Stabilization of rigid body dynamics by the Energy–Casimir method

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Abstract: We show how the Energy–Casimir method can be used to prove stabilizability of the angular momentum equations of the rigid body about its intermediate axis of inertia, by a single torque applied about the major or minor axis. We also show how this system has associated with it, a Lie–Poisson bracket which is invariant under SO(3) for small feedback, but is invariant under SO(2, 1) for feedback large enough to achieve stability.

Keywords: Stabilization; feedback; rigid body; Energy–Casimir; Hamiltonian.

Introduction

There has been much interest over the past decade in the problem of stabilizing the angular momentum equations and attitude equations of the rigid body with \( n \leq 2 \) torques. Work related in spirit to ours includes that of Baillieul [6], Bonnard [10], Brockett [11,12], Crouch [14], Aeyels [1,2], Krishnaprasad [16], Alvarez-Sanchez [4], Aeyels and Szafranski [3], Sontag and Sussmann [21], and Byrnes and Isidori [13].

Byrnes and Isidori show that with two (independent) torques (such as gas jets) the full attitude equations may be asymptotically stabilized to revolve motion about a principal axis.

In Brockett [12], it is shown by finding a Lyapunov function that the null solution of the angular velocity equations may be stabilized by two control torques. In Aeyels [2], the same result is demonstrated by center manifold theory. In Aeyels [1], it is shown that the null solution of the angular velocity equations may be "robustly" stabilized (though not asymptotically stabilized) by a single torque about the major or minor axis. This result is in fact sharp since Aeyels and Szafranski [3] show that the equations cannot be asymptotically stabilized by a single torque about a principal axis.

Another area in which there has been much fruitful activity in recent years is the analysis of the stability of coupled rigid and flexible bodies. Some new techniques for this problem were introduced in Baillieul and Levi [7] and Krishnaprasad and Marsden [16] both based on geometric formulations of Lagrangian and Hamiltonian mechanics. Specifically, the latter paper uses reduction and the Energy–Casimir method to analyze the stability of a rigid body with attached shear beam. This method, going back to Arnold [5], involves an analysis in the body (also called the convective) representation and utilizes the conserved energy and momentum. In more recent work, the energy–momentum method for analysis in the material representation, has been developed. See, for example, Simo, Posbergh and Marsden [19] and Simo, Lewis and Marsden [20]. In this method a block diagonalization theorem which can greatly simplify computations is available.

Our goal in this paper is to show that the latter type of stability analysis, in particular, the Energy–Casimir method, can be used to prove a stabilization result – namely that the angular momentum equations of the rigid body can be stabilized by feedback about the intermediate unstable axis of inertia by a single torque applied.
about its major or minor axis. The remarkable feature of this analysis is that there are still conserved quantities, even when torque is being applied. Further, our feedback and method of analysis is quite different from Aeyels's [1,2]. We show moreover, that the system under feedback is a Lie–Poisson (Hamiltonian) system and that the invariance group of the Lie–Poisson bracket associated with the system changes from SO(3) to SO(2, 1) as the magnitude of the feedback increases sufficiently to achieve stability. Some of these ideas are also discussed in Bloch and Marsden [9]. The deformation of the Poisson structure from SO(3) to SO(2, 1) (and also to SL(2, R)) is discussed for other purposes in Weinstein [22]; that paper also looks at a larger 4-dimensional Poisson manifold in which our feedback parameter k is a Casimir. It is also noted there that at the transition value of k, one gets a Heisenberg algebra times R for the transverse Lie algebra. This might be useful for the study of the dynamics near the transition value.

The present paper is motivated in part by our work on the stabilization of the dynamics near heteroclinic cycles for a model in the near wall region for the Navier–Stokes equation (see [9]). We think that the techniques developed here will be useful in this problem as well as in the problem of rotating rigid and flexible structures. In fact, in another paper, we show how these ideas can be used for the problem of a rigid body with internal rotors, as in Krishnaprasad [16].

1. Equations of motion and Poisson structure

We will work with the Hamiltonian (Poisson) formulations of the equations of motion for the rigid body, but we begin with the Lagrangian formulation and then transform.

