcal{U}\text{-}\lim_n f(x_n)$.

The easiest way to connect this definition with the previous ones is to notice that a sequence (x_n) in X is a (continuous) function x from the discrete space $\mathbb{N}$ into X; if X is a compact Hausdorff space, then this map extends uniquely to $\bar{x} : \beta\mathbb{N} \to X$, and so each $p \in \beta\mathbb{N}$ yields a point $\bar{x}(p) \in X$. Uniformity is easy to check, so an ultrafilter in the sense of Definition 2 yields one in the sense of Definition 4. Conversely, an operation as in Definition 4 can be applied to the sequence in $\beta\mathbb{N}$ whose n^{th} term is n, yielding a point $p \in \beta\mathbb{N}$, and these constructions are inverse to each other.

One can verify that $\mathcal{U}\text{-}\lim_n x_n$ is the unique point in X such that every neighborhood G of it satisfies $(\mathcal{U}n)\, x_n \in G$. In particular, it follows that $\mathcal{U}\text{-}\lim_n x_n$ is a limit point or a member of the sequence (x_n). Thus, a non-trivial ultrafilter can be regarded as a systematic way of passing to a limit point of any sequence.

The trivial ultrafilter $\hat{a}$ corresponds in Definition 2 to the point $a \in \mathbb{N} \subseteq \beta\mathbb{N}$, in Definition 3 to the "quantifier" that just substitutes a for the quantified variable, and in Definition 4 to the operation that picks out the a^{th} term from sequences.

The set $\beta\mathbb{N}$ of all ultrafilters on $\mathbb{N}$ admits a binary operation $+$, extending ordinary addition on $\mathbb{N}$ (see for example [12, 23]). In the context of Definition 4, it amounts to an iteration of limit operations:

$$(\mathcal{U} + \mathcal{V})\text{-}\lim_p x_p = \mathcal{U}\text{-}\lim_m (\mathcal{V}\text{-}\lim_n x_{m+n}).$$

Translating this into the language of quantifiers, one again finds an iteration:

$$((\mathcal{U} + \mathcal{V})p)\, \varphi(p) \iff (\mathcal{U}m)(\mathcal{V}n)\, \varphi(m+n).$$

The equivalent characterizations in terms of Definitions 1 and 2 are more complicated, at least on first sight. Definition 1 leads to

$$\mathcal{U} + \mathcal{V} = \{X \subseteq \mathbb{N} \mid \{m \mid \{n \mid m + n \in X\} \in \mathcal{V}\} \in \mathcal{U}\}.$$

And for Definition 2 we have the following description of addition. Start with ordinary addition $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Extend it by continuity to $+ : \mathbb{N} \times \beta\mathbb{N} \to \beta\mathbb{N}$, fixing the first argument in $\mathbb{N}$ and requiring continuity in the second. Then, fixing the second argument in $\beta\mathbb{N}$ and requiring continuity in the first, obtain an extension $+ : \beta\mathbb{N} \times \beta\mathbb{N} \to \beta\mathbb{N}$.

Notice that the operation $+$ on $\beta\mathbb{N}$ is a continuous function of the left summand for any fixed value in $\beta\mathbb{N}$ of the right summand, but it is not a continuous function of the right summand for a fixed left summand unless the latter is in $\mathbb{N}$ (see [23, Section 10]). I refer to continuity in the left argument as left-continuity, and I therefore call $\beta\mathbb{N}$ a left topological semigroup. (Caution: Some authors use "right" instead of "left" because the right translations are continuous, and some authors define $\mathcal{U}+\mathcal{V}$ to be what I would call $\mathcal{V}+\mathcal{U}$; authors who disagree with me on both points therefore say "left," just as I do, though they mean the opposite.)

The addition operation on $\beta\mathbb{N}$ is associative (most easily checked using the quantifier description of $+$); it is commutative as long as one of the summands is in $\mathbb{N}$, but not in general (for details, see [23, Section 10]).

2. Dynamics

Topological dynamics is concerned with the behavior of iterations of a continuous map T from a space X into itself. (Actually, it is considerably more general [14], but the preceding description covers what will be relevant here.) For the purposes of this paper, a *dynamical system* consists of a compact Hausdorff space X and a continuous function $T : X \to X$. We write T^n for the n^{th} iterate $T \circ T \circ \cdots \circ T$ of T. To study the limiting behavior of these iterates for large n, we define (as in [24]) for each ultrafilter $\mathcal{U}$ on $\mathbb{N}$

$$T^{\mathcal{U}}(x) = \mathcal{U}\text{-}\lim_n T^n(x).$$

Regarded as a function of $\mathcal{U} \in \beta\mathbb{N}$, for fixed $x \in X$, this is the continuous extension to $\beta\mathbb{N}$ of the function $\mathbb{N} \to X : n \mapsto T^n(x)$. It follows that $\{T^{\mathcal{U}}(x) \mid \mathcal{U} \in \beta\mathbb{N}\}$ is the closure in X of the forward orbit $\{T^n(x) \mid n \in \mathbb{N}\}$ of the point x.

But regarded as a function of x for fixed $\mathcal{U}$, $T^{\mathcal{U}}(x)$ need not be continuous unless $\mathcal{U}$ is principal. If $\mathcal{U}$ is the principal ultrafilter $\hat{n}$, then $T^{\mathcal{U}} = T^n$, so no confusion will be caused by identifying $\hat{n}$ with n in this context.

The notion of iteration with respect to an ultrafilter, $T^{\mathcal{U}}$, connects nicely with the addition operation on ultrafilters in that

$$T^{\mathcal{U}}(T^{\mathcal{V}}(x)) = T^{\mathcal{U}+\mathcal{V}}(x).$$

Indeed, we have

$$\begin{aligned}
T^{\mathcal{U}+\mathcal{V}}(x) &= (\mathcal{U}+\mathcal{V})\text{-}\lim_p T^p(x) \\
&= \mathcal{U}\text{-}\lim_m \mathcal{V}\text{-}\lim_n T^{m+n}(x) = \mathcal{U}\text{-}\lim_m \mathcal{V}\text{-}\lim_n T^m(T^n(x)) \\
&= \mathcal{U}\text{-}\lim_m T^m(\mathcal{V}\text{-}\lim_n T^n(x)) \qquad \text{(as } T^m \text{ is continuous)} \\
&= T^{\mathcal{U}}(T^{\mathcal{V}}(x)).
\end{aligned}$$

We next introduce some concepts from topological dynamics, i.e., concepts about the behavior of $T^n(x)$ for large n. In each definition, it is assumed that (X, T) is a dynamical system. More information about these concepts can be found in [14, 15].

