Unsplit Families, Dominating Families, and Ultrafilters

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Abstract. We study some weakenings of the finite intersection property for families of subsets of the natural numbers. The weakenings involve (1) requiring intersections for only a fixed number of sets from the family and (2) requiring the sets to have elements near each other rather than actually intersecting. These weakenings fit into a chain of implications, none of which are reversible under CH, but almost all of which are consistently reversible. We also connect these properties with weakened domination properties for families for functions on the natural numbers. For unsplit families of sets, the chain of implications collapses from infinitely many properties to just four.

1. Introduction and Background Information

We use the customary notations ω for the set of natural numbers, $\mathcal{P}(\omega)$ for its power set, $[\omega]^\omega$ for the family of all infinite subsets of ω, ${}^\omega\omega$ for the family of all functions from ω to ω, $A \subseteq^* B$ for almost inclusion (i.e., $A - B$ is finite), and $f \leq^* g$ for eventual majorization (i.e., $f(n) \leq g(n)$ for all but finitely many n).

Definition 1.1. A family $\mathcal{X} \subseteq [\omega]^\omega$ is *unsplit* if

$$(\forall A \in \mathcal{P}(\omega))(\exists X \in \mathcal{X})\,(X \subseteq^* A \text{ or } X \subseteq^* \omega - A).$$

That is, no single $A \subseteq \omega$ splits every $X \in \mathcal{X}$ into two infinite pieces.

Definition 1.2. Following the standard notation for cardinal characteristics of the continuum ([**5, 8, 3**]), we write

- $\mathfrak{c}$ for the cardinality of the continuum,
- $\mathfrak{r}$ for the smallest cardinality of any unsplit family (the unsplitting or refining or reaping number),
- $\mathfrak{u}$ for the minimum cardinality of any base for a nonprincipal ultrafilter on ω,
- $\mathfrak{d}$ for the minimum cardinality of a *dominating* family, i.e., a family $\mathcal{D} \subseteq {}^\omega\omega$ such that every $f \in {}^\omega\omega$ is $\leq^* g$ for some $g \in \mathcal{D}$ (the dominating number), and
- $\mathfrak{g}$ for the groupwise density number, for whose definition we refer to [**4**] or [**3**] since we shall not need the details here.

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Notice that a base for a nonprincipal ultrafilter on ω is the same thing as an unsplit filter base. In particular, $\mathfrak{r} \leq \mathfrak{u}$. Goldstern and Shelah [**6**] showed that $\mathfrak{r} < \mathfrak{u}$ is consistent, but Aubrey [**1**] showed that $\mathfrak{r} \geq \min\{\mathfrak{u}, \mathfrak{d}\}$.

Part of our purpose in the present paper is to make Aubrey's result more explicit by showing how to obtain, quite directly from an unsplit family, either an ultrafilter base or a dominating family of the same cardinality.

The essential difference between ultrafilters and general unsplit families is that the former (and generating families for ultrafilters) have the strong finite intersection property (the intersection of any finitely many sets from the family is infinite). We shall study the role of the strong finite intersection property by looking at some weakenings of it, both in connection with unsplit families and in more general situations.

The remainder of this introductory section is devoted to fixing some notation and terminology that will be needed later.

DEFINITION 1.3. If $\mathcal{X} \subseteq [\omega]^\omega$ and $f : \omega \to \omega$ then

$$f(\mathcal{X}) := \{f(X) : X \in \mathcal{X}\}.$$

REMARK 1.4. The notation $f(\mathcal{X})$ is often used for $\{Y \subseteq \omega : f^{-1}(Y) \in \mathcal{X}\}$. That usage seems to be preferable when dealing with families closed under supersets, but our definition above works better in the situations we shall consider, where no upward closure is required. If $\mathcal{X}$ is closed upward, then the upward closure of our $f(\mathcal{X})$ is the other $f(\mathcal{X})$.

When we use the notation $f(\mathcal{X})$, the function f will usually be finite-to-one and monotone. That is, there is a partition Π of ω into finite intervals such that f is constant on these intervals and strictly increases from one interval to the next. In this situation, $f(\mathcal{X})$ can be visualized as obtained from $\mathcal{X}$ by collapsing each interval in Π to a point.

LEMMA 1.5. *If f is finite-to-one and $\mathcal{X}$ is unsplit, then $f(\mathcal{X})$ is also unsplit.*

Proof The hypothesis that f is finite-to-one is used only to ensure that $f(\mathcal{X}) \subseteq [\omega]^\omega$. The unsplit property is immediate, for if A were to split every set in $f(\mathcal{X})$ then $f^{-1}(A)$ would split every set in $\mathcal{X}$. □

DEFINITION 1.6. For $X \in [\omega]^\omega$ and $n \in \omega$, let $\mathrm{next}(X, n)$ be the smallest element of X that is $\geq n$.

We recall from [**2**] the following weakened form of domination.

DEFINITION 1.7. For a positive integer k, a family $\mathcal{F} \subseteq {}^\omega\omega$ is *k-dominating* if every $g \in {}^\omega\omega$ is eventually dominated by the (pointwise) maximum of some k functions from $\mathcal{F}$.