The rigid body equations with a single torque about the minor axis are given by

\[ \dot{\omega}_1 = \frac{(I_2 - I_1)}{I_1} \omega_2 \omega_3, \]  
\[ \dot{\omega}_2 = \frac{(I_1 - I_2)}{I_2} \omega_3 \omega_1, \]  
\[ \dot{\omega}_3 = \frac{(I_2 - I_1)}{I_3} \omega_1 \omega_2 + u, \]

where the \( I_i \) are the principal moments of inertia and we take \( I_1 > I_2 > I_3 \). Note that the control torque is taken to be about the minor axis, but all our results carry over without essential change to the case where the torque is about the major axis.

Now we employ the feedback

\[ u = -k \frac{I_1 I_2}{I_3} \omega_1 \omega_2, \]  

where k is the feedback gain parameter. (The normalization with the \( I_i \)'s is only to make the Hamiltonian picture cleaner.) We refer to the system with this feedback as the controlled system. It is now easy to show:

**Lemma 1.1.** For the controlled system (1.1), (1.2), the quantities

\[ E_c = \frac{1}{2} \left( I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 \frac{a_3}{a_3 - k} \right), \]

and

\[ M_c^2 = \frac{1}{2} \left( I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 \frac{a_3}{a_3 - k} \right), \]

where \( a_3 = (I_1 - I_2)/I_1 I_2 \), are constants of the motion.

**Proof.** Note that

\[ \frac{d}{dt} \omega_3^2 = \frac{I_1 - I_2 - k I_1 I_2}{I_3} \omega_1 \omega_2 \omega_3 \]

\[ = \frac{I_1 - I_2}{I_3} \frac{a_3 - k}{a_3} \omega_1 \omega_2 \omega_3. \]

Then the calculations reduce to the standard rigid body calculations. □

Now make the Legendre transformation

\[ m_1 = \frac{\partial E_c}{\partial \omega_1} = I_1 \omega_1, \]  
\[ m_2 = \frac{\partial E_c}{\partial \omega_2} = I_2 \omega_2, \]  
\[ m_3 = \frac{\partial E_c}{\partial \omega_3} = I_3 \omega_3 \frac{a_3}{a_3 - k}. \]

Then the equations of motion of the system (1.1), (1.2) become

\[ \dot{m}_1 = a_3 - k \frac{a_3 - k}{a_3} m_2 m_3, \]  

(1.6a)
The controlled system (1.6) is Lie-Poisson with Hamiltonian $H$ given by (1.7), with respect to the Lie-Poisson structure

\[
\{ F, G \}_1 = -\nabla M^2_1 \cdot (\nabla F \times \nabla G)
\]  

where $M^2_1$ is given by (1.8).

**Theorem 1.2.** The controlled system (1.6) is Lie-Poisson with Hamiltonian $H$ given by (1.7), with respect to the Lie-Poisson structure

\[
\{ F, G \}_1 = -\nabla M^2_1 \cdot (\nabla F \times \nabla G)
\]  

where $M^2_1$ is given by (1.8).

**Proof.** One readily checks that

\[
\dot{m}_i = \{ m_i, H \}_1, \quad i = 1, 2, 3,
\]  

which gives the result. \( \Box \)

This result is important as it places this system in a large class of physical systems of interest which are Lie-Poisson (see [15,16,17]) and further enables us to apply the Energy-Casimir method for stability, as discussed in the next section.

Remarkably, there are three other Poisson structures for our system with corresponding conserved Hamiltonians and momenta. We list the structures in the appendix. We also discuss in the appendix the change in the invariance group of the Poisson structure and the change in the nature of level surfaces as $k$ changes.

### 2. Stability

The free rigid body equations (1.6) with $k = 0$, have relative equilibria when

\[
(m_1, m_2, m_3) = (M, 0, 0),
\]

\[
(m_1, m_2, m_3) = (0, 0, M)
\]

and when

\[
(m_1, m_2, m_3) = (0, M, 0).
\]

The first two cases, which correspond to rotation about the major or minor axis are well known to be nonlinearly stable (see, for example, [15] and the references therein), while the last case, rotation about the intermediate axis, is unstable. Here we consider the question: for $k$ sufficiently large, can we stabilize the system about its intermediate axis, i.e., about the relative equilibrium $(m_1, m_2, m_3)$?

