Homework 8 Solutions

16.2: #10

\[
\int_1^2 \int_0^1 (x + y)^{-2} \, dx \, dy = \int_1^2 (x + y)^{-1} \bigg|_0^1 \, dy = \int_1^2 (1 + y)^{-1} + y^{-1} \, dy = -\ln(1 + y) + \ln(y) \bigg|_1^2 = -\ln(3)
\]

16.2: #12

\[
\int_0^1 \int_0^1 \frac{xy}{\sqrt{x^2 + y^2 + 1}} \, dy \, dx = \int_0^1 x(x^2 + y^2 + 1)^{1/2} \bigg|_0^1 \, dx = \int_0^1 x(x^2 + 2)^{1/2} - x(x^2 + 1)^{1/2} \, dx = \frac{1}{3}(x^2 + 2)^{3/2} - \frac{1}{3}(x^2 + 1)^{3/2} \bigg|_0^1 = \frac{1}{3}(3^{3/2} - 2^{3/2} - 2^{3/2} + 1)
\]

16.2: #14

\[
\int \int_R \cos(x + 2y) \, dA = \int_0^1 \int_0^1 \cos(x + 2y) \, dy \, dx = \int_0^1 \frac{1}{2} \sin(x + 2y) \bigg|_0^1 \, dx = \int_0^1 \frac{1}{2}(\sin(x + 2) - \sin(x)) \, dx = \frac{1}{2}(\cos(3) - \cos(5) + \cos(2) - 1)
\]
16.2: #16

\[ \int \int_R \frac{1 + x^2}{1 + y^2} \, dy \, dx = \int_0^1 \int_0^1 \frac{1 + x^2}{1 + y^2} \, dy \, dx \]

\[ = \int_0^1 (1 + x^2) \arctan(y) \bigg|_0^1 \, dx \]

\[ = \int_0^1 \frac{\pi}{4} (1 + x^2) \, dx \]

\[ = \frac{\pi}{4} \left( x + \frac{x^3}{3} \right) \bigg|_0^1 \]

\[ = \frac{\pi}{4} \left( 1 + \frac{1}{3} \right) \]

16.3: #4

\[ \int_0^1 \int_x^{2-x} (x^2 - y) \, dy \, dx = \int_0^1 \left( x^2y - \frac{y^2}{2} \right) \bigg|_x^{2-x} \, dx \]

\[ = \int_0^1 x^2(2 - x) - \frac{1}{2}(2 - x)^2 - x^3 + \frac{1}{2} x^2 \, dx \]

\[ = \frac{2}{3} x^3 - \frac{1}{4} x^4 + \frac{1}{6} (2 - x)^3 - \frac{1}{4} x^4 + \frac{1}{6} x^3 \bigg|_0^1 \]

\[ = \frac{2}{3} - \frac{1}{4} + \frac{1}{6} - \frac{1}{4} + \frac{1}{6} - \frac{8}{6} \]

\[ = -\frac{5}{6} \]

16.3: #18

\[ \int \int_D 2xy \, dA = \int_0^1 \int_{2x}^{3-x} 2xy \, dy \, dx \]

\[ = \int_0^1 xy^2 \bigg|_{2x}^{3-x} \, dx \]

\[ = \int_0^1 x(3 - x)^2 - 4x^3 \, dx \]

\[ = \int_0^1 9x - 6x^2 - 3x^3 \, dx \]

\[ = \frac{9}{2} x^2 - 2x^3 - \frac{3}{4} x^4 \bigg|_0^1 \]

\[ = \frac{9}{2} - 2 - \frac{3}{4} \]

\[ = \frac{7}{4} \]

16.3: #32
\[\int \int_D 2xydA = \int_0^1 \int_{2x}^{3-x} 2xydydx\]
\[= \int_0^1 xy^2 \bigg|_{2x}^{3-x} \, dx\]
\[= \int_0^1 x(3 - x)^2 - 4x^3 \, dx\]
\[= \int_0^1 9x - 6x^2 - 3x^3 \, dx\]
\[= \frac{9}{2}x^2 - 2x^3 - \frac{3}{4}x^4 \bigg|_0^1\]
\[= \frac{9}{2} - 2 - \frac{3}{4}\]
\[= \frac{7}{4}\]

16.4: #14

\[\int \int_R ye^rdA = \int_0^5 \int_0^{\pi/2} r \sin(\theta)e^{r \cos(\theta)} r d\theta dr\]
\[= \int_0^5 -r e^{r \cos(\theta)} \bigg|_0^{\pi/2} \, dr\]
\[= \int_0^5 -r(1 - e^r) \, dr\]
\[= \int_0^5 re^r - r dr\]
\[= re^r - e^r - \frac{r^2}{2} \bigg|_0^5\]
\[= 4e^5 - 23\frac{2}{5}\]

In the fifth line we integrated by parts, but didn’t write down all the steps.
16.4: #18 If you draw the picture, you see that you need to take all theta values between 0 and 2π. So the area is given by the formula:

\[
\int_0^{2\pi} \int_0^{4+3\cos(\theta)} r \, dr \, d\theta = \int_0^{2\pi} \frac{1}{2} (4 + 3 \cos(\theta)^2) \, d\theta
\]

\[
= \frac{1}{2} \int_0^{2\pi} 16 + 24 \cos(\theta) + 9 \cos^2(\theta) \, d\theta
\]

\[
= \frac{1}{2} \int_0^{2\pi} 16 + 24 \cos(\theta) + 9 \frac{1 + \cos(2\theta)}{2} \, d\theta
\]

\[
= \frac{1}{2} \left( 16\theta + 24 \sin(\theta) + 9 \frac{\theta + \sin(2\theta)/2}{2} \right|_0^{2\pi}
\]

\[
= \frac{1}{2} (32\pi + 9\pi)
\]

\[
= \frac{41\pi}{2}
\]

16.4: #26 The two paraboloids intersect when \(3x^2 + 3y^2 = 4 - x^2 - y^2\), namely, when \(x^2 + y^2 = 1\). So the volume we’re trying to compute is given by

\[
\int \int_D (4 - x^2 - y^2) - (3x^2 + 3y^2) \, dA
\]

where \(D\) is the region in the plane \(x^2 + y^2 \leq 1\). Switching to polar coordinates,

\[
\int_0^{2\pi} \int_0^1 (4 - 4r^2) r \, dr \, d\theta = \int_0^{2\pi} 2r^2 - r^4 \left|_0^1 \right.
\]

\[
= \int_0^{2\pi} 1 \, d\theta
\]

\[
= 2\pi
\]