2. Meeting, Nearly Meeting, and Just Missing

In this section, we introduce and study some properties that refine and approximate the strong finite intersection property. Throughout this section, $\mathcal{X}$ is an arbitrary subfamily of $[\omega]^\omega$.

The first definition is a natural quantitative version of the strong finite intersection property, replacing "finite" by a specific number.

DEFINITION 2.1. Let k be a positive integer. A family $\mathcal{X} \subseteq [\omega]^\omega$ is *k-meeting* if, for all $X_1, \dots, X_k \in \mathcal{X}$, $X_1 \cap \dots \cap X_k$ is infinite.

Thus, $\mathcal{X}$ has the strong finite intersection property if and only if it is k-meeting for all k.

The next definition is a weakening of k-meeting, working modulo a possible collapse of finite intervals to points. We note that the analogous "modulo collapse of finite intervals" weakening of the strong finite intersection property has played a role, for example, in results of Laflamme [**7**].

DEFINITION 2.2. Let k be a positive integer. A family $\mathcal{X} \subseteq [\omega]^\omega$ is *nearly k-meeting* if there is a finite-to-one, monotone $f : \omega \to \omega$ such that $f(\mathcal{X})$ is k-meeting.

The next definition admittedly looks unnatural, because of its dependence on the immediate successor relation on ω. However, it will be used primarily in its "nearly" form, which turns out to be considerably more natural.

DEFINITION 2.3. Let k be a positive integer. A family $\mathcal{X} \subseteq [\omega]^\omega$ is *k-close* if, for all $X_1, \dots, X_k \in \mathcal{X}$, there are infinitely many n such that, for each $i = 1, \dots, k$ either $n \in X_i$ or $n + 1 \in X_i$.

In other words, instead of requiring the k sets X_i to meet infinitely often, we allow an "error" of 1; when they all hit a two-element interval, that's as good as actually all meeting.

DEFINITION 2.4. Let k be a positive integer. A family $\mathcal{X} \subseteq [\omega]^\omega$ is *nearly k-close* if there is a finite-to-one, monotone $f : \omega \to \omega$ such that $f(\mathcal{X})$ is k-close.

It is often useful to reformulate the "nearly" definitions in terms of the interval partitions corresponding to the finite-to-one monotone functions.

LEMMA 2.5. *$\mathcal{X}$ is nearly k-close if and only if there is a partition Π of ω into intervals $[\pi_n, \pi_{n+1})$, where $0 = \pi_0 < \pi_1 < \pi_2 < \dots$, such that, for every k sets $X_i \in \mathcal{X}$, there are infinitely many n such that all k of the X_i meet the double interval $[\pi_n, \pi_{n+2})$. Nearly k-meeting is the same except that the double interval is replaced with the single interval $[\pi_n, \pi_{n+1})$.*

In the situation of the lemma, we shall say that Π witnesses that $\mathcal{X}$ is nearly k-close or nearly k-meeting.

It is somewhat surprising that it really makes a difference whether we use the double intervals $[\pi_n, \pi_{n+2})$ or the single intervals $[\pi_n, \pi_{n+1})$, considering that the partition Π can be adjusted.

Notice that as k increases, the (nearly) meeting and closeness properties become stronger, in contrast to k-dominating, which becomes weaker. In fact, the following proposition exhibits a negative correlation between closeness and domination.

PROPOSITION 2.6. *$\mathcal{X}$ is nearly k-close if and only if $\{next(X, -) : X \in \mathcal{X}\}$ is not k-dominating.*

Proof First assume that $\mathcal{X}$ is nearly k-close and let the interval partition Π witness this. Define $g : \omega \to \omega$ by letting $g(n)$ be the left endpoint of the third interval of Π after the one containing n. So there are two consecutive Π-intervals included in $[n, g(n))$. We claim that g witnesses that $\{\mathrm{next}(X, -) : X \in \mathcal{X}\}$ is not k-dominating. To prove this, suppose toward a contradiction that $X_1, \dots, X_k$ are k sets in $\mathcal{X}$ such that the maximum of the associated functions $\mathrm{next}(X_i, -)$ eventually majorizes g. So, for all but finitely many n, we have that at least one X_i has its next element after n greater than the right endpoint of the second Π-interval after n. This means that X_i is disjoint from the two Π-intervals immediately following the one that contains n. Since n is arbitrary (provided it is large enough), we have that, with finitely many exceptions, no union of two consecutive Π-intervals intersects all the X_i. This contradicts the assumption that Π witnesses that $\mathcal{X}$ is nearly k-close, and so half of the proposition is proved.

For the converse, suppose some function $g : \omega \to \omega$ is not eventually majorized by the maximum of any k of the functions $\mathrm{next}(X, -)$ for $X \in \mathcal{X}$. Let Π be a partition of ω into intervals such that, for each n, $g(n)$ is in the next interval after n or earlier. Such a partition is easily constructed by inductively choosing the successive intervals. Given any k sets $X_1, \dots, X_k \in \mathcal{X}$, the fact that the maximum of the associated functions $\mathrm{next}(X_i, -)$ fails to eventually majorize g means that there are infinitely many n such that all k of the X_i meet the interval $[n, g(n)]$. But this interval is included in the union of two consecutive Π-intervals. Thus, there are infinitely many unions of two consecutive Π-intervals, each of which meets all k of the sets X_i. Thus $\mathcal{X}$ is nearly k-close. □

REMARK 2.7. This proposition suggests that the notion of "nearly k-close" is more natural than it looks at first sight. The proof of the proposition also provides a certain robustness of the notion. For example, where the definition of "k-close" requires each of the k sets to contain n or $n+1$, let us put instead the weaker requirement that each of these sets contains one of $n, n+1, \dots, n+p$ for a fixed p. Then the first half of the proof of the proposition still works if we just redefine $g(n)$ to be the left endpoint of the $(p+2)^{\mathrm{nd}}$ (rather than the third) Π-interval after n. Thus, although we have weakened the notion of "k-close", the notion of "nearly k-close" is unchanged. The same applies if we let p vary with n, provided we appropriately adjust the g in the proof.