Consider the system in the form (1.6) with quantities $H_1$ and $M_1$ and the bracket $\{ F, G \}_1$ in (1.9). Consider firstly the system linearized about $(0, M, 0)$. Its eigenvalues are given by the solutions of

\[
\lambda \left( \lambda^2 - a_1 a_3, k \right) = 0.
\]

Hence for $k = 0$ the system is unstable, but for $k > a_3$, we have two imaginary eigenvalues and one zero eigenvalue. Is the system stable? We shall prove that it is via the ‘Energy-Casimir’ method.

We now give a summary of the Energy-Casimir method for finite-dimensional systems for the convenience of the reader. For the infinite-dimensional case, see [15,17]. The method proceeds in the following algorithmic way:

**Step 1.** Write the equations of motion in first-order form

\[
\dot{u} = F(u)
\]

where $u \in P$, a differentiable manifold. (Often $P$ is a Poisson manifold, i.e., a manifold admitting a Poisson bracket and (2.2) is given in Poisson bracket form.) Find a conserved function $H$ for (2.2), i.e., a function $H(u)$ such that $(d/dt) H(u) = 0$ for any $C^1$ solution of (2.2). (Usually $H$ is the energy.)

**Step 2.** Find a family of constants of the motion for (2.2). (Often the $C$’s are taken to be Casimirs...
Step 3. Find a $C$ such that $H + C$ has a critical point at the (relative) equilibrium of interest.

Step 4. Definiteness of the second variation of $H + C$ at the critical point is then sufficient to prove nonlinear (Lyapunov) stability. (In the infinite-dimensional case there are additional technical conditions required, but our example is finite dimensional).

We use this method to prove the following:

**Theorem 2.1.** The rigid body equation with a single torque about the minor axis and with feedback $u = -km_1m_2$, i.e., the system (1.6), is stabilized about the relative equilibrium

$$(m_1, m_2, m_3) = (0, M, 0)$$

for $k > a_3 = (I_1 - I_2)/I_1I_2$.

**Proof.** Consider the Energy–Casimir function

$$H + C = H_1 + \phi(M_1^2)$$

where $H_1$ and $M_1^2$ are given by (1.7) and (1.8) respectively. We know $H_1$ and $M_1$ are conserved and $M_1$ is a Casimir for the bracket $\{F, G\}_1$. Here, $\phi$ is an arbitrary smooth function. Now the first variation of $H_1 + \phi(M_1^2)$ is given by

$$\delta(H_1 + \phi(M_1^2)) = \left(\frac{m_1}{I_1}\delta m_1 + \frac{m_2}{I_2}\delta m_2 + \frac{m_3}{I_3}a_3 - k\delta m_3\right)$$

$$+ \phi'(M_1^2)\left(m_1\delta m_1 + m_2\delta m_2 + m_3\frac{a_3 - k}{a_3}\right) \tag{2.3}$$

This equals zero if

$$\frac{m_1}{I_1} + \phi'm_1 = 0 \tag{2.4a}$$

$$\frac{m_2}{I_2} + \phi'm_2 = 0 \tag{2.4b}$$

$$\frac{m_3}{I_3}a_3 - k + \phi'm_3\frac{a_3}{a_3 - k} = 0 \tag{2.4c}$$

At the relative equilibrium $(m_1, m_2, m_3) = (0, M, 0)$, (2.4) will hold if $\phi' = -1/I_2$. Then

$$\delta^2(H_1 + \phi(M_1^2))$$

$$= \frac{(\delta m_1)^2}{I_1} + \frac{(\delta m_2)^2}{I_2} + \frac{(\delta m_3)^2}{I_3}a_3 - k$$

$$- \frac{1}{I_2}\left((\delta m_1)^2 + (\delta m_2)^2 + (\delta m_3)^2\frac{a_3 - k}{a_3}\right)$$

$$+ \phi''(M_1^2)M_1^2(\delta m_1)^2. \tag{2.5}$$

at the equilibrium of interest.

