Finite Controllability of Infinite-Dimensional Quantum Systems

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Abstract—**Quantum phenomena of interest in connection with applications to computation and communication often involve generating specific transfers between eigenstates, and their linear superpositions. For some quantum systems, such as spin systems, the quantum evolution equation (the Schrödinger equation) is finite-dimensional and old results on controllability of systems defined on on Lie groups and quotient spaces provide most of what is needed insofar as controllability of non-dissipative systems is concerned. However, in an infinite-dimensional setting, controlling the evolution of quantum systems often presents difficulties, both conceptual and technical. In this paper we present a systematic approach to a class of such problems for which it is possible to avoid some of the technical issues. In particular, we analyze controllability for infinite-dimensional bilinear systems under assumptions that make controllability possible using trajectories lying in a nested family of pre-defined subspaces. This result, which we call the Finite Controllability Theorem, provides a set of sufficient conditions for controllability in an infinite-dimensional setting. We consider specific physical systems that are of interest for quantum computing, and provide insights into the types of quantum operations (gates) that may be developed.**

I. Introduction

Over the last three decades, there has been a steady stream of papers in the physics/chemistry literature [1], [2], [3], [4], [5] that describe new experiments in atomic and molecular science, and new ways of thinking in which control-theoretic ideas are of central importance. More recently, the driving force has been the desire to manipulate quantum states in ways that would make possible quantum computation or quantum communication, see for example [6], [7], [8], [9], [10], [11]. Phenomena involving the interaction between electromagnetic radiation (light) and matter (e.g. ions, spin states, etc.) are especially interesting because they are possible paradigms of future quantum computing devices [12]. Many of the exciting ideas are related to the control of these systems (see for example, work on control of trapped-ion quantum states [13], [14], [15], [16], [17], [18]). Some infinite-dimensional systems can be made to be effectively finite-dimensional by either bandwidth limits imposed by the control fields [6], or by turning off specific transitions in order to truncate the Hilbert space [17], and the controllability of such systems can be analyzed using finite-dimensional methods [19], [20], [21], [22].

We are interested in the quantum systems that are modelled as finite-dimensional for quantum computing purposes, when in fact they are infinite-dimensional. The well-known paper by Huang, Tarn and Clark [1] seems to be overly pessimistic with repect to the control of infinite quantum systems by asserting that "using piecewise-constant controls, global controllability cannot be achieved with a finite number of operations". In 2000, Zuazua summarized the field aptly thus: "From a mathematical point of view this [when the finite-dimensional system approaches the infinite-dimensional Schrödinger equation] is a very challenging (and very likely difficult) open problem in this area" [23]. Recently, Turinici and Rabitz [24] adapted the classical Ball-Marsden-Slemrod results [25] to the quantum setting and showed that exact controllability does not hold in infinite-dimensional quantum systems (see also [26]). Along with an excellent review, Illner, Lange and Teismann [27] show that the Hartree equation that is well-known in quantum chemistry with bilinear control is not controllable in finite or infinite time. In spite of these negative results, most quantum computing systems that are indeed infinite-dimensional have shown themselves to be remarkably amenable to the production of a variety of states. Is it possible then to make a statement regarding the reachable set of states in such systems?

There has been recent progress in attacking this problem, and a few positive results. Ref. [28] showed that for an infinite-dimensional quantum control system with bounded control operators a quantum state could be steered only within a dense subspace of the relevant Hilbert space. In 2003 [16], we showed that one can reach any finite linear superposition of states in a two-level system coupled to a harmonic oscillator by the alternate application of control fields even when one of the control operators is unbounded. Ref. [29] presents an ingenious method of creating arbitrary finite superpositions of rotational states of molecules using the theory of loop groups.. Ref. [30] presents a scheme that is well-known in atomic physics - when an infinite-dimensional quantum system has unequally spaced bound state energy levels, single resonant fields can be used to transfer population (albeit very slowly). Ref. [31] provides a prescription for determining whether there exists a submanifold that is strongly analytic controllable, but this prescription does not identify the submanifolds. Ref. [32] presents an adiabatic method of controlling a sequentially

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connected system in which the transition couplings get weaker as one moves away from the ground state. Ref. [33] presents an algebraic framework to determine the nature of controllability of some infinite-dimensional quantum systems, specifically those with continuous spectra. Ref. [34] proves approximate controllability for bound states of quantum systems that are not equally spaced in energy even when the control matrices are unbounded.

We begin this paper in Section II by discussing the difficulties associated with applying traditional methods of analysis to infinite-dimensional control systems. We also discuss a specific class of infinite-dimensional systems — one in which it is desirable to steer a finite superposition of eigenstates to another finite superposition of eigenstates. We present a general theorem on Finite Controllability of quantum systems that exhibits the conditions needed for such transfers. For the specific quantum-computing models of interest, the trajectories are constrained to lie within subspaces. Section III presents examples of quantum-computing systems such as a model of a trapped-ion qubit and trapped-electron qubits. We show how our theorem can used to be prove finite controllability in the former case. We also discuss other similar systems where the theorem can and cannot be applied.

