VOTING RULES FOR INFINITE SETS AND BOOLEAN ALGEBRAS

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Abstract. A voting rule in a Boolean algebra B is an upward closed subset that contains, for each element $x \in B$, exactly one of x and $\neg x$. We study several aspects of voting rules, with special attention to their relationship with ultrafilters. In particular, we study the set-theoretic hypothesis that all voting rules in the Boolean algebra of subsets of the natural numbers modulo finite sets are nearly ultrafilters. We define the notion of support of a voting rule and use it to describe voting rules that are, in a sense, as different as possible from ultrafilters. Finally, we consider how much of the axiom of choice is needed to guarantee the existence of voting rules.

1. Introduction

Consider a set V of voters who must choose between two alternatives. When the number of voters is finite and odd, majority rule is the natural way to combine the individual preferences into an overall result. (Because we consider only two alternatives, we do not encounter Arrow's impossibility theorem [1] here. It and related results like the Gibbard-Satterthwaite theorem [4, 13] depend on the presence of at least three alternatives.) When the number of voters is even or infinite, there is no similarly fair voting rule, but there are plenty of possibilities if we require less (or no) fairness. We shall be concerned in this paper with voting rules that treat the two alternatives symmetrically (i.e., fairly) but may treat different voters differently (i.e., unfairly). The following definition is intended to express minimal reasonable assumptions about such a voting rule.

Definition 1. A *voting rule* on a set V is a family $\mathcal{W}$ of subsets of V, called the *winning coalitions*, which is

Monotonic: if $X \subseteq Y \subseteq V$ and $X \in \mathcal{W}$, then $Y \in \mathcal{W}$ and
Decisive: if $X \subseteq V$ then either $X \in \mathcal{W}$ or $V - X \in \mathcal{W}$ but not both.

The idea behind the definition is that, if X is the set of voters preferring a particular one of the two alternatives, and if no voters are undecided so $V - X$ is the set preferring the other alternative, then the first alternative wins the vote if and only if $X \in \mathcal{W}$. The requirement of decisiveness ensures that there is a single winner. The requirement of monotonicity means that a winner will not become a loser by attracting more voters.

A more precise terminology would be "binary, symmetric, voting rule" to indicate that the vote is always between just two alternatives and that these alternatives are treated the same way. As we will not have occasion to consider any non-binary or non-symmetric rules, we omit the extra adjectives.

Remark 2. Monotone families of sets occur naturally in many contexts and, as a result, have acquired many names. They have been called simple games [18], quantifiers (though some authors use "quantifier" for not necessarily monotone families), access structures, updeals, and semifilters. The additional requirement of decisiveness has also been expressed as "constant sum" and as "self-dual." Chapter 1 of [18] describes many properties of voting rules, or more generally of simple games, and constructions on them.

Among the voting rules are the *ultrafilters*, which can be defined as voting rules $\mathcal{W}$ that are closed under finite intersections. A major theme of this paper is the connection (or lack thereof) between ultrafilters and more general voting rules. In particular, we shall consider in Section 3 the question of characterizing sets of the form $\{X \cap Y : X, Y \in \mathcal{W}\}$. If $\mathcal{W}$ is an ultrafilter, this family is just $\mathcal{W}$ itself, but for other voting rules it can be much larger. In Section 4, we discuss the possibility that all infinitary voting rules on a countable set are nearly ultrafilters (in a sense to be made precise). We also compare in Section 5 the uses of the axiom of choice needed to produce voting rules with those needed to produce ultrafilters.

2. Basic Facts

The definition of voting rules can be generalized by replacing the subsets of V with the elements of an arbitrary Boolean algebra $\mathcal{B}$.

Definition 3. A *voting rule* in a Boolean algebra $\mathcal{B}$ is a set $\mathcal{W} \subseteq \mathcal{B}$ that is

- **Monotonic:** if $x \leq y$ in $\mathcal{B}$ and $x \in \mathcal{W}$, then $y \in \mathcal{W}$ and
- **Decisive:** for every $x \in \mathcal{B}$, either $x \in \mathcal{W}$ or $\neg x \in \mathcal{W}$ but not both.

Thus, a voting rule *on* a set V is a voting rule *in* the Boolean algebra $\mathcal{P}(V)$ of subsets of V.

We shall often be interested in those voting rules on an infinite set V such that no finite set of voters ever makes a difference.

Definition 4. A voting rule $\mathcal{W}$ on V is *infinitary* if, whenever F is finite and $X \cup F \in \mathcal{W}$ then $X \in \mathcal{W}$.

Observe that an infinitary voting rule on V amounts to a voting rule in the Boolean algebra $\mathcal{P}(V)/\text{fin}$ obtained by dividing $\mathcal{P}(V)$ by the ideal of finite subsets of V.

Definition 5. Let $\mathcal{B}$ be a Boolean algebra and k a positive integer. A subset $\mathcal{L}$ of $\mathcal{B}$ is *k-linked* if each k elements of $\mathcal{L}$ have nonempty intersection.

Definition 6. An element s of a Boolean algebra $\mathcal{B}$ *splits* another element b if both $b \wedge s$ and $b \wedge \neg s$ are not 0. A family $\mathcal{R} \subseteq \mathcal{B}$ is *unsplit* if no single $s \in \mathcal{B}$ splits all elements of $\mathcal{R}$.

Proposition 7. *For a subset $\mathcal{W}$ of a Boolean algebra $\mathcal{B}$, the following statements are equivalent.*

(1) *$\mathcal{W}$ is a voting rule.*
(2) *$\mathcal{W}$ is the upward closure of a 2-linked and unsplit family $\mathcal{L} \subseteq \mathcal{B}$.*
(3) *$\mathcal{W}$ is a maximal 2-linked family in $\mathcal{B}$.*

Proof. A voting rule is 2-linked because, if it contains a and b then it cannot contain $\neg a$ by decisiveness, and so b cannot be $\leq \neg a$ by monotonicity. Furthermore, a voting rule is unsplit, because if s split it then neither s nor $\neg s$ would be in it, contrary to decisiveness. Since voting rules are closed upward (monotonicity), we have that (1) $\implies$ (2).

To prove (2) $\implies$ (3), assume that $\mathcal{W}$ and $\mathcal{L}$ are as in (2). Then $\mathcal{W}$ is 2-linked, because this property is clearly preserved by upward closure. It remains to show that, if $\mathcal{W} \cup \{s\}$ is 2-linked, then s is already in $\mathcal{W}$. We have that $b \wedge s \neq 0$ for all $b \in \mathcal{W}$, and so, a fortiori, for all $b \in \mathcal{L}$. We must not also have that, for all $b \in \mathcal{L}$, $b \wedge \neg s \neq 0$, lest s split all elements of $\mathcal{L}$. So there is $b \in \mathcal{L}$ with $b \wedge \neg s = 0$, which means that $b \leq s$. As $b \in \mathcal{L}$, we have $s \in \mathcal{W}$, as required.