Definition. A point $x \in X$ is *recurrent* if, for each neighborhood G of x, infinitely many $n \in \mathbb{N}$ satisfy $T^n(x) \in G$. It is *uniformly recurrent* if, for each neighborhood G of x, there is $M \in \mathbb{N}$ such that $\forall n\, \exists k < M\, T^{n+k}(x) \in G$.

Thus, recurrence means that, under the iteration of T, the point x returns to each of its neighborhoods infinitely often. Uniform recurrence bounds how long the sequence of iterates can stay out of any given neighborhood; there is M depending on G such that of every M consecutive iterates at least one is in G.

Definition. Two points $x, y \in X$ are *proximal* if, for every neighborhood G of the diagonal in $X \times X$, infinitely many $n \in \mathbb{N}$ satisfy $(T^n(x), T^n(y)) \in G$.

Proximality is usually defined in the context of metric spaces by requiring that, for every positive ε, infinitely many n have the distance between $T^n(x)$ and $T^n(y)$ smaller than ε. This definition clearly makes use not of the full metric structure but only of the associated uniform structure; that is, it makes sense in any uniform space. A compact Hausdorff space has a unique uniform structure, and the definition we gave for compact Hausdorff spaces is just the specialization to this case of the general concept in uniform spaces.

The dynamical concepts just defined can be elegantly expressed in terms of ultrafilter iterations, as follows.

Theorem 1. *Let (X, T) be a dynamical system.*

(1) *A point $x \in X$ is recurrent if and only if $T^{\mathcal{U}}(x) = x$ for some non-trivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$, if and only if $T^{\mathcal{U}}(x) = x$ for some $\mathcal{U} \neq \hat{0}$.*
(2) *A point $x \in X$ is uniformly recurrent if and only if for every ultrafilter $\mathcal{V}$ on $\mathbb{N}$ there is an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ with $T^{\mathcal{U}}(T^{\mathcal{V}}(x)) = x$*
(3) *Two points $x, y \in X$ are proximal if and only if there is an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ with $T^{\mathcal{U}}(x) = T^{\mathcal{U}}(y)$, if and only if there is a non-trivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$ with $T^{\mathcal{U}}(x) = T^{\mathcal{U}}(y)$.*

Proof. (1) By definition, recurrence means that x is a limit point of the sequence $(T^n(x))$. That implies that x is in the closure $\{T^{\mathcal{U}}(x) \mid \mathcal{U} \in \beta\mathbb{N} - \{\hat{0}\}\}$ of $\{T^n(x) \mid n \in \mathbb{N} - \{0\}\}$. This in turn implies that $x \in \{T^{\mathcal{U}}(x) \mid \mathcal{U} \in \beta\mathbb{N} - \mathbb{N}\}$, i.e., that we can take $\mathcal{U}$ non-trivial. Indeed, if $\mathcal{U}$ were trivial, say $\mathcal{U} = \hat{n} \neq \hat{0}$, then $T^n(x) = x$, so $T^{nk}(x) = x$ for all k, and therefore, if we take $\mathcal{U}'$ to be a non-principal ultrafilter containing the set of multiples of n, then $T^{\mathcal{U}'}(x) = x$ also. Finally, $x \in \{T^{\mathcal{U}}(x) \mid \mathcal{U} \in \beta\mathbb{N} - \mathbb{N}\}$ implies recurrence, since every $T^{\mathcal{U}}(x)$ with non-principal $\mathcal{U}$ is a limit point of $\{T^n(x) \mid n \in \mathbb{N}\}$.

(2) Assume first that x is uniformly recurrent, and let an ultrafilter $\mathcal{V}$ be given. Temporarily fix a closed neighborhood G of x, and let M be as in the definition of uniform recurrence for this neighborhood. So $\forall n\, \exists k < M\, T^{n+k}(x) \in G$. As only finitely many k's occur and as $\mathcal{V}$ is an ultrafilter, the same k must work for $\mathcal{V}$-almost all n. Fix this k, so we have $(\mathcal{V}n)\, T^{n+k}(x) \in G$. Equivalently, $(\mathcal{V}n)\, T^n(x) \in T^{-k}(G)$. As $T^{-k}(G)$ is closed, $T^{\mathcal{V}}(x) \in T^{-k}(G)$, and so $T^k(T^{\mathcal{V}}(x)) \in G$. Now unfix G, and remember that, as X is a compact Hausdorff space, every neighborhood

of x includes a closed neighborhood. So we have shown that, for every neighborhood G of x, the set

$$Y_G = \{k \in \mathbb{N} \mid T^k(T^{\mathcal{V}}(x)) \in G\}$$

is non-empty. Clearly, $Y_{G_1} \cap Y_{G_2} = Y_{G_1 \cap G_2}$, so, as G ranges over the neighborhoods of x, the sets Y_G generate a filter. Extend it to an ultrafilter $\mathcal{U}$. Then we have, for each neighborhood G of x, $(\mathcal{U}k)\, T^k(T^{\mathcal{V}}(x)) \in G$, so $T^{\mathcal{U}}(T^{\mathcal{V}}(x)) = x$, as desired.