II. Infinite-Dimensional Controllability

Controllability results for infinite-dimensional systems are seldom just straightforward extensions of the finite-dimensional ones, and in particular this is true for bilinear systems. Recently, there has been significant interest in the class of bilinear systems because of their relevance to quantum control. In the following subsection we illustrate the limitations of applying the tools of finite-dimensional systems analysis to certain classes of infinite-dimensional systems.

A. Limitations of Lie Algebraic analysis

Examining the Lie algebraic structure often gives us insights into the controllability of a quantum system, but in the case of infinite-dimensional systems, this insight is limited. In the well-known example of a resonantly-driven quantum harmonic oscillator (see e.g. [35], [16] and [36], the evolution is given by

$$\frac{\partial \psi}{\partial t} = \left(\omega \frac{i}{2} \left(\frac{\partial^2}{\partial x^2} - x^2 \right) - iu(t)x \right) \psi. \tag{1}$$

Here, the bilinear control term $u(t)x$ arises because of the dipole interaction between the field and harmonic oscillator. The two operators of interest, $A = \frac{i}{2}\left(\frac{\partial^2 \psi}{\partial x^2} - x^2\right)$ and $B = -ix$ generate a Lie algebra of skew-hermitian operators that is just four-dimensional. Thus, we expect that the control of this system will be limited. For example, it is well-known that it is not possible to transfer the number state $x(0) = |0\rangle$ to $x(T) = |n\rangle$ for $n > 0$ [37] using this control. We return to this example in Section III.

However, even when we encounter infinite-dimensional systems for which the Lie algebra also is infinite-dimensional (such as in Ref.[38], [16]), the statements one can make about the controllability are also limited. More work is required to say with precision exactly what the reachable states are. This issue is illustrated in Section III.

In computing the span of the Lie algebra, it is necessary, of course that the domain of the operators involved be such that the bracket operations are allowable. This is the case in this paper, but we do not discuss these technical details here.

B. Finite Controllability

In this section, we prove an elementary but useful theorem about controllability on finite-dimensional subspaces of a complex Hilbert space. It is the basis for the applications we will present in the next section.

Definition 1: Given a complex Hilbert space $\mathcal{H}$ and a finite collection of skew-hermitean infinitesimal generators $\{G_1, G_2, ..., G_m\}$, we will say that the system

$$\dot{x} = \sum_{i=1}^{m} u_i G_i x$$

is *unit vector controllable* in a subspace $\mathcal{H}_\alpha$ if any unit length vector $x_0 \in \mathcal{H}_\alpha$ can be steered to any second unit length vector $x_f \in \mathcal{H}_\alpha$ in finite time, using piecewise constant controls with a finite number of discontinuities. We will say that a piecewise constant control defined on $[0, T]$ is a *pure control* if there is only one u_i that is nonzero at each point in time.

Remark: Suppose we are given the above system together with a nested set of finite-dimensional subpaces $\mathcal{H} = \{\mathcal{H}_1 \subset \mathcal{H}_2 \subset \mathcal{H}_3 ...\}$, and suppose that for each G_i and each $\mathcal{H}_j$ there is a second subspace $\mathcal{H}_k$ containing $\mathcal{H}_i$ such that $\mathcal{H}_k$ is invariant for G_i. Clearly there is no question about existence of solutions for $\dot{x} = \sum_{i=1}^{m} u_i G_i x$ provided that one restricts u to being piecewise-constant pure controls, and the initial condition lies in one of the invariant subspaces in the chain $\mathcal{H} = \mathcal{H}_1 \subset \mathcal{H}_2 \subset \mathcal{H}_3 ....$ In this case, the solution at time T is of the form of a product of exponentials $x(T) = e^{G_i \alpha_i} e^{G_j \alpha_{i=1}} \cdots e^{G_1 \alpha_1} x(0)$.