Finally, we assume (3) and prove (1) by verifying monotonicity and decisiveness. For monotonicity, suppose, toward a contradiction, that $x \leq y$, $x \in \mathcal{W}$, but $y \notin \mathcal{W}$. By maximality, $\mathcal{W} \cup \{y\}$ is not 2-linked, so there is $w \in \mathcal{W}$ with $w \wedge y = 0$. But then $w \wedge x = 0$, contrary to the assumption that $\mathcal{W}$ is 2-linked. To prove decisiveness, notice first that we cannot have both $x \in \mathcal{W}$ and $\neg x \in \mathcal{W}$ as $\mathcal{W}$ is 2-linked. Finally, suppose, toward a contradiction, that $\mathcal{W}$ contains neither x nor $\neg x$. Then, by maximality, neither $\mathcal{W} \cup \{x\}$ nor $\mathcal{W} \cup \{\neg x\}$ is 2-linked. So there are $a, b \in \mathcal{W}$ with $a \wedge x = b \wedge \neg x = 0$. But then $a \le \neg x$ and $b \le x$, and so $a \wedge b = 0$ and $\mathcal{W}$ fails to be 2-linked. This contradiction completes the proof. □

Item 3 of the proposition and Zorn's lemma immediately give the following existence theorem for voting rules.

Corollary 8. *Every 2-linked family in a Boolean algebra is included in a voting rule.*

We note that the mere existence of voting rules, in any non-degenerate Boolean algebra, is a consequence of the existence of ultrafilters, since every ultrafilter is clearly a voting rule. Indeed, inspection of definitions reveals that ultrafilters are the same thing as voting rules closed under binary intersection. The corollary, however, gives voting rules that are not ultrafilters, once the Boolean algebra is bigger than the four-element algebra $\mathcal{P}(2)$. Indeed, once $\mathcal{B}$ has three disjoint elements $a, b, c \neq 0$, it has a 2-linked family $\{a \vee b, a \vee c, b \vee c\}$ that is not included in any ultrafilter (because it is not 3-linked, whereas ultrafilters are k-linked for all k).

In view of this example, showing that voting rules can be quite different from ultrafilters (and examples later in the paper, showing a larger difference), it is amusing to observe that ultrafilters can be characterized by a seemingly minor change in item 2 of Proposition 7.

Proposition 9. *Ultrafilters in a Boolean algebra are exactly the upward closures of unsplit 3-linked families.*

Proof. For the non-obvious direction, suppose that $\mathcal{L}$ is unsplit and 3-linked, and let $\mathcal{U}$ be its upward closure. We know from Proposition 7 that $\mathcal{U}$ is a voting rule; it therefore remains only to check that $\mathcal{U}$ contains $a \wedge b$ whenever it contains a and b.

Suppose, therefore, that $a, b \in \mathcal{U}$; so there are $x, y \in \mathcal{L}$ with $x \le a$ and $y \le b$. Since $a \wedge b$ doesn't split $\mathcal{L}$, find $z \in \mathcal{L}$ such that either $z \le a \wedge b$ or $z \wedge a \wedge b = 0$. The latter case is impossible, for it would give $z \wedge x \wedge y = 0$ whereas we assumed that $\mathcal{L}$ is 3-linked. So we must have $z \le a \wedge b$ and $a \wedge b \in \mathcal{U}$, as required. □

We close this section with the observation that, if $h : \mathcal{B}_2 \to \mathcal{B}_1$ is a homomorphism of Boolean algebras and $\mathcal{W}$ is a voting rule in $\mathcal{B}_1$, then its pre-image $h^{-1}(\mathcal{W})$ is a voting rule in $\mathcal{B}_2$. In particular, if $f : V_1 \to V_2$ is any function, then we have an induced homomorphism (in the opposite direction) of power set algebras $f^{-1} : \mathcal{P}(V_2) \to \mathcal{P}(V_1)$ and thus any voting rule $\mathcal{W}$ on V_1 gives rise to a voting rule

$$\{X \subseteq V_2 : f^{-1}(X) \in \mathcal{W}\},$$

which logically should be called $(f^{-1})^{-1}(\mathcal{W})$ but which in fact is called $f(\mathcal{W})$. (The order relation defined by $f(\mathcal{W}) \leq \mathcal{W}$ for all $\mathcal{W}$ and f is called the Rudin-Keisler ordering in ultrafilter theory, and this terminology is extended in [18, Section 1.3] to voting rules.)

In terms of voting, we can think of V_1 as a set of actual voters, partitioned into blocs such that each bloc always votes together, for a single alternative. V_2 indexes the blocs, and f maps each voter to the bloc he belongs to. Then an election, conducted according to $\mathcal{W}$ among the voters in V_1 is equivalent (because each bloc votes in unison) to an election conducted among the blocs, according to $f(\mathcal{W})$.

3. Critical Sets

The notion of a critical set of voters is intended to formalize the intuitive notion of a set which, voting as a bloc, *might* hold the balance of power in an election. Here "might" means that the bloc's power depends on the other voters' choices. Although we shall give simpler characterizations below (Proposition 11), we begin with a definition designed to capture this "balance of power" concept. We formulate the definition for arbitrary Boolean algebras.

Definition 10. Let $\mathcal{W}$ be a voting rule in a Boolean algebra $\mathcal{B}$. An element $c \in \mathcal{B}$ is *critical* for $\mathcal{W}$ if there are elements $a, b \in \mathcal{B}$, disjoint from c and from each other, such that $a \vee c$ and $b \vee c$ are both in $\mathcal{W}$.

Notice that the notion of critical would be unchanged if, in the definition, we required $a \vee b \vee c = 1$, so that $\{a, b, c\}$ is a partition of 1 (except that a or b could be 0). With this additional requirement, and interpreted in a power set algebra $\mathcal{P}(V)$, our notion of critical corresponds to the "balance of power" description given above. A set C of voters is critical if the complementary set $V - C$ can be partitioned into A and B, with the members of A voting for one alternative and those of B for the other, so that if all members of C vote for a single alternative then that alternative wins. Thus, under some circumstances, namely when the other voters split as A versus B, if C votes as a bloc, it controls the outcome of the vote.

Clearly, every $c \in \mathcal{W}$ is critical for $\mathcal{W}$; we can take $a = b = 0$ in the definition. It will follow from the next proposition that the converse characterizes ultrafilters.

Proposition 11. *Let $\mathcal{W}$ be a voting rule in a Boolean algebra $\mathcal{B}$. For any $c \in \mathcal{B}$, the following are equivalent.*

(1) *c is critical for $\mathcal{W}$.*
(2) *c is the meet of two elements of $\mathcal{W}$.*

(3) *c is $\geq$ the meet of two elements of $\mathcal{W}$.*

Proof. The equivalence of 2 and 3 is immediate from the monotonicity of $\mathcal{W}$. If c is critical, then let a and b be as in the definition of critical, and observe that $a \vee c$ and $b \vee c$ are elements of $\mathcal{W}$ whose meet is c. Conversely, if c is the meet of elements $x, y \in \mathcal{W}$, then $x \wedge \neg c$ and $y \wedge \neg c$ serve as the a and b in the definition of critical. ☐

Corollary 12. *Ultrafilters are exactly those voting rules $\mathcal{W}$ for which only the elements of $\mathcal{W}$ are critical for $\mathcal{W}$.*

Proof. In view of the proposition, "only the elements of $\mathcal{W}$ are critical for $\mathcal{W}$" means exactly that $\mathcal{W}$ is closed under binary intersections. ☐

The following proposition formalizes the intuition that a non-critical set of voters has no influence at all on an election. Whether a coalition wins is unchanged if it gains or loses any votes from a non-critical set.

