1. The electrostatic potential at \((0, 0, -a)\) of a charge of constant density \(\sigma\) on the hemisphere 
\[ S : x^2 + y^2 + z^2 = a^2, \ z \geq 0 \]
is 
\[ U = \int \int_S \frac{\sigma}{\sqrt{x^2 + y^2 + (z + a)^2}} dS. \]
Show that \(U = 2\pi\sigma a(2 - \sqrt{2})\).

2. Find the value of the line integral 
\[ \int_C yzdx + xzdy + xydz \]
where \(C\) is the curve with initial point \(P_0 = (1, 0, 0)\) given by.

   a) \(x = \cos t, \ y = \sin t, \ z = t^2\) with \(t \in [0, \pi]\).

   b) The line segment between \(P_0\) and \((-1, 0, -\pi)\).

   c) The ellipse \(x^2 + 4y^2 = 1\) and \(z = 0\) parametrized clockwise.

   d) The circle \(x^2 + y^2 = 9\) and \(z = 0\) parametrized counterclockwise as viewed from above. Use the fact that the path \(C\) is the boundary of the region \(x^2 + y^2 \leq 9\) with \(z = 0\). Use Green’s Theorem to compute value of the line integral.

   e) The same path as in part d), but now using Stokes Theorem. Use the fact that the path \(C\) in this case is the boundary of the hemisphere \(x^2 + y^2 + z^2 = 9\) with \(z \leq 0\).

   f) Can you use the fundamental theorem of line integrals to compute the value of any of the above line integrals? Explain.

3. Find the surface area of the paraboloid \(z = \frac{1}{3}(x^2 + y^2)\) that lies between the plane \(z = 4\) and the sphere \(x^2 + y^2 + z^2 = 4\).

4. Find the center of mass for the solid tetrahedron with vertices \((1, 0, 0), \ (0, 1, 0), \ (0, 0, 1)\) and the origin and density function \(\rho(x, y, z) = x^2 + y^2 + z^2 + 1\).

5. Evaluate the surface integral \(\int \int_S \mathbf{F} \cdot d\mathbf{S}\) for the vector field \(\mathbf{F}(x, y, z) = xzi - yj + zk\) and the surface \(S\) being \(x^2 + y^2 + z^2 = 1\) with outward orientation.

6. If the components of the vector field \(\mathbf{F}\) have continuous second partial derivatives and \(S\) is the unit sphere centered at the origin with outward orientation. Show that \(\int \int_S \text{curl} \mathbf{F} \cdot d\mathbf{S} = 0\) by
   
   - applying Divergence Theorem.
   - considering the intersection curve of \(S\) with \(xy\)-plane and applying Stokes’ Theorem.
7. $C$ is any piecewise-smooth simple closed plane curve. Prove $\oint_C f(x) \, dx + g(y) \, dy = 0$ by

- Green’s Theorem.
- representing it as a line integral of a gradient vector field and applying Fundamental Theorem for line integrals.

8. Use the divergence theorem to calculate the surface integral

$$\iiint_S \mathbf{F} \cdot d\mathbf{S}$$

for

$$\mathbf{F} = x^4 i - x^3 z^2 j + 4xy^2 z k.$$  

$S$ is the surface bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = x + 2$ and $z = 0$.

9. Prove

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}).$$

10. Prove

$$\nabla \left( \frac{f}{g} \right) = \frac{g \nabla f - f \nabla g}{g^2}, \quad g \neq 0.$$  

11. Find the equation of the tangent plane for the surface prescribed by $x^2 + y^2 + 2z^3 = 4$ at the point $(1, 1, 1)$.

12. Consider the solid region $E$ between the cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = 2 - x^2 - y^2$

a) Sketch $E$

b) Let $V$ be the volume of $E$. Find a formula for $V$ in terms of a triple integral.

c) Find the value of $V$.

13. Let $S$ be the surface given by $z^2 = y^2 + x^2$ that lies between the planes $z = 0$ and $z = 2$.

a) Sketch the surface $S$.

b) Compute $\iint_S (z - x) dS$

14. Consider $\mathbf{F}(x, y, z) = < x^2, xy, z + 1 >$. Let $E$ be the solid enclosed by the paraboloid $z = 4 - x^2 - y^2$ and the plane $z = 0$ with normal vectors pointing outside $E$.

a) Sketch the solid $E$.

b) Write the integral expressing the flux of $\mathbf{F}$ across the boundary of $E$.

c) Compute the flux $\mathbf{F}$ across the boundary of $E$ using the Divergence Theorem.

15. A package in the shape of a rectangular box can be mailed by US parcel post if the sum of its length and girth (the perimeter of a cross section perpendicular to the length) is at most 108 inches. Find the dimensions of the box with the largest volume that can be mailed by Parcel Post. Use the following steps:

a) Make a sketch of the box and label the variables to be used to solve the problem.
b) Set up the optimization problem in terms of the variables you defined in (a).

c) Solve the problem

16. a) Let $h(x, y) = \ln(x^2 + y^2)$, compute $\nabla h$ and $\nabla \cdot \nabla h$.

b) Show that for any two functions $f(x, y)$ and $g(x, y)$,

$$\nabla \cdot (f \nabla g - g \nabla f) = f \nabla \cdot \nabla g - g \nabla \cdot \nabla f$$

17. Consider the vector field $F(x, y, z) = <xy, -y^2, z>$ and let $S$ be the surface on the paraboloid $z = x^2 + y^2$ between the planes $z = 1$ and $z = 4$ oriented with its unit normal vectors pointing inside the paraboloid.

a) Sketch $S$.

b) Indicate the induced orientation of the boundary $C$ of the surface $S$ in your sketch in (a) so that

$$\int_C F \cdot dr = \int_S \nabla \times F \cdot dS$$

holds.

c) Find a parameterization of $C$ and use it to set up

$$\int_C F \, dr$$

int terms of this parametrization (do not compute the value of the line integral).

d) Use Stokes' theorem to compute

$$\int_C F \cdot dr$$

with the appropriate surface integrals.