Since $I_1 > I_2 > I_3$ and $a_3 = (I_1 - I_2)/I_1I_2$, for $k$ sufficiently large that $a_3 - k < 0$ and choosing $\phi'' < 0$, the second variation is negative definite and we have nonlinear stability. \(\Box\)

A similar argument holds for a torque about the major axis. We remark also that the origin, $(m_1, m_2, m_3) = (0, 0, 0)$ is nonlinearly stable for the free system as can be seen by using the free Hamiltonian $(H_1$ in (1.7) with $k = 0)$ as a Lyapunov function.

We hope in future to apply this method to stabilization analysis for more complex multibody systems and some fluid mechanical systems. In particular, in another publication, we show how to apply these ideas to the rigid body with internal rotors. We also intend to explore the use of the Energy–Momentum method mentioned above, in addition to the Energy–Casimir method which should give results on attitude stabilization.

### Appendix: Other Poisson structures and their invariance group

In this appendix we first list the three other Poisson structures for our system. (These changes of variables are useful for different applications.) Then we discuss the nature of the invariance group of the Poisson structures.

The system is Poisson with the Poisson structure

$$\{F, G\}_2 = -\nabla M_2 \cdot (\nabla F \times \nabla G) \tag{A.1}$$

with conserved momentum squared

$$M_2^2 = \frac{1}{2}\left(m_1^2\frac{a_3 - k}{a_3} + m_2^2\frac{a_3 - k}{a_3} + m_3^2\left(\frac{a_3 - k}{a_3}\right)^2\right) \tag{A.2}$$
and conserved Hamiltonian
\[
H_2 = \frac{1}{2} \left( m_1^2 \frac{a_3}{T_1} + m_2^2 \frac{a_3}{T_2} + m_3^2 \frac{a_3}{T_3} \right). 
\]

Changing variables to the ‘classical’ momentum variables
\[
m_1 = I_1 \omega_1, \quad m_2 = I_2 \omega_2, \quad \hat{m}_3 = I_3 \omega_3, 
\]
the system is Hamiltonian with Lie–Poisson structure
\[
\{ F, G \}_3 = -\nabla M_3^2 \cdot (\nabla F \times \nabla G)
\]
where the conserved momentum squared is
\[
M_3^2 = \frac{1}{2} \left( m_1^2 \frac{a_3 - k}{a_3} + m_2^2 \frac{a_3 - k}{a_3} + \hat{m}_3^2 \right)
\]
and the conserved Hamiltonian is
\[
H_3 = \frac{1}{2} \left( m_1^2 \frac{\omega_1}{T_1} + m_2^2 \frac{\omega_2}{T_2} + \hat{m}_3^2 \frac{\omega_3}{T_3} \right).
\]

Finally, the system is also Hamiltonian with Lie–Poisson structure
\[
\{ F, G \}_4 = -\nabla M_4^2 \cdot (\nabla F \times \nabla G)
\]
where
\[
M_4^2 = \frac{1}{2} \left( m_1^2 + m_2^2 + \hat{m}_3^2 \frac{a_3}{a_3 - k} \right)
\]
and
\[
H_4 = \frac{1}{2} \left( m_1^2 \frac{a_3 - k}{a_3} + m_2^2 \frac{a_3 - k}{a_3} + \hat{m}_3^2 \right).
\]

In each case the quantity \( M_i \) is a Casimir – that is, it commutes with every other quantity under the bracket.

Since the brackets \( \{ F, G \} \), are Lie–Poisson for the invariance group of the quadratic function \( M_i^2 \), we see:

**Proposition A.1.** The invariance group of the quadratic functions \( M_i^2, \, i = 1, \ldots, 4 \), corresponding to each of the Lie–Poisson structures \( \{ F, G \} \), changes from SO(3) to SO(2, 1) as \( k \) becomes larger than \( a_3 \).

Note also that the level surfaces \( M_i^2 = c \) change from ellipsoids to hyperboloids as \( k \) becomes larger than \( a_3 \).

### References


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