Conversely, suppose x is not uniformly recurrent, and fix an open neighborhood G such that no M satisfies the definition of uniform recurrence. That is, for all $M \in \mathbb{N}$, the set

$$Y_M = \{n \in \mathbb{N} \mid (\forall k < M)\, T^{n+k}(x) \notin G\}$$

is nonempty. These sets Y_M generate a filter, as they form a chain, so there is an ultrafilter $\mathcal{V}$ containing all of them. For every $k \in \mathbb{N}$ we have, since $Y_{k+1} \in \mathcal{V}$, $(\mathcal{V}n)\, T^{n+k}(x) \notin G$; so $(\mathcal{V}n)\, T^n(x) \notin T^{-k}(G)$; so, as $T^{-k}(G)$ is open, $T^{\mathcal{V}}(x) \notin T^{-k}(G)$; so $T^k(T^{\mathcal{V}}(x)) \notin G$. As k is arbitrary, it follows that, for any ultrafilter $\mathcal{U}$, $T^{\mathcal{U}}(T^{\mathcal{V}}(x)) \notin G$, as desired.

(3) Suppose $T^{\mathcal{U}}(x) = T^{\mathcal{U}}(y)$. If $\mathcal{U}$ is principal, then proximality follows trivially, so suppose $\mathcal{U}$ is non-principal. To prove that x and y are proximal, let G be any neighborhood of the diagonal. Since $(T^{\mathcal{U}}(x), T^{\mathcal{U}}(y)) = \mathcal{U}\text{-}\lim_n (T^n(x), T^n(y))$ is on the diagonal by assumption, there must be infinitely many n (in fact $\mathcal{U}$-almost all n) such that $(T^n(x), T^n(y)) \in G$, as required.

Conversely, suppose x and y are proximal. So, as G ranges over neighborhoods of the diagonal, the sets

$$Y_G = \{n \in \mathbb{N} \mid (T^n(x), T^n(y)) \in G\}$$

are non-empty, and they generate a filter (because $Y_{G_1} \cap Y_{G_2} = Y_{G_1 \cap G_2}$), which we extend to an ultrafilter $\mathcal{U}$. Then we have, for all closed neighborhoods G of the diagonal, $(\mathcal{U}n)\, (T^n(x), T^n(y)) \in G$ and so $(T^{\mathcal{U}}(x), T^{\mathcal{U}}(y)) \in G$. But the intersection of all closed neighborhoods of the diagonal is just the diagonal, so we conclude that $T^{\mathcal{U}}(x) = T^{\mathcal{U}}(y)$. □

We close this section by pointing out a simpler connection between dynamical systems and ultrafilters: ultrafilters provide the universal example of a dynamical system. The compact Hausdorff space $\beta\mathbb{N}$ with the shift map $S : \beta\mathbb{N} \to \beta\mathbb{N} : \mathcal{U} \mapsto \hat{1} + \mathcal{U}$ is a dynamical system and enjoys the following universal property. If (X, T) is any dynamical system and x is any point in X, then there is a unique continuous map $f : \beta\mathbb{N} \to X$ such that $f \circ S = T \circ f$ and $f(0) = x$, namely the map defined by $f(\mathcal{U}) = T^{\mathcal{U}}(x)$. Thus, $(\beta\mathbb{N}, S)$ may be regarded as the free dynamical system on one generator.

Henceforth, when we refer to $\beta\mathbb{N}$ as a dynamical system, we mean $(\beta\mathbb{N}, S)$.

3. Dynamics = Algebra

The addition operation defined for $\beta\mathbb{N}$ in Section 1 is just the ultrafilter iteration of the shift map, i.e., of the universal dynamical system. Indeed, we have, for any ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$

$$S^{\mathcal{U}}(\mathcal{V}) = \mathcal{U}\text{-}\lim_n S^n(\mathcal{V}) = \mathcal{U}\text{-}\lim_n(n+\mathcal{V}) = \mathcal{U}+\mathcal{V},$$

where at the last step we used that $\mathcal{U}$-lim commutes with continuous maps, like addition as a function of its left argument, and that $\mathcal{U}\text{-}\lim_n n = \mathcal{U}$.

Of course, this equivalence between iteration in $(\beta\mathbb{N}, S)$ and addition allows us to reformulate Theorem 1 algebraically. We do so in the following theorem, adding some more reformulations in terms of subsemigroups (i.e., non-empty subsets closed under addition) and ideals in the semigroup $(\beta\mathbb{N}, +)$. A *left ideal* is a non-empty set $I \subseteq \beta\mathbb{N}$ such that if $\mathcal{U} \in I$ and $\mathcal{V} \in \beta\mathbb{N}$ then $\mathcal{V}+\mathcal{U} \in I$; *right ideals* and *two-sided ideals* are defined analogously.

Theorem 2.

1. *An ultrafilter $\mathcal{U}$ is recurrent in $\beta\mathbb{N}$ if and only if $\mathcal{V}+\mathcal{U} = \mathcal{U}$ for some $\mathcal{V} \neq \hat{0}$.*
2. *An ultrafilter $\mathcal{U}$ is uniformly recurrent in $\beta\mathbb{N}$ if and only if, for each $\mathcal{V}$, there is $\mathcal{W}$ with $\mathcal{W}+\mathcal{V}+\mathcal{U} = \mathcal{U}$, if and only if $\mathcal{U}$ belongs to a minimal (closed) left ideal in $\beta\mathbb{N}$.*
3. *Two ultrafilters $\mathcal{U}_1$ and $\mathcal{U}_2$ are proximal in $\beta\mathbb{N}$ if and only if there is an ultrafilter $\mathcal{V}$ such that $\mathcal{V}+\mathcal{U}_1 = \mathcal{V}+\mathcal{U}_2$.*
4. *An ultrafilter $\mathcal{U}$ generates a minimal closed subsemigroup of $\beta\mathbb{N}$ if and only if it is idempotent, i.e., $\mathcal{U}+\mathcal{U} = \mathcal{U}$.*

Proof. Parts (1), (3), and the first part of (2) are immediate consequences of the corresponding parts of Theorem 1 and the fact that $S^{\mathcal{U}}(\mathcal{V}) = \mathcal{U}+\mathcal{V}$.