Proposition 13. *Let $\mathcal{W}$ be a voting rule in a Boolean algebra $\mathcal{B}$. An element $c \in \mathcal{B}$ is not critical if and only if, for all $x \in \mathcal{B}$, and all y with $x \wedge \neg c \leq y \leq x \vee c$,*

$$x \in \mathcal{W} \iff y \in \mathcal{W}.$$

Notice that the condition $x \wedge \neg c \leq y \leq x \vee c$, expressing the idea that y is obtained from x by adding or removing parts of c, is equivalent to $x \triangle y \leq c$ (where $\triangle$ means symmetric difference) and to $x \vee c = y \vee c$. These reformulations make it obvious that the condition is symmetrical between x and y.

Proof. Suppose first that c is critical, and let a and b be as in the definition. Then $x = a \vee c$ and $y = a$ satisfy $x \wedge \neg c \leq y \leq x \vee c$, yet $x \in \mathcal{W}$ and $y \notin \mathcal{W}$ (the latter because y is disjoint from $b \vee c \in \mathcal{W}$).

Conversely, suppose that x and y satisfy $x \wedge \neg c \leq y \leq x \vee c$, yet $\mathcal{W}$ contains exactly one of x and y. Thanks to the symmetry noted above, we may assume that $\mathcal{W}$ contains x but not y. Then $a = x \wedge \neg c$ and $b = (\neg x) \wedge (\neg c)$ witness that c is critical. Indeed, a, b, c are obviously pairwise disjoint. $a \vee c \geq x \in \mathcal{W}$, and $b \vee c \geq (\neg x) \vee c \geq (\neg y) \in \mathcal{W}$, where the last inequality uses that $x \wedge \neg c \leq y$. ☐

Corollary 14. *Let $\mathcal{W}$ be a voting rule in a Boolean algebra $\mathcal{B}$. The non-critical elements of $\mathcal{B}$ constitute an ideal in $\mathcal{B}$. That is,*

(1) *if c is non-critical then so is every $a \leq c$, and*
(2) *if c_1 and c_2 are non-critical, then so is $c_1 \vee c_2$.*

Proof. Assertion (1) follows immediately from part (3) of Proposition 11. For assertion (2), suppose $x \triangle y \leq c_1 \vee c_2$ where neither c_1 nor

c_2 is critical, and suppose that $x \in \mathcal{W}$; we must show $y \in \mathcal{W}$. Write d_i for $c_i \wedge (x \triangle y)$, so $x \triangle y = d_1 \vee d_2$ and therefore $y = x \triangle (d_1 \vee d_2)$. Let $z = x \triangle d_1$. Then $x \triangle z = d_1 \leq c_1$ and therefore $z \in \mathcal{W}$. Furthermore,

$$z \triangle y = x \triangle d_1 \triangle x \triangle (d_1 \vee d_2) = d_1 \triangle (d_1 \vee d_2) \leq d_2 \leq c_2,$$

and so $y \in W$. □

Definition 15. The *non-critical ideal* of a voting rule in $\mathcal{B}$ is the ideal of all non-critical elements of $\mathcal{B}$. The dual filter is called the *support filter*, and its elements are called supports of the voting rule.

In view of Proposition 13, if s is a support of a voting rule $\mathcal{W}$ in $\mathcal{B}$ then to tell whether some x belongs to $\mathcal{W}$ it is enough to know whether $x \wedge s \in \mathcal{W}$. In particular, $\mathcal{W}$ is completely determined by the elements $\leq s$ that it contains. This is the reason for the terminology "support".

The characterization of criticality in Proposition 11 immediately gives the following characterization of supports.

Corollary 16. *Let $\mathcal{W}$ be a voting rule in $\mathcal{B}$. An element $s \in \mathcal{B}$ is a support of $\mathcal{W}$ if and only if every intersection of two elements of $\mathcal{W}$ has non-zero intersection with s.*

It follows that the supports of $\mathcal{W}$ are those s such that the "restriction" of $\mathcal{W}$ to s, namely

$$\mathcal{W} \restriction s = \{a \wedge s : a \in \mathcal{W}\},$$

is a voting rule in the Boolean algebra $\mathcal{B} \restriction s = \{x \in \mathcal{B} : x \leq s\}$.

Which filters can occur as the support filter of a voting rule? Our earlier remarks imply that if $\mathcal{W}$ is an ultrafilter then its support filter is $\mathcal{W}$ itself. The next two propositions charcterize the support filters that are very close to being ultrafilters in the sense that they are the intersections of finitely many ultrafilters.

Proposition 17. *The support filter of a voting rule is never the intersection of two distinct ultrafilters.*

Proof. Suppose, toward a contradiction, that the support filter of $\mathcal{W}$ is the intersection of two distinct ultrafilters $\mathcal{U}$ and $\mathcal{V}$. Then the critical elements are those that are in at least one of these ultrafilters. Fix some $x \in \mathcal{U} - \mathcal{V}$. Being in $\mathcal{U}$, x is critical, so it is the meet of two elements of $\mathcal{W}$, at least one of which — call it y — is outside $\mathcal{V}$ (as $x \notin \mathcal{V}$). Similarly, $\neg x$, being in $\mathcal{V}$, is critical, so it is the meet of two elements of $\mathcal{W}$, at least one of which — call it z — is outside $\mathcal{U}$ (as $(\neg x) \notin \mathcal{U}$). Then $y \wedge z$, as the meet of two sets from $\mathcal{W}$, is critical. Yet it is in neither of $\mathcal{U}$ and $\mathcal{V}$ (because $y \notin \mathcal{V}$ and $z \notin \mathcal{U}$). □

Proposition 18. *Let k be an integer ≥ 3. Any intersection of k distinct ultrafilters, in any Boolean algebra $\mathcal{B}$, is the support filter of a voting rule in $\mathcal{B}$.*

Proof. Let $\mathcal{U}_1, \dots, \mathcal{U}_k$ be distinct ultrafilters in $\mathcal{B}$, and notice, for future reference, that, given any subset S of $\{1, \dots, k\}$, there is an $x \in \mathcal{B}$ that belongs to $\mathcal{U}_i$ for all $i \in S$ and for no $i \notin S$. Indeed, if we choose, for each pair $i \neq j$, an element $y_{ij} \in \mathcal{U}_i - \mathcal{U}_j$, then

$$x = \bigvee_{i \in S} \bigwedge_{j \notin S} y_{ij}$$

does the job.

We treat first the case that k is odd, say $k = 2r - 1$. In this case, define $\mathcal{W}$ by a majority vote among the given ultrafilters; that is,

$$\mathcal{W} = \{x \in \mathcal{B} : |\{i : x \in \mathcal{U}_i\}| \geq r\}$$

This is clearly a voting rule. If $x, y \in \mathcal{W}$ then the sets $\{i : x \in \mathcal{U}_i\}$ and $\{i : x \in \mathcal{U}_i\}$ must have at least one element i in common, as each of them contains at least r of the $2r - 1$ possible values of i. For such a common element i, we have that $\mathcal{U}_i$ contains both x and y and therefore also $x \wedge y$. Thus, every element critical for $\mathcal{W}$, being $x \wedge y$ for some $x, y \in \mathcal{W}$ by Proposition 11, is in at least one $\mathcal{U}_i$.

