Review Sheet for Second Midterm Exam

1. Let \( \mathbf{r}(t) = \langle \cos t, \sin t, t \rangle \). Find \( \mathbf{T}(\pi) \), \( \mathbf{N}(\pi) \), and \( \mathbf{B}(\pi) \).

2. At what point do the curves \( \mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle \) and \( \mathbf{r}_2(s) = \langle 2s + 1, s^2 + 1, e^s \rangle \) intersect? Find \( \cos \theta \), where \( \theta \) is the angle of intersection.

3. Consider the Frenet-Serret formulas \( \frac{d\mathbf{T}}{ds} = \kappa \mathbf{N} \), \( \frac{d\mathbf{N}}{ds} = -\kappa \mathbf{T} + \tau \mathbf{B} \), and \( \frac{d\mathbf{B}}{ds} = -\tau \mathbf{N} \). Here, \( \tau(s) \) is the torsion. (a) Using the definition of the curvature \( \kappa = |d\mathbf{T}/ds| \), derive the first formula. (b) Deduce the second formula from the first and third formulas.

4. Prove (a) \( \mathbf{r}'' = s'' \mathbf{T} + \kappa (s')^2 \mathbf{N} \), (b) \( \mathbf{r}' \times \mathbf{r}'' = \kappa (s')^3 \mathbf{B} \), (c) \( \mathbf{r}''' = [s''' - \kappa^2(s')^3] \mathbf{T} + [3\kappa s' s'' + \kappa'(s')^2] \mathbf{N} + \kappa \tau(s')^3 \mathbf{B} \), and finally (d) \( \tau = (\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''/|\mathbf{r}' \times \mathbf{r}''|^2 \).

5. Find the tangential and normal components of the acceleration vector for the position vector \( \mathbf{r}(t) = (1 + t) \mathbf{i} + \sin 3t \mathbf{j} + \cos 3t \mathbf{k} \).

6. Find the limit \( \lim_{(x,y) \to (1,2)} xy \cos(2x - y) \) using the \( \varepsilon-\delta \) definition.

7. Find the limit \( \lim_{(x,y) \to (0,0)} (x^2 + y^2)^3 \ln(x^2 + y^2)^3 \).

8. Find \( \partial z/\partial x \) and \( \partial z/\partial y \) for \( yz = x \ln(y + z) \).

9. Find partial derivative \( f_{trrs} \) for \( f(r, s, t) = r^2 t \ln(r s^2 e^t) \).

10. Determine whether each of the following advection diffusion equation \( u_t - u_{xx} + u_x = 0 \). (a) \( u = e^{t+x} \), (b) \( u = e^{2t+2x} \), (c) \( u = e^{2t-x} \), (d) \( u = e^{t-2x} \).

11. Find an equation of the tangent plane to the surface \( z = e^{2-x^2-y^2} \ln(1 + x^2) \) at the point \((1, -1, \ln 2)\).
12. The dimensions of a quarter (United States coin) are measured to be 24.4 mm (diameter) and 1.8 mm (thickness). Each measurement is correct to within 0.2 mm. Use differentials to estimate the largest possible error when the volume of the quarter is calculated from these measurements. (Use $\pi \approx 3.14$.)

13. Prove that if $f(x, y)$ is differentiable at $(a, b)$, then $f(x, y)$ is continuous at $(a, b)$.

14. Let $f(x, y) = x^2 y/(x^4 + y^2)$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Is the function $f$ differentiable at $(0, 0)$? Explain.

15. Find $dw/dt$ for $w = xye^z$, $x = t$, $y = t^2$, $z = \ln t$.

16. Find $dy/dx$ for $y^4 + x\cos y = y + \ln x$.

17. Suppose that the equation $F(x, y, z) = 0$ implicitly defines each of the three variables $x$, $y$, and $z$ as functions of the order two: $z = f(x, y)$, $y = g(x, z)$, $x = h(y, z)$. If $F$ is differentiable and $F_x$, $F_y$, and $F_z$ are all nonzero, show that \[ \frac{\partial z}{\partial x} \frac{\partial x}{\partial y} \frac{\partial y}{\partial z} = -1 . \]

18. Near a buoy, the depth of a lake at the point with coordinates $(x, y)$ is $z = 10 + x^2 - y^3 + 20y$, where $x$, $y$, and $z$ are measured in meters. A fisherman in a small boat starts at the point $(4, 3)$ and moves toward the buoy, which is located at $(0, 0)$. Is the water under the boat getting deeper or shallower when he departs? Explain.

19. If $g(x, y) = x^2 + y^2 + 2xy + 2x - 2y$, find the gradient vector $\nabla g(1, -1)$ and use it to find the tangent line to the level curve $g(x, y) = 4$ at the point $(1, -1)$. Sketch the level curve, the tangent line, and the gradient vector.