REMARK 2.8. To avoid confusion, we point out a potential ambiguity in the phrase "nearly k-close for all k". Its intended meaning is that, for each k there is a finite-to-one monotone f_k (or equivalently an interval partition Π_k) witnessing k-closeness. But one can easily imagine the same phrase as obtained by prefixing "nearly" to the compound adjective "k-close for all k". That would mean that a single f (or Π) works for all k simultaneously, an apparently stronger statement than the previous one. Fortunately, the following lemma shows that the two possible meanings of this phrase are equivalent.

LEMMA 2.9. *For any $\mathcal{X} \subseteq [\omega]^\omega$, the following three statements are equivalent.*

1. *For every $k \in \omega$, $\mathcal{X}$ is nearly k-close.*
2. *There is a finite-to-one $f : \omega \to \omega$ such that, for every $k \in \omega$, $f(\mathcal{X})$ is k-close.*
3. *There is a function $g : \omega \to \omega$ that is not eventually majorized by the maximum of any finitely many of the functions next$(X, -)$ for $X \in \mathcal{X}$.*

Proof Since assertion (1) says that for each k there is a finite-to-one monotone f_k such that $f_k(\mathcal{X})$ is k-close and assertion (2) says that a single f works for all k simultaneously, it is clear that (2) implies (1).

Next, we assume (1) and prove (3). By Proposition 2.6, there is for each $k \in \omega$ some $g_k : \omega \to \omega$ that is not eventually majorized by the maximum of any k of the functions $\mathrm{next}(X, -)$ for $X \in \mathcal{X}$. Let $g : \omega \to \omega$ eventually majorize all the g_k; for example we could take $g(n) = \max\{g_k(n) : k \leq n\}$. Then g is as required in (3).

Finally, we assume that g is as in (3) and we prove (2). This proof is exactly like the second half of the proof of Proposition 2.6, with the fixed k replaced by an arbitrary finite number, so we do not repeat the argument here. □

REMARK 2.10. It is easy to see that any $2k$-close family is nearly k-meeting, witnessed by one of the two functions $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$. Indeed, if $X_1, \dots, X_k$ are k sets in $\mathcal{X}$ such that only finitely many of the two-point intervals $[2n, 2n+1]$ meet all the X_i, and if $Y_1, \dots, Y_k$ are k sets in $\mathcal{X}$ such that only finitely many $[2n+1, 2n+2]$ meet all the Y_i, then there are only finitely many m such that each of the $2k$ sets X_i and Y_i contains m or $m+1$. That is, $\mathcal{X}$ is not $2k$-close.

Of course, it follows immediately that any nearly $2k$-close family is nearly k-meeting. Indeed, if Π witnesses "nearly $2k$-close", then we can witness "nearly k meeting" with a partition obtained by merging pairs of consecutive intervals from Π.

In this context with "nearly", the following theorem lets us improve $2k$ to $k+1$ by merging (perhaps) considerably longer blocks of intervals.

THEOREM 2.11. *Every nearly $(k+1)$-close family is nearly k-meeting.*

Proof It suffices to prove that every $(k+1)$-close family $\mathcal{X}$ is nearly k-meeting, for if $\mathcal{X}$ is only nearly $(k+1)$-close then we can apply the argument to a $(k+1)$-close family of the form $f(\mathcal{X})$.

Assume therefore that $\mathcal{X}$ is $(k+1)$-close. Consider the partition of ω into the intervals of length two, $[2n, 2n+1]$. If every $X \in \mathcal{X}$ meets all but finitely many of these intervals, then this partition witnesses that $\mathcal{X}$ is nearly k-meeting (and in fact nearly l-meeting for all $l \in \omega$).

So we may assume, without loss of generality, that there is some $Z \in \mathcal{X}$ that misses infinitely many of the intervals $[2n, 2n+1]$. Fix such a Z, and enumerate as $n_1 < n_2 < \dots$ the integers n such that $Z \cap [2n, 2n+1] = \varnothing$. Let Π be the partition of ω into the intervals $[2n_i+1, 2n_{i+1}]$, along with the initial interval $[0, 2n_1]$. In other words, the partition Π breaks ω in the middle of each of the intervals $[2n, 2n+1]$ that Z misses. We intend to show that Π witnesses that $\mathcal{X}$ is nearly k-meeting, i.e., that given any k elements of $\mathcal{X}$, we can find an interval $[2n_i+1, 2n_{i+1}] \in \Pi$ that meets all k of them.