Conversely, if $z \in \mathcal{U}_i$ for some i, then z is critical for $\mathcal{W}$. To see this, apply the observation at the beginning of this proof to obtain some $a \in \mathcal{B}$ that belongs to exactly r of the $\mathcal{U}_j$'s, including $\mathcal{U}_i$. Then $a \in \mathcal{W}$ but $a \wedge \neg z \notin \mathcal{W}$.

Thus, the set of critical elements is the union of the k given ultrafilters. It follows that the filter of supports is the intersection of these ultrafilters.

It remains to consider the case that k is even, say $k = 2r$. We again define $\mathcal{W}$ by a majority vote among the $\mathcal{U}_i$'s, but we must add a tie-breaker clause in case there is no majority. We use the very simple tie-breaker that gives the first ultrafilter a little extra voting power. That is, let

$$\mathcal{W} = \{x : |\{i : x \in \mathcal{U}_i\}| > r\} \cup \{x : |\{i : x \in \mathcal{U}_i\}| = r \text{ and } x \in \mathcal{U}_1\}.$$

Again, any $x, y \in \mathcal{W}$ must belong to at least one $\mathcal{U}_i$ simultaneously. This is clear by the same counting argument as for odd k if at least one of x and y is in at least $r + 1$ of the ultrafilters; in the contrary case, both are in $\mathcal{U}_1$. Therefore, as before, all critical elements are in at least one of the ultrafilters.

The converse is proved just as in the case of k odd except that a should be chosen so that the r ultrafilters that contain it include not only $\mathcal{U}_i$ but also $\mathcal{U}_1$. (This is where we need that $k \geq 3$, so that $r > 1$.) □

It seems more difficult to determine whether a filter can be a support filter if it is not the intersection of finitely many ultrafilters. We give some partial results, beginning with an example where the support filter is as far as posisble from being an ultrafilter.

Proposition 19. *On any set V of cardinality different from 2, there is a voting rule for which all nonempty subsets of V are critical. Thus the support filter consists of only the set V.*

Proof. For finite V, apply Proposition 18 to the set of all ultrafilters on V (which are automatically principal), to get a voting rule for which the critical sets are those that are in at least one of these ultrafilters, i.e., all the nonempty subsets of V.

From now on, suppose V is infinite; let κ be its cardinality. A well-known theorem of Hausdorff [5] provides a family of 2^κ independent subsets of κ. That is, we have sets $A_i \subseteq V$, for $i < 2^\kappa$, such that, whenever F and G are disjoint, finite subsets of 2^κ,

$$\bigcap_{i\in F} A_\alpha \cap \bigcap_{j\in G} (V - A_\alpha) \neq \varnothing.$$

Let all the nonempty subsets of V be enumerated in a sequence of length 2^κ, and let X_α denote the α^{th} set in this enumeration.

The family

$$\mathcal{L} = \{X_\alpha \cup A_\alpha, X_\alpha \cup (V - A_\alpha) : \alpha < 2^\kappa\}$$

is 2-linked in the power set $\mathcal{P}(V)$. Indeed, if two elements of $\mathcal{X}$ come from the same α, i.e., if they are $X_\alpha \cup A_\alpha$ and $X_\alpha \cup (V - A_\alpha)$, then their intersection is $X_\alpha \neq \varnothing$, while if they come from different α and β then their intersection is nonempty because A_α and A_β are independent.

By Zorn's lemma, extend $\mathcal{L}$ to a maximal 2-linked family, i.e., to a voting rule $\mathcal{W}$. We saw above that every X_α occurs as the intersection of two sets, $X_\alpha \cup A_\alpha$ and $X_\alpha \cup (V - A_\alpha)$, that are in $\mathcal{W}$. Thus, all nonempty subsets of V are critical for $\mathcal{W}$. □

The proof of Proposition 19 can be applied in numerous other contexts. For example, the definition of independence easily implies that the intersections $\bigcap_{i\in F} A_\alpha \cap \bigcap_{j\in G}(V - A_\alpha)$ in the proof are not only nonempty but infinite. Taking (X_α) to enumerate the infinite subsets of V instead of all the nonempty subsets, we can repeat the proof in the Boolean algebra $\mathcal{P}(V)/\text{fin}$, obtaining the following variant of the proposition.

Corollary 20. *On any infinite set V there is an infinitary voting rule for which all infinite subsets of V are critical. Thus the support filter consists of only the cofinite subsets of V.*

To state our most general result in this direction, we need two definitions.

Definition 21. A subset $\mathcal{I}$ of a Boolean algebra $\mathcal{B}$ is *pairwise independent* if it has at least two elements and, for every two distinct elements $x, y \in \mathcal{I}$, all four of $x \wedge y$, $x \wedge (\neg y)$, $(\neg x) \wedge y$, and $(\neg x) \wedge (\neg y)$ are non-zero.

This is the special case of independence (as defined in the proof of Proposition 19 but in general Boolean algebras) where instead of arbitrary finite F and G we use only those with $|F \cup G| = 2$. Another way to view it is that $\mathcal{I} \cup \{\neg x : x \in \mathcal{I}\}$ is 2-linked apart from the obvious exception that $x \wedge (\neg x) = 0$.

Definition 22. A set $\mathcal{D}$ of non-zero elements in a Boolean algebra $\mathcal{B}$ is *dense* if its upward closure contains all non-zero elements of $\mathcal{B}$.

Proposition 23. *Let $\mathcal{F}$ be a filter in the Boolean algebra $\mathcal{B}$. Suppose the quotient algebra $\mathcal{B}/\mathcal{F}$ has a pairwise independent subset $\mathcal{I}$ and a dense subset $\mathcal{D}$ with $|\mathcal{I}| \geq |\mathcal{D}|$. Then there is a voting rule $\mathcal{W}$ in $\mathcal{B}$ whose support filter is $\mathcal{F}$.*

Proof. It suffices to find a voting rule $\mathcal{W}'$ in $\mathcal{B}/\mathcal{F}$ for which every non-zero element of $\mathcal{B}/\mathcal{F}$ is critical. Indeed, once we have such a $\mathcal{W}'$, we can obtain the required $\mathcal{W}$ as its pre-image under the quotient projection from $\mathcal{B}$ to $\mathcal{B}/\mathcal{F}$.

