Problem Set 9

Section 16.5:

1) The boundary of a lamina consists of the semicircles $y = \sqrt{1 - x^2}$, $y = \sqrt{4 - x^2}$ together with the parts of the positive $x$-axis that joins them. Find the center of mass of the lamina if the density at any point is proportional to its distance from the origin.

**Solution:**

Best to do in polar coordinates.

$\rho = kr$

\[
m = \int_0^\pi \int_1^2 kr^2 dr d\theta = \frac{7}{3}\pi k
\]

$M_y = \int \int x \rho dA = 0$ by symmetry

$M_x = \int \int_D y \rho dA = \int_0^\pi \int_1^2 (r \sin \theta)(kr)r dr d\theta = \frac{15}{2}k$

Thus $(\bar{x}, \bar{y}) = (0, \frac{45}{14\pi})$.

2) Consider the lamina bounded by $y = 1 - x^2$ and $y = 0$ with density $\rho(x, y) = ky$. Find the moments of inertia $I_x, I_y, I_0$.

**Solution**

\[
I_x = \int \int_D y^2 \rho dA = \int_{-1}^1 \int_{0}^{1-x^2} y^2 ky dy dx = (64/215)k
\]
\[ I_y = \int \int_D x^2 \rho dA = \int_{-1}^{1} \int_0^{1-x^2} kx^2dydx = (8/105)k \]

\[ I_0 = I_x + I_y = (\pi/12)k. \]

**Section 16.6:**

3) Find the area of the part of the surface \( z = xy \) that lies within the cylinder \( x^2 + y^2 = 1 \).

**Solution**

Since \( f_x = y \) and \( f_y = x \)

\[ A(S) = \int \int_D \sqrt{1 + x^2 + y^2}dA = \int_0^{2\pi} \int_0^1 \sqrt{1 + r^2}da = \frac{2\pi}{3}(2\sqrt{2}-1) \]

4) Find the area of the finite part of the paraboloid \( y = x^2 + z^2 \) cut off by the plane \( y = 25 \).

**Solution:**

Surface lies above the disk \( x^2 + z^2 \) in the \( xz \) plane.

\[ A(S) = \int \int_D \sqrt{f_x^2 + f_z^2}dA = \int \int \sqrt{4x^2 + 4y^2 + 1}da \]

Converting to polar coords get

\[ \int_0^{2\pi} \int_0^5 \sqrt{4r^2 + 1}rdrd\theta = \pi/8(101\sqrt{101} - 1). \]

**Section 16.7:**
5) Find $\int \int \int_D xydV$ where $E$ is the region bounded by the parabolic cylinders $y = x^2$ and $x = y^2$ and the planes $z = 0$ and $z = x + y$.

**Solution:**

$E$ is the solid region above the Type I region in the plane between the curves $y = x^2$ $y = \sqrt{x}$ and below the plane $z = x + y$.

Hence we get

$$\int \int \int xydV = \int_0^1 \int_0^{\sqrt{y}} \int_0^{x+y} xydzdvdx = \int_0^1 \int_0^{\sqrt{y}} (x^2 + xy^2)dydx$$

This equals $3/28$.

**Section 16.8:**

6) Find the volume of the solid that lies between the paraboloid $z = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 2$.

**Solution:**

In cylindrical coordinates $E$ is bounded below by the cone $z = r$ and above by the sphere $z^2 + r^2 = 2$.

The cone and the sphere intersect when $r = 1$ so

$$E = (r, \theta, z) | 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, r \leq z \leq \sqrt{2 - r^2}$$

This gives volume

$$\int \int \int_E dV = \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{2-r^2}} rdzdrd\theta = 4/3\pi(\sqrt{2} - 1)$$
7) Compute \( \int \int \int_B (x^2 + y^2 + z^2)^2 \, dV \) where \( B \) is the ball with center the origin and radius 5.

**Solution:**

\[
\int \int \int_B (x^2 + y^2 + z^2)^2 \, dV = \int_0^{2\pi} \int_0^\pi \int_0^5 (\rho^2)^2 \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi
\]

This is approximately 140.

**Section 16.9:**

8) Use the transformation \( u = xy, v = xy^2 \) to compute \( \int \int_R y^2 \, dA \) where \( R \) is the region bounded by the curves \( xy = 1, xy = 2, xy^2 = 1, xy^2 = 2 \).

**Solution:**

Have \( y = u/v \) and \( x = u^2/v \) so

\[
\frac{\partial(x, y)}{\partial(u, v)} = 1/v
\]

and \( R \) is a square with vertices \((1, 1), (1, 2), (2, 2), (1, 2)\) so that

\[
\int \int_R y^2 \, dA = \int_1^2 \int_1^2 \frac{v^2}{u^2} \left( \frac{1}{v} \right) \, dudv = 3/4
\]