So let any k sets $X_1, \dots, X_k \in \mathcal{X}$ be given. Apply the assumption that $\mathcal{X}$ is $(k+1)$-close to the $k+1$ sets $Z, X_1, \dots, X_k$. We obtain infinitely many m such that the interval $[m, m+1]$ meets Z and all k of the X_i. Such an m cannot be of the form $2n_i$, for then the interval $[m, m+1] = [2n_i, 2n_i+1]$ would miss Z (by definition of n_i). Therefore, $[m, m+1]$ lies entirely within one of the intervals of Π, and so that interval of Π meets all the X_i. Since this happens for infinitely many m, the proof is complete. $\square$

In view of the theorem, we have the following chain of implications between the "nearly" meeting and close properties.

$$
\begin{aligned}
(1) \quad \dots \implies \text{nearly } (k+1)\text{-close} \implies& \\
&\text{nearly } k\text{-meeting} \implies \text{nearly } k\text{-close} \implies \dots \\
\implies \text{nearly 4-close} \implies& \text{ nearly 3-meeting} \implies \text{nearly 3-close} \implies \\
&\text{nearly 2-meeting} \implies \text{nearly 2-close} \implies \text{nearly 1-meeting.}
\end{aligned}
$$

Of course, the last item in the chain is always true, since we have $\mathcal{X} \subseteq [\omega]^\omega$. At the other end of the chain, we can consider the conjunction of all the properties listed. That conjunction is item (1) from Lemma 2.9. We can now extend that lemma by including properties involving meeting rather than closeness.

COROLLARY 2.12. *For any $\mathcal{X} \subseteq [\omega]^\omega$, the following are equivalent.*

(1) *The three statements from Lemma 2.9.*
(2) *For every $k \in \omega$, $\mathcal{X}$ is nearly k-meeting.*
(3) *There is a finite-to-one $f : \omega \to \omega$ such that, for every $k \in \omega$, $f(\mathcal{X})$ is k-meeting.*

Proof The chain of implications above immediately shows that item (2) here and item (1) of Lemma 2.9 are equivalent, and it is trivial that in the present corollary (3) implies (2). To complete the proof, it suffices to deduce the present (3) from item (2) of Lemma 2.9.

For this purpose, notice first that, in the proof of Theorem 2.11, the partition witnessing that $\mathcal{X}$ is nearly k-meeting was obtained without reference to the value of k. Thus, if $\mathcal{X}$ is $(k+1)$-close for all k, then there is a function g, independent of k, such that $g(\mathcal{X})$ is k-meeting for all k. Now if (2) of Lemma 2.9 holds, then we have an f such that $f(\mathcal{X})$ is $(k+1)$-close for all k, and so we get g such that $gf(\mathcal{X})$ is k-meeting for all k. That is, we get (3) of the present corollary, as required. $\square$

3. Consistency Results

In this section, we consider the reversibility of the implications in the chain (1). The last implication in the chain is not reversible, for $[\omega]^\omega$ is 1-meeting but not nearly 2-close. All the other reversals, however, are independent of ZFC, as the following theorems show. In connection with the first of these theorems, we recall the result from [**4**] that the inequality $\mathfrak{u} < \mathfrak{g}$ is consistent relative to ZFC.

THEOREM 3.1. *Assume $\mathfrak{u} < \mathfrak{g}$. Then all the implications except the last in the chain (1) are reversible. Thus, if $\mathcal{X} \subseteq [\omega]^\omega$ is nearly 2-close then there is a finite-to-one, monotone $f : \omega \to \omega$ such that $f(\mathcal{X})$ has the strong finite intersection property.*

Proof Under the assumption $\mathfrak{u} < \mathfrak{g}$, it is shown in [**2**, Theorem 6.5] that every k-dominating family of monotone functions from ω to ω, for any $k < \omega$, is 2-dominating. By Proposition 2.6, it follows that every nearly 2-close family $\mathcal{X} \subseteq [\omega]^\omega$ is nearly k-close for all $k \in \omega$. This immediately gives the first conclusion of the theorem. The second follows by Corollary 2.12. $\square$

The next two theorems show that the continuum hypothesis (or Martin's axiom, or even weaker hypotheses) gives a picture diametrically opposed to the picture in Theorem 3.1 under $\mathfrak{u} < \mathfrak{g}$.

THEOREM 3.2. *Assume $\mathfrak{r} \geq \mathfrak{d}$. Then there is, for any $k \geq 1$, a k-meeting family that is not nearly $k+1$-close.*

Before beginning the proof, we recall some information from [**3**] and relate it to the present situation. Consider two interval partitions (i.e., partitions of ω into finite intervals), Π and Π'. We say that Π *dominates* Π' if each interval in Π, with finitely many exceptions, includes an interval in Π'. It is easy to see (and explicitly shown in [**3**]) that there is a family of $\mathfrak{d}$ interval partitions such that every interval partition is dominated by one from this family.

