INTERPRETING EXTENDED SET THEORY IN CLASSICAL SET THEORY

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Abstract. We exhibit an interpretation of the Extended Set Theory proposed by Dave Childs in classical Zermelo-Fraenkel set theory with the axiom of choice and an axiom asserting the existence of arbitrarily large inaccessible cardinals. In particular, if the existence of arbitrarily large inaccessible cardinals is consistent with ZFC, then Childs's Extended Set Theory is also consistent.

1. Introduction

In [1], Childs axiomatizes a set theory, XST, that differs from traditional Zermelo-Fraenkel set theory with choice (ZFC) in two major ways (and several minor ways).

The biggest difference between XST and ZFC is that the fundamental membership relation of XST is a ternary relation, written $x \in_s y$ and read "x is an element of y with *scope* s." Most of the axioms of XST are analogs of traditional ZFC axioms, modified to acount for membership scopes, and allowing atoms (also called urelements). This aspect of XST can be interpreted in ZFC quite straightforwardly, as we shall see below.

The second major difference is that XST contains the so-called Klass axiom, which permits the formation of certain very large sets. Specifically, any sets in which (hereditarily) all scopes are below a particular bound can be collected into a set, with higher scopes. The natural interpretation of this in classical set theory, which we present below, uses the assumption of arbitrarily large inaccessible cardinals.

2. Sets with Scopes

We begin by interpreting in ZFC the idea of sets with scopes. This will establish the relative consistency of XST minus the Klass axiom. (The Klass axiom will be treated later.) The interpretation is quite straightforward. An XST set a is very similar to a binary relation in

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ZFC; the primitive relation $x \in_s a$ of XST corresponds to the (non-primitive) $\langle x, s\rangle \in a$ of classical set theory. Accordingly, our interpretation will simply model in ZFC a cumulative hierachy of binary relations.

We shall begin our construction of this hierarchy with any number of atoms. Strictly speaking, we could do without atoms, since XST does not require the existence of any atoms. Nevertheless, it permits the existence of atoms, and there are hints in [1] that Childs might want, at least for some purposes, to require some atoms, for example infinitely many natural names, i.e., the elements in the domain of η on page 2 of [1].

Within ZFC, we shall use the standard Kuratowski coding of ordered pairs, $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$. We also use the standard von Neumann coding of ordinal numbers, where each ordinal number is the set of all strictly smaller ordinal numbers. In particular, the empty set $\varnothing$ is identified with the ordinal 0, and it is a member of every non-zero ordinal. We note that an ordered pair neither equals nor contains $\varnothing$, so no ordered pair is an ordinal number. Furthermore, no relation (i.e., set of ordered pairs) except $\varnothing$ is an ordinal number.

Let $\mathcal{A}$ be an arbitrary set of non-zero ordinal numbers. The *cumulative hierarchy of relations* over $\mathcal{A}$ is defined as follows.

$$\begin{aligned} C_0 &= \mathcal{A} \\ C_{\alpha+1} &= \mathcal{A} \cup \mathcal{P}(C_\alpha \times C_\alpha) \\ C_\lambda &= \bigcup_{\alpha<\lambda} C_\alpha && \text{for limit } \lambda \\ C &= \bigcup_{\text{ordinals } \alpha} C_\alpha && . \end{aligned}$$

Here $\times$ is the usual cartesian product of two sets; $X \times Y$ is the set of ordered pairs $\langle x, y\rangle$ with $x \in X$ and $y \in Y$. We also used the usual notation $\mathcal{P}$ for power set, the collection of all subsets of a set. Thus, $\mathcal{P}(C_\alpha \times C_\alpha)$ is the set of all binary relations on C_α.

For each ordinal α, C_α is a set, but C, the union over all α, is a proper class. It is easy to check, by tranfinite induction, that $C_\alpha \subseteq C_\beta$ whenever $\alpha < \beta$. For each $x \in C - \mathcal{A}$, there is a unique ordinal α such that $x \in C_{\alpha+1} - C_\alpha$. This α will be called the *height* of x.

An equivalent way to describe C (which we won't need, but which may be helpful for intuition) is that it is the smallest class that has, among its elements, all the members of $\mathcal{A}$ and all sets of ordered pairs whose two components are in C.