To finish the proof of (2), notice first that every ultrafilter $\mathcal{U}$ generates a left ideal, namely $\beta\mathbb{N}+\mathcal{U} = \{\mathcal{V}+\mathcal{U} \mid \mathcal{V} \in \beta\mathbb{N}\}$. It follows that a minimal left ideal, being the ideal generated by any of its elements $\mathcal{U}$, is closed, for it is the image of the compact space $\beta\mathbb{N}$ under the continuous map adding $\mathcal{U}$ on the right. That is why "closed" is parenthesized in (2); putting it in or leaving it out doesn't affect the statement. Now to say that a particular ultrafilter $\mathcal{U}$ is in a minimal left ideal is to say that the left ideal $\beta\mathbb{N}+\mathcal{U}$ that it generates is minimal or, equivalently, is generated by each of its elements. That is, for every ultrafilter $\mathcal{V}$, the ideal $\beta\mathbb{N}+\mathcal{V}+\mathcal{U}$ generated by $\mathcal{V}+\mathcal{U}$ must be all of $\beta\mathbb{N}+\mathcal{U}$. Equivalently, $\beta\mathbb{N}+\mathcal{V}+\mathcal{U}$ must contain the generator $\mathcal{U}$ of $\beta\mathbb{N}+\mathcal{U}$. But that means that, for every $\mathcal{V}$, we can express $\mathcal{U}$ as $\mathcal{W}+\mathcal{V}+\mathcal{U}$ by suitably choosing $\mathcal{W}$. This completes the proof of (2).

(4), which is included in the theorem because of its analogy to (2), is trivial in one direction, as $\{\mathcal{U}\}$ is a closed subsemigroup if $\mathcal{U}$ is idempotent. To prove the non-trivial direction (due, as far as I know, to Ellis [14]), let C be a minimal closed subsemigroup of $\beta\mathbb{N}$ and let $\mathcal{U} \in C$. Then $C+\mathcal{U}$ is also closed (being the image of the compact set C under a continuous map) and a subsemigroup of C, so by minimality it equals C. In particular it contains $\mathcal{U}$. So the set $D = \{\mathcal{V} \in C \mid \mathcal{V}+\mathcal{U} = \mathcal{U}\}$ is nonempty. It is closed (being the pre-image of $\{\mathcal{U}\}$ under a continuous map) and also a subsemigroup of C, so it equals C and therefore contains $\mathcal{U}$. That is, $\mathcal{U}+\mathcal{U} = \mathcal{U}$, and the proof is complete. (It follows, of course, by minimality, that $C = \{\mathcal{U}\}$.) $\square$

The information in Theorem 2 about ideals and subsemigroups can be used to give quick existence proofs for the corresponding sorts of ultrafilters.

Corollary. *There exist uniformly recurrent ultrafilters. There exist non-trivial idempotent ultrafilters.*

Proof. The intersection of a chain of closed subsemigroups of $\beta\mathbb{N}$ is again a closed subsemigroup; it is non-empty by compactness, and it is obviously closed topologically and closed under addition. By Zorn's Lemma, there are minimal closed subsemigroups of $\beta\mathbb{N}-\mathbb{N}$. Their elements are idempotent by (4) of the theorem. The same argument applied to closed left ideals yields uniformly recurrent points. □

The concepts characterized in Theorem 2 are related to each other as follows.

Theorem 3. *Each of the following statements about an ultrafilter $\mathcal{U}$ and the dynamical system $\beta\mathbb{N}$ implies the next.*

(1) *$\mathcal{U}$ is uniformly recurrent and proximal to 0.*
(2) *$\mathcal{U}$ is idempotent.*
(3) *$\mathcal{U}$ is recurrent and proximal to 0.*

Proof. (1) $\Longrightarrow$ (2) Let $\mathcal{U}$ be uniformly recurrent and proximal to 0. By Theorem 2(3), fix $\mathcal{V}$ with $\mathcal{V}+\mathcal{U}=\mathcal{V}+0=\mathcal{V}$. By Theorem 2(2), fix $\mathcal{W}$ with

$$\mathcal{W}+\mathcal{V}+\mathcal{U}=\mathcal{U}.$$

Combining these two equations, we get $\mathcal{U}=\mathcal{W}+\mathcal{V}$, and substituting this into the displayed equation we get $\mathcal{U}+\mathcal{U}=\mathcal{U}$.

(2) $\Longrightarrow$ (3) If $\mathcal{U}$ is idempotent, then the requirement $\mathcal{V}+\mathcal{U}=\mathcal{U}$ for recurrence and the requirement $\mathcal{V}+\mathcal{U}=\mathcal{V}$ for proximality to 0 (see Theorem 2) are satisfied by taking $\mathcal{V}=\mathcal{U}$. □

The preceding results connect the algebraic properties of $\beta\mathbb{N}$ with its dynamical properties, but in fact, thanks to the universality of $\beta\mathbb{N}$ among dynamical systems, we easily get connections between the algebra of $\beta\mathbb{N}$ and arbitrary dynamical systems.

Theorem 4. *Let (X,T) be a dynamical system and let $x\in X$. If $\mathcal{U}$ is (uniformly) recurrent in $\beta\mathbb{N}$ then $T^{\mathcal{U}}(x)$ is (uniformly) recurrent in X. If $\mathcal{U}_1$ and $\mathcal{U}_2$ are proximal in $\beta\mathbb{N}$ then $T^{\mathcal{U}_1}(x)$ and $T^{\mathcal{U}_2}(x)$ are proximal in X.*

Proof. Each part is proved by combining the corresponding parts of Theorems 1 and 2 with the fact that $T^{\mathcal{U}+\mathcal{V}}=T^{\mathcal{U}}\circ T^{\mathcal{V}}$.

Suppose $\mathcal{U}$ is recurrent in $\beta\mathbb{N}$. So by Theorem 2(1) there is $\mathcal{V}$ with $\mathcal{V}+\mathcal{U}=\mathcal{U}$. Then $T^{\mathcal{V}}(T^{\mathcal{U}}(x))=T^{\mathcal{V}+\mathcal{U}}(x)=T^{\mathcal{U}}(x)$, so $T^{\mathcal{U}}(x)$ is recurrent by Theorem 1(1).