LEMMA 3.3. *Suppose Π dominates Π'. Then, with finitely many exceptions, every union of two consecutive intervals from Π' is included in the union of two consecutive intervals from Π. Thus, if Π' witnesses that a family $\mathcal{X}$ is nearly k-close for a certain k, then so does Π.*

Proof If the first conclusion failed, then, infinitely often, the union of some two consecutive intervals from Π' would meet three or more consecutive intervals from Π. Then any of these three or more Π-intervals except the first and last would include no entire interval from Π'. This contradicts the assumption that Π dominates Π', so the first conclusion is proved. The second conclusion follows by the definition of what it means to witness near closeness. □

LEMMA 3.4. *Given any natural number $n \geq 2$ and any family $\mathcal{A}$ of fewer than $\mathfrak{r}$ infinite subsets of ω, we can partition ω into n pieces each of which has infinite intersection with each set in $\mathcal{A}$.*

Proof For $n = 2$, this is just the definition of $\mathfrak{r}$. If the result is true for n, then to get it for $n + 1$, first partition ω into n pieces of the desired sort, then choose one of the pieces P, and apply the definition of $\mathfrak{r}$ within P, splitting P into two pieces each meeting infinitely all the sets $P \cap A$ for $A \in \mathcal{A}$. □

Proof of Theorem 3.2 Fix $k \geq 1$, and fix a family $\{\Pi_\alpha : \alpha < \mathfrak{d}\}$ of $\mathfrak{d}$ interval partitions dominating all interval partitions. We shall produce a k-meeting family $\mathcal{X}$ such that no Π_α witnesses that $\mathcal{X}$ is nearly $(k+1)$-close. According to Lemma 3.3, no other partition can witness that $\mathcal{X}$ is nearly $(k + 1)$-close, and so the proof will be complete.

The construction of $\mathcal{X}$ is an induction of length $\mathfrak{d}$. At stage α, we shall put into $\mathcal{X}$ some $k + 1$ sets $A_0^\alpha, \ldots, A_k^\alpha$ such that no two consecutive intervals of Π_α meet all $k + 1$ of them. Thus, stage α will ensure that Π_α does not witness that $\mathcal{X}$ is nearly $(k + 1)$-close. During the construction, we shall take care that every k sets that we put into $\mathcal{X}$ have infinitely many points in common, so the final $\mathcal{X}$ will be k-meeting.

We now describe stage α. Since α will be fixed during this description, we omit it from the notation, writing simply Π for Π_α, and writing A_i for the $k + 1$ sets to be added. We also write I_n for the n^{th} interval of Π. Let $\mathcal{Y}$ be the family of sets already put into $\mathcal{X}$ at earlier stages. We assume, as an induction hypothesis, that $\mathcal{Y}$ is k-meeting. Since only finitely many sets are added to $\mathcal{X}$ at any stage, and since our induction has length $\mathfrak{d}$, we have $|\mathcal{Y}| < \mathfrak{d}$.

To each $(\leq k)$-element subfamily $\mathcal{K} \subseteq \mathcal{Y}$, associate the set $B_\mathcal{K}$ of those $n \in \omega$ such that the intersection of the sets in $\mathcal{K}$ meets the n^{th} interval I_n of Π. Since $\mathcal{Y}$ is k-meeting, each $B_\mathcal{K}$ is infinite. Furthermore, the number of such sets $B_\mathcal{K}$ is, like $|\mathcal{Y}|$, smaller than $\mathfrak{d}$ and therefore, by the hypothesis of the theorem, smaller than $\mathfrak{r}$.

We shall need a co-infinite set $Z \subseteq \omega$ that meets each $B_\mathcal{K}$ infinitely often. Since the number of $B_\mathcal{K}$'s is $< \mathfrak{r}$, we can simply split ω into two pieces that each meet each $B_\mathcal{K}$ infinitely, and then let Z be one of the pieces. (In fact, $\mathfrak{r}$ isn't really relevant here; for any family of $< \mathfrak{c}$ infinite subsets of ω there is a coinfinite set meeting them all infinitely.) Fix such a Z, and view it as the union of some intervals J_n separated by members of $\omega - Z$. Since Z is infinite and co-infinite, the intervals J_n are finite and there are infinitely many of them. Let $C_\mathcal{K}$ be the set of those $n \in \omega$ such that $B_\mathcal{K}$ meets J_n. Our choice of Z ensures that each $C_\mathcal{K}$ is infinite. If we write $\bar{J}_n$ for $\bigcup_{m \in J_n} I_m$ then $C_\mathcal{K}$ is the set of n such that $\bar{J}_n$ meets all of the sets in $\mathcal{K}$.

Since the $C_\mathcal{K}$'s form a family of $< \mathfrak{r}$ infinite subsets of ω, Lemma 3.4 lets us partition ω into $k + 1$ pieces $Z_0, \ldots, Z_k$, each meeting every $C_\mathcal{K}$ infinitely often.

Define, for $0 \leq i \leq k$,

$$A_i = \bigcup_{n \in Z - Z_i} \bar{J}_n.$$

(Note that the union is not over n in Z_i but rather over n in the union of the other k pieces Z_j.) These A_i's are the sets that we add to $\mathcal{X}$ at the current stage of the construction. It remains to verify that they have the required properties.

Consider any two consecutive intervals of Π, say I_m and I_{m+1}; we must check that the A_i do not all meet $I_m \cup I_{m+1}$. Our construction of Z and the J_n ensures that, if m and $m+1$ are both in Z then they are in the same J_n. So we can fix an n such that each of m and $m+1$ is either in J_n or outside Z. Let i be the (unique) index such that $n \in Z_i$, and observe that A_i does not meet $\bar{J}_n$, nor does A_i meet any I_p whose index p is outside Z. It follows that $I_m \cup I_{m+1}$ cannot meet A_i, which is what we wanted to check.