Our interpretation of XST minus the Klass axiom has C as its universe of discourse, and the primitive ternary predicate is interpreted as

$$x \in_s a \text{ is } \langle x, s\rangle \in a.$$

The $\varnothing$ of XST can (conveniently) be taken to be the $\varnothing$ of ZFC, the empty relation, which is of course in C_1 and thus in C.

Recalling that the elements of $\mathcal{A}$, being non-zero ordinal numbers, are not binary relations, we see that they serve as the atoms in our interpretation of XST.

Although it is not explicitly indicated in [1], there is another primitive concept in this theory, namely the *total scope* $\mathcal{S}_*(X)$ of a set X. Axiom 2.2 describes this operation but does not specify it uniquely. We shall make this specification unique by understanding it as a recursive definition of $\mathcal{S}_*$; this seems to be consonant with Childs's intention.

More explicitly, we invoke the familiar metatheorem on definitions by recursion in ZFC to obtain a definable operation $\mathcal{S}_*$ such that (provably in ZFC), for all $a \in C$:

- If $a \in C_0 = \mathcal{A}$ then $\mathcal{S}_*(a) = \varnothing$.
- If $a \in C_{\alpha+1} - C_\alpha$ then
$$\mathcal{S}_*(a) = \{\langle s, s\rangle : \langle x, s\rangle \in a\} \cup \bigcup_{\langle x,s\rangle \in a} \big(\mathcal{S}_*(x) \cup \mathcal{S}_*(s)\big).$$

With these definitions for the universe of discourse and the primitive notions of XST, it is straightforward to work out the defined terms of XST and to check that the axioms in Sections 2.2 and 2.3 hold in our interpretation. Explicitly, this means that, when these axioms are translated into the language of ZFC, using the definitions above, the translated assertions are theorems of ZFC.

3. The Klass Axiom

Childs's *Klass axiom*, Axiom 3.1 of [1] asserts that, for each entity τ, there exists a set Y with the following two properties:

- The only scope for memebrship in Y is τ, i.e., if $a \in_s Y$ then $s = \tau$.
- For all a, we have $a \in_\tau Y$ if and only if $\mathcal{S}_*(a) \subseteq \tau$.

In other words, the members of Y are exactly the entities with total support included in τ, and the scope for their membership in Y is τ.

In [1], the Klass axiom is formulated so that the members of Y are not all the entities with total support included in τ but only those that satisfy a given formula Φ. This looks more general; our version is the special case where Φ is always true. But the general case follows immediately from this special case, thanks to the axiom schema of separation. We therefore concentrate on the special case, the Klass axiom as formulated above.

If we tried to verify the Klass axiom in the interpretation from the preceding section, we would need (in ZFC)

$$Y = \{\langle a, \tau\rangle : \mathcal{S}_*(a) \subseteq \tau\},$$

but this Y is a proper class, so it is not available to serve as a witness. We must therefore modify the interpretation, and the following seems to be the most natural way to accomplish this.

We work in the theory ZFC+, which is ZFC plus the assumption that there are arbitrarily large inaccessible cardinals. This assumption is equivalent to the assumption that every set is a member of some Grothendieck universe; in this form it is often used in category theory and its applications to algebraic geometry. From a set-theorist's point of view, it is a rather mild large-cardinal assumption. From now on, we work in the theory ZFC+.

Notation 1. Fix an enumeration $\langle i_\alpha\rangle$, in increasing order, of some inaccessible cardinals such that $i_\alpha > |\mathcal{A}|$ and $i_\alpha > \alpha$, for all α.

Note that the inequalities imposed on i_α can both be achieved by just skipping some elements in the increasing enumeration of all the inaccessible cardinals.

Definition 2. An element a of C is *scope-bounded* if, for each ordinal α, the number of $\langle x, s\rangle \in a$ with s of height $\leq \alpha$ is $< i_{2\alpha+1}$.

So a scope-bounded set could have arbitrarily many elements, but, if the number of elements is large enough, then most of them must have scopes of great height. Note that every element of $\mathcal{A}$ is scope-bounded, because it has no elements of the form $\langle x, s\rangle$.

Let B be the subclass of C consisting of the hereditarily scope-bounded elements. Here a set is called *hereditarily scope-bounded* if not only it but also all members of its transitive closure are scope-bounded.