To get $\mathcal{W}'$, we proceed essentially as in the proof of Proposition 19. Let $\kappa = |\mathcal{I}|$, enumerate $\mathcal{I}$ (without repetitions) as $(a_\alpha)_{\alpha<\kappa}$, and, since $|\mathcal{D}| \leq \kappa$, enumerate $\mathcal{D}$ (possibly with repetitions) as $(x_\alpha)_{\alpha<\kappa}$. Then the family $\{x_\alpha \vee a_\alpha, x_\alpha \vee \neg a_\alpha : \alpha < \kappa\}$ is 2-linked, just as in the earlier proof, so it can be extended to a voting rule $\mathcal{W}'$. All elements of $\mathcal{D}$ are critical for $\mathcal{W}'$, just as in the earlier proof; therefore so are all elements of the upward closure of $\mathcal{D}$, and these are all the non-zero elements of $\mathcal{B}/\mathcal{F}$. □

Remark 24. The use of "pairwise independent" rather than "independent" in the hypothesis of Proposition 23 may make a difference. For example, in interval Boolean algebras, all independent families are countable; see [9, Corollary 15.15]. On the other hand, an interval algebra can have a large pairwise independent family. Consider, for example, the interval algebra obtained from the unit interval $[0, 1]$ with its usual ordering. All the half-open subintervals $[x, x + l)$ of a fixed length $l > 1/2$ form a pairwise independent family of the cardinality of the continuum. Using the same idea, it is easy to construct linear orderings of any uncountable cardinality κ in whose interval algebra there is a pairwise independent family of the same cardinality κ as the whole algebra.

4. Near Ultrafilters

In this section, we concentrate on infinitary voting rules on ω, and we consider possible relations between them and non-principal ultrafilters. Of course, non-principal ultrafilters are a special case of voting rules. We saw in the proof of Proposition 18 how finitely many ultrafilters can be combined to produce voting rules that are not just ultrafilters. Our concern in the present section will be with the reverse direction; is there an easy way to "extract" an ultrafilter from a voting rule? It turns out to be consistent with ZFC (though not provable in ZFC) that there is a very simple way to do this.

We consider the following statement, which we call $(V \to U)$ because it converts **V**oting rules to **U**ltrafilters:

> For every infinitary voting rule $\mathcal{W}$ on ω, there is a finite-to-one function $f : \omega \to \omega$ such that $f(\mathcal{W})$ is an ultrafilter.

It seems surprising that such a tight connection can exist between arbitrary infinitary voting rules and the rather special ones given by ultrafilters, but it turns out to be consistent.

Proposition 25. *$(V \to U)$ follows from $\mathfrak{u} < \mathfrak{g}$, and it is therefore consistent relative to ZFC.*

We omit here the definitions of the cardinal characteristics $\mathfrak{u}$ and $\mathfrak{g}$ since we shall not need them. We can treat the inequality $\mathfrak{u} < \mathfrak{g}$ as a black box, citing some known information about it in the following proof. For more information about this inequality, see [2].

Proof. We need two facts about the inequality $\mathfrak{u} < \mathfrak{g}$. The first is that it is consistent relative to ZFC; this is proved in [3] and it justifies the "therefore" in the proposition. The second is a theorem of Laflamme [10]: Assume that $\mathfrak{u} < \mathfrak{g}$ and that $\mathcal{X}$ is an upward closed family of subsets of ω closed under finite modifications (i.e., if $X \in \mathcal{X}$ and $X \triangle Y$ is finite, then $Y \in \mathcal{X}$). Then there is a finite-to-one $f : \omega \to \omega$ such that $f(\mathcal{X})$ is one of the following three possibilities: (1) the family of all infinite subsets of ω, (2) a non-principal ultrafilter on ω, and (3) the family of cofinite subsets of ω. Here $f(\mathcal{X})$ is defined just as it was for voting rules, namely $\{X \subseteq \omega : f^{-1}(X) \in \mathcal{X}\}$.

Laflamme's theorem applies in particular when $\mathcal{X}$ is an infinitary voting rule on ω. In this case, as we observed earlier, $f(\mathcal{X})$ is again a voting rule. This excludes two of the three alternatives in Laflamme's theorem. The family of all infinite subsets of ω and the family of cofinite subsets of ω are not voting rules, because, when X is an infinite, coinfinite subset of ω (like the set of even numbers), the former contains both X and $\omega - X$ while the latter contains neither. The only surviving alternative is that $f(\mathcal{X})$ is an ultrafilter. □

The inequality $\mathfrak{u} < \mathfrak{g}$ is a fairly strong set-theoretic statement; rather few of the familiar models of set theory satisfy it. So it is reasonable to ask whether its consequence $(V \to U)$ is also strong in the same sense. Or was the assumption $\mathfrak{u} < \mathfrak{g}$ in Proposition 25 overkill? We do not have a definitive answer to this question, but the next result will show that $(V \to U)$ gives at least one of the familiar consequences of $\mathfrak{u} < \mathfrak{g}$, one which also holds in rather few models. To state it, we must recall the concept of near coherence, in the special case of ultrafilters on ω.

Definition 26. Two non-principal ultrafilters $\mathcal{U}_1$ and $\mathcal{U}_2$ on ω are *nearly coherent* if there is a finite-to-one function $f : \omega \to \omega$ such that $f(\mathcal{U}_1) = f(\mathcal{U}_2)$. Near coherence is an equivalence relation, and the equivalence classes are called *near-coherence classes*.

Proposition 27. $(V \to U)$ *implies that there are at most two near-coherence classes of non-principal ultrafilters on* ω.

Proof. Assume that there are three non-principal ultrafilters $\mathcal{U}_1$, $\mathcal{U}_2$, and $\mathcal{U}_3$, no two of which are nearly coherent. Let $\mathcal{W}$ be the voting rule consisting of those $X \subseteq \omega$ that are in at least two of these three ultrafilters. Let $f : \omega \to \omega$ be any finite-to-one function. Then $f(\mathcal{W})$ consists of those $X \subseteq \omega$ that belong to at least two of the three image ultrafilters $f(\mathcal{U}_1)$, $f(\mathcal{U}_2)$, and $f(\mathcal{U}_3)$. These three ultrafilters are distinct because the $\mathcal{U}_i$ were not nearly coherent. But then the support filter of $\mathcal{W}$ is the intersection of $f(\mathcal{U}_1)$, $f(\mathcal{U}_2)$, and $f(\mathcal{U}_3)$, as was shown in the proof of Proposition 18 (for the case of odd k). Since the support filter of an ultrafilter is that ultrafilter itself, it follows that $f(\mathcal{W})$ is not an ultrafilter. Thus, $\mathcal{W}$ is a counterexample to $(V \to U)$. □

It is known that $\mathfrak{u} < \mathfrak{g}$ implies the principle of *filter dichotomy* (FD), which asserts that, for every filter $\mathcal{F}$ on ω, there is a finite-to-one $f : \omega \to \omega$ such that $f(\mathcal{F})$ is an ultrafilter. FD in turn implies the principle of near-coherence of filters (NCF), which asserts that there is only one near-coherence class of non-principal ultrafilters on ω, which in turn trivially implies the concluson of Proposition 27. It is an old unsolved problem whether either implication, from $\mathfrak{u} < \mathfrak{g}$ to FD or from FD to NCF, can be reversed. We now have additional open problems: Does $(V \to U)$ imply FD, or at least NCF? Is it implied by NCF, or at least by FD?

5. Axiom of Choice

In this final section, we discuss the extent to which the axiom of choice is needed for proving the existence of voting rules, and we relate it to the analogous extent for ultrafilters. We begin by showing that, as with ultrafilters, we cannot hope to establish the existence of voting rules in general Boolean algebras, or even in the specific case of $\mathcal{P}(\omega)/\text{fin}$, without using the axiom of choice. The proof is essentially the same as the standard proof of the corresponding result for ultrafilters, combining results from [16, 17, 15]. .