Theorem 2 (Finitely Controllable Infinite Dimensional Systems): Consider a complex Hilbert space $\mathcal{X}$ together with a nested set of finite-dimensional subpaces $\mathcal{H} = \{\mathcal{H}_1 \subset \mathcal{H}_2 \subset \mathcal{H}_3 ...\}$. Consider

$$\dot{x} = (\sum_{i=1}^{m} u_i G_i) x.$$

Assume that $\mathcal{H}_1$ is an invariant subspace for a subset $\mathcal{G}_1$ of the set $\{G_i\}$ and that the system is unit vector controllable on $\mathcal{H}_1$ using only this subset of the G_i. If for each $\mathcal{H}_\alpha$ $\alpha \neq 1$ there is a subset $\mathcal{G}_\alpha$ of $\{G_i\}$ that leaves $\mathcal{H}_\alpha$ invariant and if for any unit vector in $\mathcal{H}_\alpha$ at least one of the solutions generated using pure controls associated with elements of $(\mathcal{G}_\alpha)$ contains a point in one of the lower dimensional subspaces $\mathcal{H}_\beta$, then any unit vector in any of the $\mathcal{H}_i$ can be steered to any other unit vector in any other $\mathcal{H}_j$ using a finite number of piecewise constant controls.

Remark: Given a system and a nested set of finite dimensional subspaces compatable with it as described above, it will be said to be *finitely controllable* if it can be transfered from any point in one of the subpaces to any point in any other one of the subspaces with a trajectory lying entirely within the smallest subspace in the nested set that contains both.

Proof: Suppose that $x_0 \in \mathcal{H}_\alpha$ and $x_f \in \mathcal{H}_\beta$ are unit vectors representing the initial value of x and the desired final value, respectively. Then, by assumption, either there exists a subset of the $\{G_i\}$ that leaves $\mathcal{H}_\alpha$ invariant and steers x_0 to a point in some $\mathcal{H}_\beta$ with the dimension of $\mathcal{H}_\beta$ being strictly less than that of $\mathcal{H}_\alpha$, or else, $x_0 \in \mathcal{H}_1$ and can be steered to any other point in $\mathcal{H}_1$. A finite induction on the index of the set $\mathcal{H}_i$ then shows that x_0 can be steered to a unit vector in the controllable subspace $\mathcal{H}_1$. To finish the proof, observe that if one can reach $x_a \in \mathcal{H}$ from $x_b \in \mathcal{H}$ then the standard time reversal argument using the fact that the G_i are skew-Hermetian shows that it is possible to reach $x_b \in \mathcal{H}$ from $x_a \in \mathcal{H}$. Thus a second application of the steps given above implies that it is possible to reach $x(T) \in \mathcal{H}_\beta$. ■

III. Physical Systems

In this section, we apply the Finite Controllability Theorem to determine the reachable set of states of four infinite-dimensional quantum systems, namely, the quantum harmonic oscillator, a spin-half system coupled to a harmonic oscillator (model of a trapped-ion qubit), an N-level atom coupled to harmonic oscillator, and a spin-half system coupled to two harmonic oscillators (model of a trapped electron qubit). Note that in all sections below we use atomic/scaled/dimensionless units so that the evolution equations do not explicitly contain Planck's constant $\hbar$, the charge of the electron e or the mass of the electron m_e.

A. System 1: Quantum harmonic oscillator

We discuss this well-known system first. The controllability algebra is finite-dimensional and, in particular, the system does not satisfy the conditions needed for the application of the Finite Rank Controllability Theorem. The discussion, however, is key to setting up the formalism used in subsequent examples that are finitely controllable.

The problem of controlling the harmonic oscillator has been discussed many times (see e.g. [39] and other references as discussed above). If the control is a sinusoidal resonant driving field (of frequency equal to the harmonic oscillator frequency ω_m) as shown in the transfer graph Fig. 1, then the evolution is via equation (1) at frequency ω_m.

Here, the control term $u(t)x$ arises because of the dipole interaction between the field and harmonic oscillator. The operators of interest are $A = \frac{i}{2}\left(\frac{\partial^2}{\partial x^2} - x^2\right)$ and $B = -ix$. A and B generate a Lie algebra of skew-hermitian operators that is just four-dimensional ($C = [A, B] = \frac{\partial}{\partial x}$, $D = [B, C] = iI$, where I is the identity operator). This in itself tells us that the resonantly driven harmonic oscillator is not controllable.

As is well-known, the spectrum of A is discrete. If we describe the evolution in terms of an eigenfunction expansion, with the basis being $|n\rangle$'s, the eigenfunctions of $\partial^2/\partial x^2 - x^2$, then the evolution is via

$$\begin{aligned} \dot{x}_n &= -i\omega_m(n + \frac{1}{2})x_n \\ &\quad -u(t)\frac{i}{\sqrt{2}}(\sqrt{n-1}x_{n-1} - \sqrt{n}x_{n+1}). \end{aligned} \tag{2}$$

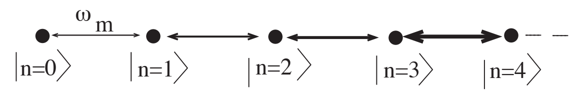

Fig. 1. Graphical representation of the quantum harmonic oscillator driven by a sinusoidal resonant field. Note that while the strengths of the transition couplings increase as the square root of the quantum number n as shown by the boldness of the connections between energy levels, the transition frequency between each level is the same. It is this latter property that leads to a lack of controllability.