Suppose $\mathcal{U}$ is uniformly recurrent. By Theorem 2(2), for every $\mathcal{V}$ there is $\mathcal{W}$ with $\mathcal{W}+\mathcal{V}+\mathcal{U}=\mathcal{U}$ and therefore $T^{\mathcal{W}}(T^{\mathcal{V}}(T^{\mathcal{U}}(x)))=T^{\mathcal{U}}(x)$. By Theorem 1(2), $T^{\mathcal{U}}(x)$ is uniformly recurrent.

Finally, suppose $\mathcal{U}_1$ and $\mathcal{U}_2$ are proximal. By Theorem 2(3), there is $\mathcal{V}$ with $\mathcal{V}+\mathcal{U}_1=\mathcal{V}+\mathcal{U}_2$. Then $T^{\mathcal{V}}(T^{\mathcal{U}_1}(x))=T^{\mathcal{V}}(T^{\mathcal{U}_2}(x))$. By Theorem 1(3), $T^{\mathcal{U}_1}(x)$ and $T^{\mathcal{U}_2}(x)$ are proximal. □

As an application of these connections between dynamics and algebra, we give a short proof of the Auslander-Ellis Theorem [15].

Theorem 5. *Let (X, T) be a dynamical system. For each $x \in X$, there exists a uniformly recurrent y proximal to x.*

Proof. By the corollary of Theorem 2, there exists a uniformly recurrent $\mathcal{V} \in \beta\mathbb{N}$. It follows immediately that every ultrafilter of the form $\mathcal{W} + \mathcal{V}$ is uniformly recurrent. The set $\beta\mathbb{N} + \mathcal{V}$ of such ultrafilters is a closed subsemigroup of $\beta\mathbb{N}$. By Zorn's Lemma, it includes a minimal closed subsemigroup. By Theorem 2(2), there is an idempotent $\mathcal{U} \in \beta\mathbb{N} + \mathcal{V}$. Then $\mathcal{U}$, being uniformly recurrent and idempotent, is also proximal to 0 by Theorem 3.

Now for X, T, and x as in the theorem, let $y = T^{\mathcal{U}}(x)$. Then, by Theorem 4, y is uniformly recurrent and proximal to $T^0(x) = x$. □

The property of ultrafilters, "uniformly recurrent and proximal to 0," which played a key role in the proof of Theorem 5, has alternative algebraic descriptions that will be useful later. To introduce them, we first define a partial ordering of the idempotent ultrafilters by

$$\mathcal{U} \leq \mathcal{V} \iff \mathcal{U} + \mathcal{V} = \mathcal{V} + \mathcal{U} = \mathcal{U}.$$

This definition and parts of the next theorem are from [4]. When we refer to an idempotent ultrafilter as *minimal*, we mean with respect to this ordering.

Theorem 6. *The following three assertions are equivalent, for any $\mathcal{U} \in \beta\mathbb{N}$.*

(1) *$\mathcal{U}$ is uniformly recurrent and proximal to 0.*
(2) *$\mathcal{U}$ is idempotent and belongs to some minimal left ideal of $\beta\mathbb{N}$.*
(3) *$\mathcal{U}$ is a minimal idempotent.*

Furthermore, these equivalent conditions imply that $\mathcal{U}$ belongs to every two-sided ideal of $\beta\mathbb{N}$. Finally, every idempotent ultrafilter $\mathcal{U}$ is $\geq$ a minimal idempotent.

Proof.

The equivalence of (1) and (2) is immediate from Theorem 2(2) and Theorem 3.

To prove (2) $\implies$ (3), assume (2), and suppose $\mathcal{V}$ is an idempotent $\leq \mathcal{U}$. Since $\mathcal{U}$ is uniformly recurrent by Theorem 2, choose $\mathcal{W}$ so that $\mathcal{W} + \mathcal{V} + \mathcal{U} = \mathcal{U}$, which reduces, in view of $\mathcal{V} \leq \mathcal{U}$, to $\mathcal{W} + \mathcal{V} = \mathcal{U}$. Using this, the idempotence of $\mathcal{V}$, and again $\mathcal{V} \leq \mathcal{U}$, we compute

$$\mathcal{V} = \mathcal{U} + \mathcal{V} = \mathcal{W} + \mathcal{V} + \mathcal{V} = \mathcal{W} + \mathcal{V} = \mathcal{U},$$

so $\mathcal{U}$ is minimal.

We next show that, if $\mathcal{U}$ is idempotent and $I \subseteq \beta\mathbb{N} + \mathcal{U}$ is a minimal left ideal, then there is an idempotent $\mathcal{V} \leq \mathcal{U}$ in I. Since we already know that (2) implies (3), this gives the last sentence of the theorem; it will also be useful in establishing (3) $\implies$ (2). So let such $\mathcal{U}$ and I be given. Being a closed subsemigroup of $\beta\mathbb{N}$, I contains an idempotent $\mathcal{W}$ by the same argument as in the proof of Theorem 5. Being in $\beta\mathbb{N} + \mathcal{U}$, this $\mathcal{W}$ satisfies $\mathcal{W} + \mathcal{U} = \mathcal{W}$ because $\mathcal{U}$ is idempotent. Let

$\mathcal{V} = \mathcal{U} + \mathcal{W}$. Then $\mathcal{V}$ belongs to the left ideal I because $\mathcal{W}$ does. From $\mathcal{W} + \mathcal{U} = \mathcal{W}$ and the idempotence of $\mathcal{W}$ and $\mathcal{U}$, we infer

$$\mathcal{V} + \mathcal{U} = \mathcal{U} + \mathcal{W} + \mathcal{U} = \mathcal{U} + \mathcal{W} = \mathcal{V},$$

$$\mathcal{U} + \mathcal{V} = \mathcal{U} + \mathcal{U} + \mathcal{W} = \mathcal{U} + \mathcal{W} = \mathcal{V},$$

and

$$\mathcal{V} + \mathcal{V} = \mathcal{U} + \mathcal{W} + \mathcal{U} + \mathcal{W} = \mathcal{U} + \mathcal{W} + \mathcal{W} = \mathcal{U} + \mathcal{W} = \mathcal{V},$$

which mean that $\mathcal{V} \leq \mathcal{U}$, as desired.