It is convenient (and traditional) to identify $\mathcal{P}(\omega)$ with the set 2^ω of infinite sequences of 0's and 1's (identifying each subset of ω with its characteristic function), to equip 2^ω (and thus $\mathcal{P}(\omega)$) with the product topology obtained from the discrete topology on 2, and to equip it with the product measure obtained from the probability measure on 2 that gives each of the two elements probability 1/2. In what follows, topological and measure-theoretic notions are to be understood in the light of these conventions.

Recall that a set in a topological space is *meager* if it is covered by countably many closed sets with empty interiors, and it is said to have the *Baire property* if it differs from some open set by a meager set. Recall also the Baire category theorem, that a complete metric space (e.g., 2^ω) is not meager in itself.

Proposition 28. *An infinitary voting rule on ω is not measurable and does not have the Baire property.*

Proof. The proof for measurability is just as in [16]. If an infinitary voting rule $\mathcal{W}$ were measurable, then its measure would have to be 0 or 1 because $\mathcal{W}$ is invariant under finite changes; this zero-one law is proved in [16] and in [11, Theorem 21.3]. But the measure of $\mathcal{W}$ would also have to be 1/2 because the measure-preserving map $X \mapsto \omega - X$ interchanges $\mathcal{W}$ with $\mathcal{P}(\omega) - \mathcal{W}$.

The proof for the Baire property is similar, but we give more detail (even though this zero-one law can also be found in [11, Theorem 21.4]) because this part of the proposition is more crucial later. Suppose $\mathcal{W}$ were an infinitary voting rule but had the Baire property. We claim first that $\mathcal{W}$ is either meager or comeager in $\mathcal{P}(\omega)$.

To prove the claim, let $\mathcal{U}$ be an open set that differs from $\mathcal{W}$ by a meager set. If $\mathcal{U} = \varnothing$, then $\mathcal{W}$ is meager and we are done. So assume from now on that $\mathcal{U}$ is nonempty and therefore includes a basic open set, i.e., a set of the form

$$\mathcal{N}(s) = \{x \in 2^\omega : s \text{ is an initial segment of } x\},$$

for some finite sequence s of 0's and 1's. Then $\mathcal{W} \cap \mathcal{N}(s)$ is comeager in $\mathcal{N}(s)$. Because $\mathcal{W}$ is infinitary, it follows that $\mathcal{W} \cap \mathcal{N}(t)$ is comeager in $\mathcal{N}(t)$ for all sequences t of the same length as s. But the union of these $\mathcal{N}(t)$'s is all of 2^ω, so the claim is established.

The map $X \mapsto \omega - X$ is a homeomorphism from $\mathcal{P}(\omega)$ onto itself, and it sends $\mathcal{W}$ to $\mathcal{P}(\omega) - \mathcal{W}$. Thus, if $\mathcal{W}$ is meager, then so is $\mathcal{P}(\omega) - \mathcal{W}$, which is absurd as $\mathcal{P}(\omega)$ cannot be covered by two meager sets. Similarly, if $\mathcal{W}$ is comeager then so is $\mathcal{P}(\omega) - \mathcal{W}$, which is absurd as two comeager subsets of $\mathcal{P}(\omega)$ cannot be disjoint. So we have contradictions in both cases. □

Corollary 29. *An infinitary voting rule on ω is not a Borel (or analytic, or coanalytic) subset of $\mathcal{P}(\omega)$.*

Proof. Borel (or analytic, or coanalytic) sets are measurable and have the Baire property; see [8, Section 29B]. □

The corollary can be extended to larger classes known to enjoy the Baire propery or measurability. Classically, this includes the C-sets, the smallest σ-algebra of sets closed under Souslin's operation $\mathcal{A}$; see [8, Section 29.D]. Under additional set-theoretic hypotheses, the corollary can be extended, with the same proof, to wider classes of sets; for example projective determinacy implies that infinitary voting rules on ω cannot be projective. The same idea gives the following consistency result.

Corollary 30. *It consistent with ZF plus dependent choice that there are no infinitary voting rules on ω.*

Proof. Solovay [17] showed that it is consistent with ZF plus dependent choice that all subsets of $\mathcal{P}(\omega)$ are measurable and have the Baire property. His proof assumed the consistency of ZFC plus the existence of an inaccessible cardinal, but Shelah [15] eliminated the need for the inaccessible in the case of the Baire property. □

We turn next to the question of existence of voting rules in arbitrary (nondegenerate) Boolean algebras in set theory without the axiom of choice. The preceding corollary shows that some choice principle is needed, and indeed that the principle of dependent choice is not sufficient. On the other hand, the Boolean prime ideal theorem is obviously sufficient, since the complement of a prime ideal is an ultrafilter and thus a voting rule. Which weak forms of choice are sufficient, and which are necessary, for the existence of voting rules? Partial answers to these questions were given by Schrijver [14].

Definition 31.

- LA denotes the *linking axiom*: "Every non-degenerate Boolean algebra has a voting rule."
- OP denotes the *ordering principle*: "Every set admits a linear ordering."
- OEP denotes the *order extension principle*: "Every partial ordering can be extended to a linear ordering of the same set."
- *Choice from pairs* means the statement "Every family of two-element sets has a choice function." Similarly with "three-element sets", "finite sets", etc., in place of "pairs".

The linking axiom was introduced by Schrijver; the name refers to the characterization of voting rules as maximal 2-linked families. Schrijver [14] proved the following two implications; note that the first improves on the remark above about the Boolean prime ideal theorem, since this theorem is known to be strictly stronger than OEP.

Proposition 32 (Schrijver)**.** *OEP implies LA, and LA implies choice from pairs.*

Proof. For the first implication, let an arbitrary non-degenerate Boolean algebra $\mathcal{B}$ be given, and use OEP to extend its ordering relation to a linear order $\preceq$. It is easy to check that $\{x \in \mathcal{B} : (\neg x) \preceq x\}$ is a voting rule in $\mathcal{B}$.

For the second implication, let a family $\mathcal{F}$ of two-element sets be given. Without loss of generality, the sets in $\mathcal{F}$ are pairwise disjoint. Let $\mathcal{B}$ be the Boolean algebra generated by the union of all the pairs in $\mathcal{F}$, subject to (only) the relations that $x = \neg y$ when $\{x, y\} \in \mathcal{F}$. This algebra is non-degenerate, i.e., the given relations don't entail $0 = 1$. (This is easy to see because a deduction from the relations would use only finitely many of them, and any finitely many of the relations are easily satisfied in the two-element Boolean algebra — just choose one element from each of the finitely many relevant pairs and map the chosen elements to 1.) So LA gives a voting rule $\mathcal{W}$ in $\mathcal{B}$. Then $\{x \in \bigcup \mathcal{F} : x \in \mathcal{W}\}$ contains exactly one element from each pair in $\mathcal{F}$. □

Recall the chain of implications

$$OEP \implies OP \implies \text{(Choice from finite sets)} \implies \text{(Choice from pairs)},$$

in which none of the implications are reversible; see [6]. Since Schrijver's theorem puts LA between the first and last statements in this chain, it is natural to ask how it relates to the middle two statements. The following result gives part of the answer.