We define a new interpretation of the language of XST in ZFC+ by taking B as its universe of discourse. (So we are shrinking the universe from C down to B.) The primitive, ternary, membership predicate of XST is interpreted as it was in C, and so is the operation $\mathcal{S}_*$. We omit the straightforward verification that B is closed under $\mathcal{S}_*$, so that we have a legitimate interpretation.

To verify the interpretation of the Klass axiom in B, let an arbitrary $\tau \in B$ be given, and let β be its height. Define

$$Y = \{\langle x, \tau\rangle : x \in B \text{ and } \mathcal{S}_*(x) \subseteq^B \tau\}.$$

Here $\subseteq^B$ means the subset relation of XST (Definition 2.2 in [1]) as interpreted in B; it is easy to check, by chasing through the definitions, that this agrees with the subset relation of ZFC+.

If we can show that this Y is in B, then it clearly has the properties required in the Klass axiom for the given τ. So we concentrate on showing that Y is hereditarily scope-bounded. In fact, we can concentrate on just showing that Y is scope-bounded. The "hererditarily" part follows automatically, because each element of the transitive closure of Y either has cardinality at most 2 (if it is an ordered pair $\langle x, \tau\rangle$ or a member of such a pair) or is in the transitive closure of $\{x\}$ or $\{\tau\}$ for some $\langle x, \tau\rangle \in Y$. Either way, such an element is scope-bounded because the relevant x's and τ are in B.

So our task reduces to estimating (from above) the number of elements of Y of the form $\langle x, s\rangle$ with s of a given height α. By definition of Y, the only relevant α is the height β of τ, so what we must show is that the number of $x \in B$ with $\mathcal{S}_*(x) \subseteq \tau$ is strictly less than $i_{2\beta+1}$.

Lemma 3. *If $x \in B$ and $\mathcal{S}_*(x) \subseteq \tau$ with τ of height β, then $x \in C_{i_{2\beta}}$.*

Proof. We fix τ and (therefore) β and proceed by induction on the height of x. Given x, consider an arbitrary $\langle z, s\rangle \in x$. We have $\mathcal{S}_*(z) \subseteq \mathcal{S}_*(x) \subseteq \tau$, so by the induction hypothesis $z \in C_{i_{2\beta}}$; similarly $s \in C_{i_{2\beta}}$. Furthermore, $s \in_s \tau$ (by definition of $\mathcal{S}_*$), and so the number of possible s's is bounded by $|\tau| < i_\beta \leq i_{2\beta}$ (since i_β is inaccessible and strictly greater than the height β of τ).

Temporarily fix one such s and ask how many z's there might be with $\langle z, s\rangle \in x$. Since $s \in_s \tau$, the height of s is some $\alpha < \beta$. Since x is scope-bounded, the number of z's under consideration is $< i_{2\alpha+1}$.

Now un-fix s. The α of the preceding paragraph depends on s, but, since the number of possible s's is smaller than $i_{2\beta}$ and the latter is a regular cardinal, the total number of pairs $\langle z, s\rangle$ in x is strictly below $i_{2\beta}$.

Since $i_{2\beta}$ is a limit ordinal (because it is an infinite cardinal), $C_{i_{2\beta}}$ is the union of the sets C_ξ for $\xi < i_{2\beta}$. So each $\langle z, s\rangle \in x$ is in some such C_ξ, and a single $\xi < i_\beta$ will work for all of them, since there are fewer than $i_{2\beta}$ of them and $i_{2\beta}$ is regular. (In the degenerate case that x is a set of atoms so ξ could be 0, use $\xi = 1$ instead.) Then $x \in C_{\xi+1} \subseteq C_{i_{2\beta}}$, as required. □

In view of the lemma, the number of $x \in B$ with $\mathcal{S}_*(x) \subseteq \tau$ is bounded above by the number of elements of $C_{i_{2\beta}}$. That is strictly less than any inaccessible cardinal larger than $i_{2\beta}$, in particular $i_{2\beta+1}$.

This completes the verification that the B interpretation satisfies the Klass axiom of XST. Of course, it must still be checked that, in shrinking C down to B, we did not violate any of the other axioms. That checking is straightforward, using the fact that all the i_α are inaccessible.

References

[1] D. L. Childs, "Axioms for an extended set theory," Integrated Information Systems, Ann Arbor, MI (2010). http://xsp.xegesis.org/X_axioms.pdf

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