The proof of (3) $\Longrightarrow$ (2) is now easy. If $\mathcal{U}$ is a minimal idempotent, apply Zorn's Lemma to get a minimal left ideal $I \subseteq \beta\mathbb{N} + \mathcal{U}$ as in the preceding paragraph, and let $\mathcal{V}$ be obtained as there. Being $\leq \mathcal{U}$, this $\mathcal{V}$ must be equal to $\mathcal{U}$ by minimality. So $\mathcal{U} \in I$.

Finally, we must prove that every ultrafilter satisfying (2) belongs to every two-sided ideal. In fact, every minimal left ideal I is included in every two-sided ideal J. To see this, let $\mathcal{U} \in I$ and $\mathcal{V} \in J$. Then $I \cap J$ is non-empty because it contains $\mathcal{V} + \mathcal{U}$. So $I \cap J$ is a left ideal, and it must equal I because I is minimal. So $I \subseteq J$. $\square$

4. Combinatorics

In this section, we apply the results obtained above to give relatively easy proofs of some highly non-trivial combinatorial theorems. The first of these is Hindman's Theorem, first proved in [22]. A simpler proof was given by Baumgartner [1], but we shall give two yet simpler (given the preceding machinery) arguments, one due to Furstenberg [15] and the other to Galvin and Glazer [12, 19, 23].

Theorem 7. *If $\mathbb{N}$ is partitioned into finitely many pieces, then there is an infinite $H \subseteq \mathbb{N}$ such that all finite sums of distinct members of H lie in the same piece.*

Furstenberg's Proof. Let the given partition have k pieces, and regard it as a function $\mathbb{N} \to K$, where K is a k-element set. Let X be the set of all functions $\mathbb{N} \to K$, topologized by giving K the discrete topology and then giving X the product topology. Thus, K is a compact Hausdorff space, and the given partition is a point $x \in X$. Let $T : X \to X$ be the shift map, defined by $T(y)(n) = y(n+1)$; it is clearly continuous, so we have a dynamical system. By Theorem 5, let $y \in X$ be uniformly recurrent and proximal to x. We write out what these properties of y mean for our specific X and T. Uniform recurrence means that, given any $n \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that the initial segment $(y(0), y(1), \dots, y(n-1))$ of y recurs at least once in every segment $(y(r), \dots, y(r+N-1))$ of y of length N. Proximality means that, given any N, there are infinitely many intervals of length N where x and y agree, $(x(r), \dots, x(r+N-1)) = (y(r), \dots, y(r+N-1))$.

Let $c = y(0)$. We intend to complete the proof by finding infinitely many natural numbers, all of whose finite sums are mapped to c by x.

By uniform recurrence, find N_0 such that c occurs at least once among every N_0 consecutive terms in y. By proximality, find a place, beyond term 0, where N_0

consecutive terms of x coincide with those of y and therefore contain a c. So we can fix $h_0 > 0$ with $x(h_0) = c$. This h_0 will be the first member of our H.

By uniform recurrence, find N_1 such that among every N_1 consecutive terms of y there are $h_0 + 1$ consecutive terms that coincide with $y(0), \dots, y(h_0)$; in particular, among every N_1 consecutive terms, there are two terms a distance h_0 apart where y has the value c (the same as at 0 and h_0). By proximality, there are two places a distance h_0 apart where x has value c, say $x(h_1) = x(h_1 + h_0) = c$, with $h_1 > h_0$. h_1 will be the next member of H.

Repeating this process, we inductively choose h_i so that, for all sums s of zero or more elements of $\{h_0, \dots, h_{i-1}\}$, we have $x(s + h_i) = y(s + h_i) = c$. This is done by finding N such that every N consecutive terms of y contain a segment that coincides with the initial segment of y up to the largest s, and then finding a segment of length N beyond h_{i-1} where x and y coincide.

The set of all finite sums of distinct h_n's is clearly included in the partition piece corresponding to c. □

Galvin's and Glazer's Proof. This proof uses ultrafilters directly rather than applying them via Theorem 5 to dynamics on other spaces. Let $\mathcal{U}$ be any idempotent non-trivial ultrafilter on $\mathbb{N}$ (by the corollary to Theorem 2), and let C be the piece of the partition that is in $\mathcal{U}$ (by clause (6) of Definition 1). So we have $(\mathcal{U}n)\, n \in C$. As $\mathcal{U}$ is idempotent, we also have (cf. the quantifier form of the definition of $+$ in $\beta\mathbb{N}$) $(\mathcal{U}n)\,(\mathcal{U}k)\, n + k \in C$. As ultrafilter quantifiers respect propositional connectives,

$$(\mathcal{U}n)\,[n \in C \wedge (\mathcal{U}k)\, n + k \in C].$$

So we can fix h_0 with $h_0 \in C$ and (re-naming variables) $(\mathcal{U}n)\, h_0 + n \in C$. Using again that $\mathcal{U}$ is idempotent and respects connectives, we find

$$(\mathcal{U}n)\,[n \in C \wedge (\mathcal{U}k)\, n + k \in C \wedge h_0 + n \in C \wedge (\mathcal{U}k)\, h_0 + n + k \in C].$$

So we can fix h_1 having the four properties listed for n inside the brackets. In particular, $h_1 \in C$ and $h_0 + h_1 \in C$.

Repeating this process, we inductively choose h_i so that, for all sums s of zero or more elements of $\{h_0, \dots, h_{i-1}\}$, we have $s + h_i \in C$ and $(\mathcal{U}n)\, s + h_i + n \in C$. The latter property, when expanded by idempotence, ensures that it is possible to choose h_{i+1} to keep the induction going. (In fact, $\mathcal{U}$-almost all numbers can serve as h_{i+1}.) Clearly, all finite sums of distinct members of $H = \{h_n \mid n \in \mathbb{N}\}$ are in C. □

Notice that the Galvin-Glazer proof shows that the piece that contains the homogeneous H can be taken to be any piece of the given partition that belongs to some idempotent ultrafilter.