Proposition 33. *LA does not imply choice from finite sets; in fact, it does not even imply choice from countably many three-element sets (in ZFA).*

Here ZFA means the variant of ZF that allows atoms (also called urelements). It occurs here because the model we construct is a permutation model, in the style of Fraenkel and Mostowski (see [6, Chapter 4]). We believe that similar methods will produce a symmetric model in the style of Cohen ([6, Chapter 5]), so ZFA could be replaced by ZF in the proposition, but we have not checked the details. It does not seem possible to transfer the independence result from ZFA to ZF "automatically" by the methods of Jech and Sochor [7] or Pincus [12], because LA is not sufficiently bounded.

For the proof of Proposition 33, we shall need an extension of the concept of voting rule to a context broader than Boolean algebras. The main point here is that only a little of the structure of a Boolean algebra enters into the definition of voting rules. We need the ordering relation and we need complementation, but we do not need the join and meet operations (directly). Thus, we can make the following definitions.

Definition 34. A *poset with involution* is a partially ordered set $(\mathcal{X}, \preceq)$ together with a function $\neg : \mathcal{X} \to \mathcal{X}$ with the properties

Involutive: $\neg(\neg(x)) = x$ for all $x \in \mathcal{X}$.
Anti-Monotone: If $x \preceq y$ then $\neg(y) \preceq \neg(x)$.
Non-Degenerate: $\neg(x) \neq x$ for all $x \in \mathcal{X}$.

The order relation and the complementation operation of any non-degenerate Boolean algebra make it into a poset with involution. (The non-degeneracy requirement in the definition of posets with involution is so called because it excludes the degenerate Boolean algebra. It also excludes some other things, but no other Boolean algebras.)

Now we can copy the definition of voting rules into this new context.

Definition 35. A *voting rule* in a poset with involution $\mathcal{X}$ is a set $\mathcal{W} \subseteq \mathcal{X}$ that is

Monotonic: if $x \preceq y$ in $\mathcal{X}$ and $x \in \mathcal{W}$, then $y \in \mathcal{W}$ and

- **Decisive:** for every $x \in \mathcal{X}$, either $x \in \mathcal{W}$ or $\neg x \in \mathcal{W}$ but not both.

With the axiom of choice, we get an existence theorem for voting rules in posets with involution.

Lemma 36. *Every poset with involution has a voting rule (assuming the axiom of choice).*

Proof. Let $\mathcal{X}$ be a poset with involution and let $\mathbb{P}$ be the family of all subsets $\mathcal{Z}$ of $\mathcal{X}$ such that

- $\mathcal{Z}$ is closed upward in $\mathcal{X}$ and
- there is no $x \in \mathcal{Z}$ for which $\neg(x) \in \mathcal{Z}$.

Partially order $\mathbb{P}$ by inclusion and observe that it is closed under unions of chains. By Zorn's lemma, let $\mathcal{W}$ be a maximal element in $\mathbb{P}$. We claim that $\mathcal{W}$ is a voting rule. Since monotonicity holds by definition of $\mathbb{P}$, we need only check decisiveness, and half of this requirement is also covered by the definition of $\mathbb{P}$. We check the other half.

Suppose, therefore, that there were some $x \in \mathcal{X}$ such that neither x nor $\neg(x)$ is in $\mathcal{W}$. By maximality of $\mathcal{W}$, the upward closure of $\mathcal{W} \cup \{x\}$ must contain some y together with $\neg(y)$. As $\mathcal{W}$ is closed upward, we have

$$(y \in \mathcal{W} \text{ or } y \succeq x) \quad \text{and} \quad (\neg(y) \in \mathcal{W} \text{ or } \neg(y) \succeq x).$$

We analyze the possible combinations here. We cannot have $y \in \mathcal{W}$ and $\neg(y) \in \mathcal{W}$ because $\mathcal{W} \in \mathbb{P}$. If we had $y \in \mathcal{W}$ and $\neg(y) \succeq x$, then the properties of $\neg$ give us $y \leq \neg(x)$ and so $\neg(x) \in \mathcal{W}$, contrary to our assumption. The combination $y \succeq x$ and $\neg(y) \in \mathcal{W}$ is similarly excluded because it yields $\neg(y) \preceq \neg(x)$ and so $\neg(x) \in \mathcal{W}$. There remains only one possibility: $y \succeq x$ and $\neg(y) \succeq x$.

Arguing similarly with the roles of x and $\neg(x)$ interchanged, we obtain $y \succeq \neg(x)$ and $\neg(y) \succeq \neg(x)$. Using again the properties of $\neg$, we infer that $x \preceq \neg(y)$ and $x \preceq y$.

But now we have both $x \preceq y$ (from the last paragraph) and $y \succeq x$ (from the paragraph before that), and so $x = y$. But we also have $x \preceq \neg(y)$ and $\neg(y) \succeq x$, and so $x = \neg(y)$. Thus, $y = \neg(y)$, contrary to the non-degeneracy clause in the definition of posets with involution. □

Remark 37. The lemma can be proved using less than the full axiom of choice. It is fairly routine to give a proof using the Boolean prime ideal theorem. In fact, with more work one can show that LA itself suffices. That is, if all non-degenerate Boolean algebras have voting rules then so do all posets with involution. We omit the proof here, because we shall need the lemma only in a situation where the axiom of choice is available.

Proof of Proposition 33. We construct a permutation model of ZFA and LA in which there is a countable family of three-element sets with no choice function. Begin with a universe satisfying ZFCA (i.e., ZFA plus the axiom of choice) with a countable set A of atoms. Partition A into a countable sequence of 3-element sets T_n ($n \in \omega$). For each n, choose a cyclic ordering of T_n, and let G_n be the group of permutations of T_n that respect this ordering. Thus, G_n is a cyclic group of order 3. Let G be the direct product $\prod_{n\in\omega} G_n$, acting on A by letting G_n act on T_n for each n. Let M be the permutation model of sets hereditarily symmetric with respect to the group G and the filter of subgroups given by finite supports. Thus, a set x is symmetric if there is a finite $E \subseteq A$ such that every $\pi \in G$ that fixes all elements of E also fixes x, and a set is in M if and only if it and all sets in its transitive closure are symmetric. It is easy to check that the sets T_n, the family of these sets, and the enumeration $n \mapsto T_n$ are hereditarily symmetric but that there is no symmetric set containing exactly one element from each T_n. Thus, in M, $\{T_n : n \in \omega\}$ is a countable family of three-element sets with no choice function. (In fact, no infinite subfamily has a choice function.) It remains to check that, in M, every non-degenerate Boolean algebra has a voting rule.

So let $\mathcal{B}$ be a non-degenerate Boolean algebra in M, and let $E \subset A$ be a finite support of it. Here we mean that E supports not only the set $\mathcal{B}$ but also the Boolean algebra structure on it. Thus, each $\pi \in G$ that fixes E pointwise acts on $\mathcal{B}$ as a Boolean automorphism.