Finally, we must check that the k-meeting property of $\mathcal{Y}$ is preserved when we adjoin $A_0, \dots, A_k$. Any family of k sets from $\mathcal{Y} \cup \{A_0, \dots, A_k\}$ is the union of some $(\leq k)$-element subfamily $\mathcal{K}$ of $\mathcal{Y}$ and a subfamily $\mathcal{A}$ of $\{A_0, \dots, A_k\}$ that omits at least one A_i. Fix such $\mathcal{K}$ and i. Our choice of Z_i ensures that it contains infinitely many elements of $C_{\mathcal{K}}$. For each of these elements n, $\bar{J}_n$ meets the intersection of the sets in $\mathcal{K}$ and is a subset of all the sets in $\mathcal{A}$. Thus, all the sets in $\mathcal{K} \cup \mathcal{A}$ have a common element in $\bar{J}_n$. Since this happens for infinitely many n, the proof is complete. $\square$

The preceding theorem says that under the hypothesis $\mathfrak{r} \geq \mathfrak{d}$, and thus in particular under CH, half of the implications in the chain (1) cannot be reversed. The next theorem does the same for the other half, but it uses a stronger hypothesis, still weaker than CH.

THEOREM 3.5. *Assume $\mathfrak{r} = \mathfrak{c}$. Then there is, for each $k \geq 2$, a k-close family that is not nearly k-meeting.*

Proof Fix k and list all interval partitions of ω as Π_α, indexed by $\alpha < \mathfrak{c}$. We shall construct, in an induction of length $\mathfrak{c}$, a k-close family $\mathcal{X} \subseteq [\omega]^\omega$ that is not nearly k-meeting. At each stage α, we shall ensure that Π_α does not witness that $\mathcal{X}$ is nearly k-meeting by adding to $\mathcal{X}$ some k sets $A_1, \dots, A_k$ that do not all meet any interval of Π_α.

We now describe stage α. Since α will be fixed during this description, we omit it from the notation, writing simply Π for Π_α, and writing A_i for the k sets to be added. We also write I_n for the n^{th} interval of Π. Let $\mathcal{Y}$ be the family of sets already put into $\mathcal{X}$ at earlier stages. We assume, as an induction hypothesis, that $\mathcal{Y}$ is k-close. Since only finitely many sets are added to $\mathcal{X}$ at any stage, and since our induction has length $\mathfrak{c}$, we have $|\mathcal{Y}| < \mathfrak{c}$.

To each $(\leq k)$-element subfamily $\mathcal{K} \subseteq \mathcal{Y}$, associate the set $B_{\mathcal{K}}$ of those $n \in \omega$ such that some two-point interval $[p, p+1]$ with $p \in I_n$ meets all of the sets in $\mathcal{K}$. Since $\mathcal{Y}$ is k-close, each $B_{\mathcal{K}}$ is infinite. Furthermore, the number of such sets $B_{\mathcal{K}}$ is, like $|\mathcal{Y}|$, smaller than $\mathfrak{c}$ and therefore, by the hypothesis of the theorem, smaller than $\mathfrak{r}$.

By Lemma 3.4, partition ω into k sets $Z_1, \dots Z_k$ each meeting each $B_{\mathcal{K}}$ infinitely often. Define

$$A_i = \bigcup_{n \notin Z_i} I_n.$$

These are the k sets to be added to $\mathcal{X}$ at the current stage. We must check that they have the required properties.

It is clear that no I_n meets all the A_i; indeed, if $n \in Z_i$ then I_n is disjoint from A_i.

So it remains to prove that $\mathcal{Y} \cup \{A_1, \dots, A_k\}$ is k-close. Consider therefore any k members of this family. If they are all in $\mathcal{Y}$, then the induction hypothesis gives what we want. If they are the k newly added A_i's then we find infinitely many two-point intervals meeting them all as follows. Since the Z_i's partition ω and are all infinite, there are infinitely many n such that n is in one Z_i and $n+1$ is in a different Z_j. For such an n, the last point p in the interval I_n belongs (like that whole interval) to all the A's except A_i, while $p+1$, being in I_{n+1}, belongs to all the A's except A_j. Thus, each of the A's meets $[p, p+1]$, as required.

Finally, we consider the case of k sets, some of which are from $\mathcal{Y}$ and some from the newly added A's. So these k sets form the union of some ($\leq k$)-element subfamily $\mathcal{K}$ of $\mathcal{Y}$ and some subfamily of $\{A_1, \dots, A_k\}$ that omits at least one A_i. Fix such $\mathcal{K}$ and i. By our construction, there are infinitely many elements n in $Z_i \cap B_{\mathcal{K}}$. For each such n, there is, by definition of $B_{\mathcal{K}}$, some $p \in I_n$ such that $[p, p+1]$ meets all the sets in $\mathcal{K}$. Since I_n is also included in all the A's except A_i, it follows that $[p, p+1]$ meets all the k sets we began with, and so the proof is complete. $\square$

The contrast between the reversibility of all but the last of the implications in (1) under $\mathfrak{u} < \mathfrak{g}$ (Theorem 3.1) and their irreversibility under $\mathfrak{r} = \mathfrak{c}$ (Theorems 3.2 and 3.5) suggests questions about partial reversibility. For example, the referee asked about the consistency of "there is a 7-meeting family that is not 8-close but every 5-meeting family is 6-close." I do not know the answers to such questions. I also do not know whether the hypothesis $\mathfrak{r} = \mathfrak{c}$ in Theorem 3.5 can be replaced by the weaker hypothesis $\mathfrak{r} \geq \mathfrak{d}$. For example, in the Sacks model, where $\mathfrak{r} = \mathfrak{d} = \aleph_1 < \mathfrak{c} = \aleph_2$, is every k-close family nearly k-meeting?