20. Show that the function $f(x, y) = (xy)^{1/3}$ is continuous and the partial derivatives $f_x$ and $f_y$ exist at the origin but the directional derivatives in all other directions do not exist.

21. Find the local maximum and minimum values and saddle point(s) of the function $f(x, y) = xye^{-x-2y}$.
22. A rectangular house is designed to minimize heat loss. The east and west walls lose heat at a rate of 9 units/m² per day, the north and south walls at a rate of 8 units/m² per day, the floor at a rate of 1 units/m² per day, and the roof at a rate of 5 units/m² per day. Each wall must be at least 20 m long, the height must be at least 2 m, and the volume must be exactly 1000 m³. Find the dimensions that minimize heat loss. Is it possible to design a house with even less heat loss if the restrictions on the lengths of the walls were removed?

23. The plane $\frac{\sqrt{5}}{2}x - y + \frac{11}{2}z = 2$ intersects the cone $z^2 = x^2 + y^2$ in an ellipse. Find the highest and lowest points on the ellipse.

24. Evaluate the double integral $\iint_D \left(1 - \sqrt{x^2 + y^2}\right) dA$, $D = \{(x, y)|-1 \leq x \leq 1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}\}$.

25. Evaluate the double integral $\iint_D \left(1 - \sqrt{x^2 + y^2}\right) dA$, $D = \{(x, y)|0 \leq \sqrt{x^2 + y^2} \leq 1\}$ by first identifying it as the volume of a solid.

26. Evaluate the integral $\iint_D \left(1 - \sqrt{x^2 + y^2}\right) dA$ where $D = \{(x, y)|0 \leq x^2 + y^2 \leq 1\}$ by changing to polar coordinates.

27. Calculate the double integral $\iint_R \sin(3x + 2y) dA$, $R = \{(x, y)|0 \leq x \leq \pi/2, 0 \leq y \leq \pi/2\}$

28. Find the volume of the solid that lies under the hyperbolic paraboloid $z = 2 + x^2 - 3y^2$ and above the square $R = [1, 2] \times [-1, 1]$.

29. Find the average value of $f(x, y) = \ln(x + y + 1)$ over the rectangle $R = [0, 2] \times [0, 3]$.

30. Let $f(x, y) = xy(x^2 - y^2)/(x^2 + y^2)^3$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Show that Fubini’s theorem does not hold and $\int_0^1 \int_0^2 f(x, y) dxdy \neq \int_0^2 \int_0^1 f(x, y) dydx$. 

3
31. Evaluate the iterated integral \( \int_{0}^{1} \int_{x}^{1-x} (x - y^2) \, dy \, dx \)

32. Evaluate the double integral \( \iint_{D} (x + y) \, dA \), \( D \) is bounded by \( y = x^{1/3} \) and \( y = x^3 \)

33. Evaluate the double integral \( \iint_{D} xy \, dA \), \( D \) is the triangular region with vertices \((0, 0)\), \((3, 2)\), and \((0, 4)\)

34. Find the volume of the solid under the surface \( z = x + y^2 \) and above the region bounded by \( x = \sqrt{y} \) and \( x = y^3 \).

35. Show that \( \iint_{D} (x^2 + y^2 - 4x - 2y + 5)^{-3/2} \, dA \leq \pi/2 \), where \( D \) is the disk with center \((2, 1)\) and radius 2.

36. Find the volume of the solid inside the sphere \( x^2 + y^2 + z^2 = 9 \) and outside the cylinder \( x^2 + z^2 = 1 \).

37. Evaluate the iterated integral \( \int_{0}^{1} \int_{x}^{\sqrt{4x-x^2}} (x^2 + y^2) \, dy \, dx \)

38. Show the Gaussian integral \( \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \).

39. Find the mass \( m \) and center of mass \((\bar{x}, \bar{y})\) of the lamina that occupies the Cardioid \( D = \{(r, \theta) | 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1 + \sin \theta\} \) when the density function is \( \rho(x, y) = \sqrt{x^2 + y^2} \).

40. Suppose \( X \) and \( Y \) are random variables with joint density function
\[
f(x, y) = \sqrt{\frac{2}{\pi}} e^{-(x-1)^2/2} e^{-2y} \text{ for } -\infty < x < \infty, 0 \leq y < \infty.
\]
Find \( P(Y \geq \ln 3) \), and the expected values of \( X \) and \( Y \).

41. Find the area of the surface which is the part of the sphere \( x^2 + y^2 + z^2 = 9 \) that lies within the cylinder \( x^2 + y^2 - 3y = 0 \).

42. Evaluate the triple integral \( \iiint_{E} x^3 e^y \, dV \), where \( E \) is bounded by the parabolic cylinder \( z = 1 - 2y^2 \) and the planes \( z = 0, x = 0, \) and \( x = 1 \).