We turn next to an application of these ideas in the context of words over a finite alphabet, rather than natural numbers. We shall prove the Hales-Jewett Theorem [20, 19], but first we need some definitions and notational conventions.

Let Σ be a finite set, which we call an alphabet, and let W be the set of words on Σ, i.e., the set of finite sequences of members of Σ. Let v be an object, called a variable, that is not in Σ; let A be the set of words on $\Sigma \cup \{v\}$; and let $V = A - W$. So V is the set of words on $\Sigma \cup \{v\}$ in which v actually occurs; these are often called *variable words* over Σ. For each $a \in \Sigma$, we define a function $\hat{a} : A \to W$ sending each $x \in A$ to the result of substituting a for v in x; we call $\hat{a}(x)$ an *instance* of x. Notice that if $x \in W$ then $\hat{a}(x) = x$.

With this notation, the Hales-Jewett Theorem [20] is as follows.

Theorem 8. *If W is partitioned into finitely many pieces, then there is an $x \in V$ whose instances all lie in the same piece of the partition.*

The proof is a special case of arguments from [3].

Proof. First, observe that A is a semigroup under the operation $^\frown$ of concatenation, that W is a subsemigroup, that V is a two-sided ideal in A, and that each $\hat{a}$ is a homomorphism $A \to W$. The operation $^\frown$ can be extended to the Stone-Čech compactification βA just as addition was extended to $\beta\mathbb{N}$.

$$(\mathcal{U}^\frown\mathcal{V}x)\,\varphi(x) \iff (\mathcal{U}y)\,(\mathcal{V}z)\,\varphi(y^\frown z).$$

It is easy to verify that, in the compact left-topological semigroup βA, βW is a closed subsemigroup, βV is a closed, two-sided ideal, and the continuous extension of $\hat{a}$, which we still call $\hat{a} : \beta A \to \beta W$, is a homomorphism.

The algebraic results about $\beta\mathbb{N}$ proved earlier generalize easily to semigroups like βA and βW. We apply the analogs in this context of several parts of Theorem 6. In particular, there is a minimal idempotent $\mathcal{W} \in \beta W$. In βA, this $\mathcal{W}$ is idempotent but not necessarily minimal. (In fact, we shall see in a moment that it is definitely not minimal.) There is a minimal idempotent $\mathcal{V} \leq \mathcal{W}$ in βA. It belongs to every two-sided ideal, so $\mathcal{V} \in \beta V$. (In particular, $\mathcal{V} \neq \mathcal{W}$.)

For any $a \in \Sigma$, since $\hat{a}$ is a homomorphism $\beta A \to \beta W$, we can infer from $\mathcal{V} \leq \mathcal{W}$ that $\hat{a}(\mathcal{V}) \leq \hat{a}(\mathcal{W}) = \mathcal{W}$ (the last equality because $\hat{a}$ is the identity on W and hence on βW). By minimality of $\mathcal{W}$ in βW, it follows that $\hat{a}(\mathcal{V}) = \mathcal{W}$.

Now let W be partitioned into finitely many pieces, and let X be the piece that is in the ultrafilter $\mathcal{W}$. For each $a \in \Sigma$, we have $\hat{a}^{-1}(X) \in \mathcal{V}$ because $X \in \mathcal{W} = \hat{a}(\mathcal{V})$. So $\bigcap_{a\in\Sigma} \hat{a}^{-1}(X)$ is non-empty. (In fact it is in $\mathcal{V}$.) Any element x of this intersection clearly serves as the x required in the theorem. $\square$

The proof actually establishes a stronger theorem, obtained by broadening the notion of "instance" to allow a specified, finite set of words in W (not merely single letters) as the a's being substituted for v. Unlike Theorem 8, this stronger form is non-trivial even in the case where Σ consists of just one letter; indeed this case amounts to van der Waerden's theorem on arithmetic progressions. (Van der Waerden's theorem is usually deduced from Theorem 8 for a b-element alphabet by using base b expansions of natural numbers; see [19].)

5. P-Points

In this section, we briefly discuss the connections between the ultrafilters discussed earlier and other, perhaps more familiar (from [2, 3, 6, 8, 10, 13, 25, 26] for example), special ultrafilters. We begin with a pair of definitions.

Definition. A non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is *selective* if every function on $\mathbb{N}$ becomes one-to-one or constant when restricted to a suitable set in $\mathcal{U}$.

Definition. A non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is a *P-point* if every function on $\mathbb{N}$ becomes finite-to-one or constant when restricted to a suitable set in $\mathcal{U}$.

Both of these definitions have numerous equivalent forms. The versions above were chosen to make it obvious that all selective ultrafilters are P-points. The definition of P-point is just a combinatorial reformulation of the usual topological notion of P-point specialized to the space $\beta\mathbb{N}-\mathbb{N}$: a point such that the intersection of any countably many neighborhoods is again a (not necessarily open) neighborhood.

Selectivity also has a topological formulation, based on a theorem of Kunen [10] that characterizes selective ultrafilters $\mathcal{U}$ as those enjoying the following Ramsey property. If the set $[\mathbb{N}]^2$ of two-element subsets of $\mathbb{N}$ is partitioned into two pieces, then there is a set $H \in \mathcal{U}$ all of whose two-element subsets lie in one piece. (The corresponding statement for $[\mathbb{N}]^k$ holds for all finite k, and there are even infinitary generalizations; see [26].) If we identify $[\mathbb{N}]^2$ with the "above diagonal" subset $\{(a,b) \mid a < b\}$ of $\mathbb{N}^2$, then this Ramsey property says that $[\mathbb{N}]^2$ together with the sets $H\times H$ for $H \in \mathcal{U}$ generate an ultrafilter on $\mathbb{N}\times\mathbb{N}$. Let $\tau : \beta(\mathbb{N}\times\mathbb{N}) \to \beta\mathbb{N}\times\beta\mathbb{N}$ be the continuous extension of the inclusion map $\mathbb{N}\times\mathbb{N} \to \beta\mathbb{N}\times\beta\mathbb{N}$. Then ultrafilters on $\mathbb{N}\times\mathbb{N}$ that contain $H\times H$ for all $H \in \mathcal{U}$ are precisely those sent by τ to $(\mathcal{U},\mathcal{U})$. Thus (cf. [5, Section 10]), the Ramsey property is equivalent to saying that $\tau^{-1}(\mathcal{U},\mathcal{U})$ consists of exactly three points, namely the ultrafilter on $\mathbb{N}\times\mathbb{N}$ mentioned above, a symmetrical one "below diagonal," and an isomorphic copy of $\mathcal{U}$ concentrated on the diagonal. (Hindman [21] has shown that there are P-points $\mathcal{U}$ such that $\tau^{-1}(\mathcal{U},\mathcal{U})$ has the same cardinality, $2^{2^{\aleph_0}}$, as $\beta\mathbb{N}$.)