To simplify notation, assume from now on that $E = \varnothing$. This involves no real loss of generality, because if $E \neq \varnothing$ then the following argument works with G replaced by the subgroup fixing E pointwise.

So we have G acting on $\mathcal{B}$ as a group of Boolean automorphisms. Let $\mathcal{X}$ be the set of orbits of this action, i.e., the set of equivalence classes $[x]$ where elements $x, y \in \mathcal{B}$ are considered equivalent if there is $\pi \in G$ with $\pi(x) = y$. We claim that, for any two of these orbits, say $[x]$ and $[y]$, the following are equivalent:

(1) Some element $x' \in [x]$ is $\leq$ (in $\mathcal{B}$) some element $y' \in [y]$.
(2) For every $x' \in [x]$ there is some $y' \in [y]$ with $x' \leq y'$.
(3) For every $y' \in [y]$ there is some $x' \in [x]$ with $x' \leq y'$.

Statement (1) follows trivially from either of the other two statements. Conversely, assume (1), say witnessed by x' and y', and let an arbitrary $x'' \in [x]$ be given. Since x' and x'' are in the same orbit $[x]$, there is $\pi \in G$ with $\pi(x') = x''$. From $x' \leq y'$, we obtain, since π preserves the ordering of $\mathcal{B}$, that $x'' \leq \pi(y')$. And $\pi(y')$ is in the same orbit $[y]$ as y'. So we have (2). The proof that (1) implies (3) is entirely analogous.

Write $[x] \preceq [y]$ to mean that the three equivalent statements hold. Version (2) (or (3)) of the definition of $\preceq$ immediately shows that $\preceq$ is transitive. To prove that $\preceq$ is a partial ordering of the set $\mathcal{X}$ of orbits, it remains, since reflexivity is trivial, to prove antisymmetry. Consider, therefore, two orbits, each $\preceq$ the other. Let x be a representative of one of these orbits and choose, by version (3) of the definition, a representative y of the other orbit with $x \leq y$. Use version (3) again to choose a representative $\pi(x)$ of the first orbit with $y \leq \pi(x)$ and $\pi \in G$. So we have $x \leq \pi(x)$ and, since π is an automorphism,

$$x \leq y \leq \pi(x) \leq \pi^2(x) \leq \pi^3(x).$$

But π^3 is the identity, so all the elements in the displayed inequality chain are equal. In particular, $[x] = [y]$, as required. Thus, $\preceq$ partially orders the set $\mathcal{X}$ of orbits.

Because the action of G on $\mathcal{B}$ preserves the Boolean operations, in particular negation, we can unambiguously define an operation $\neg$ on $\mathcal{X}$ by $\neg([x]) = [\neg x]$ (where the $\neg$ inside the brackets is that of $\mathcal{B}$). We claim that $\preceq$ and $\neg$ make $\mathcal{X}$ into a poset with involution. The involutive and anti-monotone properties are clear, in view of the corresponding properties of negation in $\mathcal{B}$. We must, however, check the non-degeneracy condition, $\neg([x]) \neq [x]$. That is, we must check that no $\pi \in G$ and $x \in \mathcal{B}$ can satisfy $\pi(x) = \neg x$. The essential point here is that all elements $\pi \in G$ have order 3. So if we had $\pi(x) = \neg x$, then we would have

$$x = \pi^3(x) = \pi^2(\neg x) = \pi(\neg(\pi(x))) = \pi(\neg\neg x) = \pi(x) = \neg x.$$

But $x = \neg x$ immediately implies $0 = 1$ in $\mathcal{B}$, contrary to our assumption that $\mathcal{B}$ is a non-degenerate Boolean algebra.

So we have a poset with involution $\mathcal{X}$. Applying Lemma 36 in the original universe (where choice is available), we obtain a voting rule $\mathcal{W}_0$ in $\mathcal{X}$. Then

$$\mathcal{W} = \{x \in \mathcal{B} : [x] \in \mathcal{W}_0\}$$

is a voting rule in $\mathcal{B}$. Consisting of entire orbits of the action of G, this $\mathcal{W}$ is symmetric and thus in M. Therefore, $\mathcal{B}$ has, in M, a voting rule $\mathcal{W}$, as required. □

Essentially the same proof establishes the following variant of Proposition 33.

Corollary 38. *Let k be a positive integer different from 1, 2, and 4. Then LA does not imply choice from countably many k-element sets.*

Proof. The only properties of the number 3 used in the proof of Proposition 33 are that it is odd and different from 1. The non-degeneracy of the poset with involution $\mathcal{X}$ depended on the fact that 3 is odd, and the lack of a symmetric choice function for $\{T_n : n \in \omega\}$ depended on the fact that $3 \neq 1$. Thus, the same proof works for any odd $k \neq 1$.

For even $k > 4$, we fix two odd numbers k' and k'', both ≥ 3, with sum $k' + k'' = k$. We again use a set of atoms partitioned into k-element subsets T_n. We split each T_n into two pieces T'_n and T''_n of cardinalities k' and k'', we fix a cyclic ordering of each T'_n and each T''_n, and we let G_n be the group of permutations of T_n that preserve the subsets T'_n and T''_n and their chosen cyclic orderings. Thus G_n is the direct product of cyclic groups of orders k' and k''. Let G be the direct product $\prod_{n\in\omega} G_n$, acting on the atoms by letting G_n act on T_n for each n. Note that all elements of G have odd orders, namely factors of the product $k'k''$. Let M be the permutation model of sets hereditarily symmetric with respect to the group G and the filter of subgroups given by finite supports. The same proof as for Proposition 33 shows that M satisfies LA and that it contains the sequence of k-element sets T_n but contains no choice function for this sequence. □

Remark 39. Since choice from pairs follows from LA and since choice from singletons is trivial, two of the three exceptions in Corollary 38 are obviously needed. The third exception, for 4, is needed also. According to a theorem of Tarski (see [6, Example 7.12]), choice from pairs implies choice from 4-element sets.

Remark 40. It is likely that OP does not imply LA. A reasonable candidate for a model of OP violating LA is the model obtained from a model of ZFC by first adjoining a family $\mathcal{B}$ of Cohen reals indexed by the countable, atomless Boolean algebra $\mathcal{C}$ and then taking the symmetric submodel determined by the set of Boolean automorphisms of $\mathcal{C}$ (acting in the obvious way on the forcing conditions) and finite supports. The resulting symmetric model contains $\mathcal{B}$ with the Boolean algebra structure that it inherits from $\mathcal{C}$, but it does not contain any voting rule in $\mathcal{B}$. The ordering principle ought to hold in this model, by a "least support" argument roughly analogous to that in [6, Section 5.5]. A key ingredient of such an argument would be a lemma saying that, if an automorphism α of $\mathcal{C}$ pointwise fixes the intersection of two finite Boolean subalgebras $\mathcal{A}_1$ and $\mathcal{A}_1$ of $\mathcal{C}$, then α can be expressed as the product of some automorphisms each of which pointwise fixes either $\mathcal{A}_1$ or $\mathcal{A}_2$. I have a proof of this lemma, but there remain technical difficulties in carrying out the rest of the least support argument in this context. It therefore remains a conjecture that this model satisfies OP and therefore OP does not imply LA.

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