4. Unsplit Families

In this section, we investigate what happens to the previous results under the additional hypothesis that the family $\mathcal{X}$ is unsplit. The first result shows that most of the chain (1) collapses.

THEOREM 4.1. *Any 3-meeting unsplit family is an ultrafilter base, modulo finite sets.*

Proof Let $\mathcal{X}$ be a 3-meeting unsplit family. It suffices to show that it is a filter base modulo finite sets, i.e., that the intersection of any two members of $\mathcal{X}$ almost includes another member of $\mathcal{X}$. The "ultra" part of the conclusion then follows immediately from the "unsplit" assumption.

So let $A, B \in \mathcal{X}$. Since $\mathcal{X}$ is unsplit, it contains some C that is either almost included in $A \cap B$ or almost disjoint from $A \cap B$. The former alternative is what we want. The latter is absurd as A, B, C would be a counterexample to the "3-meeting" assumption. $\square$

COROLLARY 4.2. *For unsplit families, 3-meeting implies k-meeting, and therefore nearly 3-meeting implies nearly k-meeting, for all k.*

With a little more work, we can collapse one more implication in the chain (1).

THEOREM 4.3. *For unsplit families, nearly 3-close implies nearly 4-close.*

Proof It suffices to show that every 3-close unsplit family is nearly 4-close, for if we are given an unsplit family $\mathcal{X}$ that is only nearly 3-close then we can work instead with a 3-close family of the form $f(\mathcal{X})$.

So assume $\mathcal{X}$ is unsplit and 3-close. We shall show that the partition Π of ω into two-point intervals $[2n, 2n+1]$ witnesses that $\mathcal{X}$ is nearly 4-close. Assume, toward a contradiction, that we have four sets $A, B, C, D \in \mathcal{X}$ such that only finitely many unions $[2n, 2n+3]$ of two consecutive intervals of Π meet all four of A, B, C, D.

Let P be the union of those two-point intervals $[p, p+1]$ that meet both A and B, and similarly let Q be the union of those two-point intervals $[q, q+1]$ that meet both C and D. Then P and Q are disjoint, for if n were in both then the 3-point interval $[n-1, n+1]$ would meet all four of A, B, C, D, and that is impossible as every 3-point interval is included in a 4-point interval starting at an even number, i.e., in one of the intervals $[2n, 2n+3]$ that we assumed do not meet all of A, B, C, D.

Since $\mathcal{X}$ is unsplit, it must contain some set E almost disjoint from one of P and Q. But if $E \cap P$ is finite, then there are only finitely many 2-point intervals meeting all three of A, B, E, and similarly if $E \cap Q$ is finite, then there are only finitely many 2-point intervals meeting all three of C, D, E. In either case, we have contradicted the assumption that $\mathcal{X}$ is 3-close, and so the proof is complete. □

The following corollary combines the preceding two theorems.

COROLLARY 4.4. *For unsplit families, "nearly 3-close" is equivalent to all the earlier properties in the chain* (1) *of implications and to "ultrafilter base modulo finite".*

Thus, for unsplit families, the implication chain (1) collapses to

(2) ultrafilter base mod finite $\iff$ nearly 3-close $\implies$
nearly 2-meeting $\implies$ nearly 2-close $\implies$ nearly 1-meeting

The next corollary is our amplification of the result from [**1**] that $\mathfrak{r} \geq \min\{\mathfrak{u}, \mathfrak{d}\}$.

COROLLARY 4.5. *If $\mathcal{X}$ is unsplit, then either there is a finite-to-one monotone $f : \omega \to \omega$ such that $f(\mathcal{X})$ is an ultrafilter base modulo finite sets, or the family $\{next(X, -) : X \in \mathcal{X}\}$ is 3-dominating.*

Proof If $\mathcal{X}$ is nearly 3-close, then we have the first alternative in the conclusion, by Corollary 4.4. If $\mathcal{X}$ is not nearly 3-close, then we have the second alternative by Proposition 2.6. □

5. Consistency Results for Unsplit Families

In this section we consider the question of reversing the arrows in the shorter chain (2) for unsplit families. Our old counterexample, $[\omega]^\omega$, to the reversal of the last arrow still works in the present situation, as $[\omega]^\omega$ is obviously unsplit. Under the assumption $\mathfrak{u} < \mathfrak{g}$, the two implications other than the last are reversible by Theorem 3.1 even without the "unsplit" assumption. The question that remains is whether these two implications are reversible in ZFC. We shall show that they are not; in particular the reversals fail under CH.