P-points also have a Ramsey-like property [2, Theorem 2.3]: If $\mathcal{U}$ is a P-point and if $[\mathbb{N}]^2$ is partitioned into two pieces, then there is $H \in \mathcal{U}$ and there is a function $f : \mathbb{N} \to \mathbb{N}$ such that one piece of the partition contains all the two-element subsets $\{a,b\}$ of H for which $f(a) < b$.

Although P-points and selective ultrafilters, like the ultrafilters discussed in the previous sections, have interesting combinatorial and topological properties, they are quite different in several respects, of which we list three.

First, since $\mathcal{U}+\mathcal{V} = \mathcal{U}\text{-}\lim_n(n+\mathcal{V})$, while P-points are, in view of their topological description, never limit points of a countable set of other non-principal ultrafilters, it follows that no P-point can be of the form $\mathcal{U}+\mathcal{V}$ with non-principal $\mathcal{U}$ and $\mathcal{V}$. In particular, no P-point can be recurrent. So the family of P-points and, a fortiori, the subfamily of selective ultrafilters are disjoint from the families of ultrafilters studied in the preceding sections (recurrent, idempotent, etc.).

Second, the ultrafilters in the preceding sections are proved to exist on the basis of the usual axioms of set theory (Zermelo-Fraenkel axioms, including the axiom of choice). In contrast, the existence of P-points or of selective ultrafilters is independent of these axioms. More precisely, the continuum hypothesis (as well as weaker assumptions, like Martin's axiom) implies the existence of many selective ultrafilters and also many P-points that are not selective, but there are models of set theory with no selective ultrafilters [25] and even with no P-points [27, 28].

Finally, where Ramsey ultrafilters have $\tau^{-1}(\mathcal{U},\mathcal{U})$ as small as possible, namely of size 3, the following theorem shows that idempotent ultrafilters have it as large as possible.

Theorem 9. *If $\mathcal{U}$ is an idempotent non-trivial ultrafilter on $\mathbb{N}$, then there are $2^{2^{\aleph_0}}$ ultrafilters $\mathcal{V}$ on $\mathbb{N}\times\mathbb{N}$ with $\tau(\mathcal{V})=(\mathcal{U},\mathcal{U})$.*

Proof. Observe first that, for each natural number n, the set of multiples of n is in $\mathcal{U}$. Indeed, as there are only finitely many congruence classes modulo n, any ultrafilter must contain one of them, so we can fix j such that $0\leq j<n$ and $(\mathcal{U}x)\,x\equiv j \pmod n$. Then $(\mathcal{U}x)(\mathcal{U}y)\,x+y\equiv 2j \pmod n$, so for idempotent $\mathcal{U}$ it follows that $(\mathcal{U}x)\,x\equiv 2j \pmod n$. But then $j\equiv 2j \pmod n$ and so $j=0$ as claimed.

In particular, for any n, $\mathcal{U}$-almost all numbers x are divisible by 2^n and therefore have 0's as the last n digits in their binary expansions.

Using this, we can proceed as in Galvin's and Glazer's proof of Theorem 7 to find, in any set $C\in\mathcal{U}$, a sequence $h_0,h_1,\dots$ such that

(1) All finite sums of distinct h_i's are in C.
(2) For each i, h_{i+1} is divisible by a power of 2 that is larger than h_i.

Note the following consequence of (2). If a and b are each a sum of distinct h_i's and if no h_i occurs in both sums, then the 1's in the binary expansions of a and b occur in disjoint sets of positions. We define the meshing number $m(a,b)$ to measure the amount of intermeshing between these disjoint sets; that is, $m(a,b)$ is the length l of the longest sequence $s_1,\dots,s_l$ such that for all odd (resp. even) i, there is a 1 in position s_i of the binary expansion of a (resp. b). It is clear that every integer $l\geq 2$ occurs as $m(a,b)$ with a and b sums of different h_i's; just take the first l of the h_i's and let a (resp. b) be the sum of the odd (resp. even) numbered ones.

In view of (1), this means that each of the infinitely many sets $m^{-1}\{l\}$ meets each set of the form $C\times C$ for $C\in\mathcal{U}$. So each $m^{-1}\{l\}$ supports an ultrafilter containing all these $C\times C$ and therefore mapping to $(\mathcal{U},\mathcal{U})$ by τ. This proves that $\tau^{-1}(\mathcal{U},\mathcal{U})$ is infinite. But every infinite closed subset of $\beta\mathbb{N}$ (or the homeomorphic $\beta(\mathbb{N}\times\mathbb{N})$) has cardinality $2^{2^{\aleph_0}}$; see [13], page 424, or[18], Chapter 9. □

In spite of all these differences between P-points and selective ultrafilters on the one hand and recurrent and idempotent ultrafilters on the other, it is possible, using the continuum hypothesis (or Martin's axiom) to construct idempotent ultrafilters with strong connections to selective ultrafilters. For example, one can arrange that an idempotent ultrafilter be mapped to a selective one by the map $\mathbb{N}\to\mathbb{N}$ that sends each natural number a to the position of the rightmost (or the leftmost) 1 in its binary expansion. For more information about such matters and for combinatorial applications, see [7, 9].

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