PROPOSITION 5.1. *Any 2-meeting family can be enlarged to a 2-meeting unsplit family.*

Proof Given a 2-meeting family, use Zorn's lemma to enlarge it to a maximal 2-meeting family $\mathcal{X}$. We show that $\mathcal{X}$ is unsplit. So let any potential splitting set

$S \subseteq \omega$ be given. If $S \in \mathcal{X}$ then S doesn't split $\mathcal{X}$ because it doesn't split itself. So we may assume that $S \notin \mathcal{X}$ and therefore, by maximality, $\mathcal{X} \cup \{S\}$ is not 2-meeting. This means, since $\mathcal{X}$ is 2-meeting, that some $X \in \mathcal{X}$ is almost disjoint from S. But then S again fails to split $\mathcal{X}$. □

COROLLARY 5.2. *If $\mathfrak{r} \geq \mathfrak{d}$ then there is an unsplit, 2-meeting family that is not nearly 3-close.*

Proof By Theorem 3.2, there is a 2-meeting family that is not nearly 3-close. By Proposition 5.1, we can enlarge this family to become unsplit, while keeping it 2-meeting. Enlargement obviously preserves the property of not being nearly 3-close, so the proof is complete. □

COROLLARY 5.3. *Whether "3-dominating" in Corollary 4.5 can be improved to "2-dominating" is independent of ZFC. The improvement is correct if $\mathfrak{u} < \mathfrak{g}$ but incorrect if $\mathfrak{r} \geq \mathfrak{d}$.*

THEOREM 5.4. *If $\mathfrak{r} = \mathfrak{c}$ then there is an unsplit, 2-close family that is not nearly 2-meeting.*

Proof We use the proof of Theorem 3.5, for $k = 2$, with some extra work inserted into the inductive construction of the family $\mathcal{X}$ to ensure that it is unsplit. Let Π_α be as in that earlier proof. List all subsets of ω as S_α, indexed by $\alpha < \mathfrak{c}$. As in the proof of Theorem 3.5, we construct $\mathcal{X}$ in an inductive process of length $\mathfrak{c}$, ensuring at stage α that (as before) Π_α does not witness that $\mathcal{X}$ is nearly 2-meeting, and also (the new work) that S_α does not split $\mathcal{X}$. As before, we shall add only finitely many sets to $\mathcal{X}$ at each stage, and we shall take care that $\mathcal{X}$ remains 2-close.

We now describe stage α, omitting the subscripts α as before. Letting $\mathcal{Y}$ be the part of $\mathcal{X}$ already constructed, known to be 2-close by induction hypothesis, begin by adding A_0 and A_1 exactly as before to ensure that Π will not witness that $\mathcal{X}$ is 2-meeting. Let $\mathcal{Y}'$ be the resulting family of sets, still 2-close by construction. Next, we ensure that S will not split $\mathcal{X}$ by putting either S or its complement $\omega - S$ into $\mathcal{X}$. The choice of which set to put into $\mathcal{X}$ is dictated by the need to keep $\mathcal{X}$ 2-close (as in the earlier proof). So we must ensure that the newly added set, S or $\omega - S$, and any set Y previously added both meet infinitely many 2-point intervals. To complete the proof, we suppose that neither choice for the newly added set works, and we deduce a contradiction as follows.

Since adding S doesn't work, there is $Y_1 \in \mathcal{Y}$ such that only finitely many 2-point intervals meet both S and Y_1. Similarly, there is $Y_2 \in \mathcal{Y}$ such that only finitely many 2-point intervals meet both $\omega - S$ and Y_2. Since $\mathcal{Y}'$ is 2-close, find infinitely many 2-point intervals $[n, n+1]$ meeting both Y_1 and Y_2. If infinitely many of these n's are in S, then those 2-point intervals $[n, n+1]$ contradict our choice of Y_1. If not, then infinitely many of these n's are in $\omega - S$ and we similarly contradict our choice of Y_2. □

References

[1] Jason Aubrey, *Pseudo-Dominating Families of Functions*, Ph.D. thesis, University of Michigan (2003).

[2] Andreas Blass, "Nearly adequate sets," in *Logic and Algebra*, ed. Y. Zhang, Contemporary Mathematics 302 (2002) 33–48.

[3] Andreas Blass, "Combinatorial cardinal characteristics of the continuum," to appear in *Handbook of Set Theory*, ed. M. Foreman, A. Kanamori, and M. Magidor.

[4] Andreas Blass and Claude Laflamme, "Consistency results about filters and slenderness classes of groups," *J. Symbolic Logic* 64 (1989) 50–56.
[5] Erik K. van Douwen, "The integers and topology," in *Handbook of Set Theoretic Topology* eds. K. Kunen and J. Vaughan, North-Holland (1984) 111–167.
[6] Martin Goldstern and Saharon Shelah, "Ramsey ultrafilters and the reaping number — Con($\mathfrak{r} < \mathfrak{u}$)," *Ann. Pure Appl. Logic* 49 (1990) 121–142.
[7] Claude Laflamme, "Equivalence of families of functions on the natural numbers," *Trans. Amer. Math. Soc.* 330 (1992) 307–319.
[8] Jerry Vaughan, "Small uncountable cardinals and topology," in *Open Problems in Topology* eds. J. van Mill and G. Reed, North-Holland (1990) 195–218.

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