In this example the subspaces $\mathcal{H}_i$ of Theorem 2 are identified with the linear span of the eigenspaces of the harmonic oscillator up to quantum number i.

Although the eigenstates of the harmonic oscillator can be written as an infinite set of nested finite-dimensional subspaces, it is seen that the operator B connects space $\mathcal{H}_i$ to *both* $\mathcal{H}_{i-1}$ and $\mathcal{H}_{i+1}$. Thus finite superpositions of eigenstates may not be reached by resonantly driving the harmonic oscillator, consistent with the fact that the requirements of the Finite Controllability Theorem are not met. Physically, this is due to the degeneracy of spacings between the eigenstates and the fact that the control vector field simultaneously illuminates all states.

B. System 2: Spin-half particle in a quadratic potential

In contrast to the harmonic oscillator, the model of a spin-half particle coupled to a harmonic oscillator with suitable controls turns out to be finitely controllable. This model is a good representation of an ion with two essential internal states trapped in a quadratic potential. We show below that this system satisfies the conditions of the Finite Controllability Theorem of Section 2. Moreover, one can also provide an algorithm for explicit control. The spin-$\frac{1}{2}$ model represents a two-level atomic ion with an energy splitting $\hbar\omega_0$, where the frequency $\omega_0/2\pi$ is in the several GHz range. The atomic levels are coupled to the motion of the ion in a harmonic trap [40]. These quantized vibrational energy levels are separated by a frequency $\omega_m/2\pi$ in the MHz range.

In a frequently cited paper, Law and Eberly [13] showed that when properly interpreted, this system has interesting controllability properties, quite different from the properties of the harmonic oscillator alone. In fact, by coupling the harmonic oscillator with a two-level system it is possible to arrive at a system which is much more controllable than the harmonic oscillator. At an intuitive level, this can be seen simply as a consequence of the fact that the addition of a spin degree of freedom breaks the infinite degeneracy associated with the harmonic oscillator and allows the system to resonate with more than one frequency. This allows the transfer of population from any eigenstate to any other eigenstate by sequentially applying the two frequencies. We now analyze this system from a controllability viewpoint.

An eigenstate of the spin-half system coupled to a quantum harmonic oscillator is denoted by $|S, n\rangle$, where the first index refers to the "spin" state of the system, and the second index is the number state of the harmonic oscillator. An applied field causes transitions between the eigenstates of the coupled spin-oscillator system. A monochromatic field of angular frequency

$\omega = \omega_0$ causes resonant transitions between states $|\downarrow, n\rangle$ and $|\uparrow, n\rangle$ (carrier or spin-flip transitions). A monochromatic field of angular frequency $\omega = \omega_0 - \omega_m$ causes resonant transitions between states $|\downarrow, n\rangle$ and $|\uparrow, n-1\rangle$, i.e., produces so called red sideband (that is with angular frequency $\omega = \omega_0 - \omega_m$) transitions.

These transitions are graphically depicted in Fig. 2 with the thickness of the edges qualitatively representing the strength of the coupling between the states. As pointed out in Ref. [18], when both fields (carrier and red sideband) are applied *simultaneously*, the eigenstates of the system are sequentially connected. Therefore, we look at the trapped-ion model controlled only by these two fields.

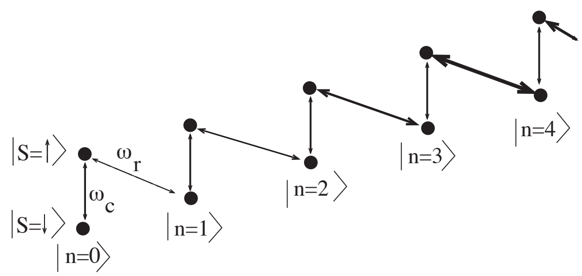

Fig. 2. Graphical representation of the coupled spin-half quantum harmonic oscillator system driven by sinusoidal resonant fields of angular frequency ω_c and ω_r as shown. When $\eta \ll 1$, the strengths of the ω_c transition couplings are independent of the harmonic oscillator quantum number n, whereas the strengths of the ω_r transition couplings increase as the square root of n as shown by the boldness of the coupling lines. Note that there is no direct coupling between two consecutive oscillator states with fixed spin.

Now we write the evolution equation of the spin-half coupled to harmonic oscillator driven by two fields that drive the carrier and red sideband transitions. The amplitudes corresponding to the fields that cause the carrier and red transitions are dubbed E_c and E_r respectively. As detailed in Ref. [41], in the interaction picture and in the energy eigenbasis, the evolution equation is written as

$$\dot{Y} = (u(t)B_c + v(t)B_r)Y. \tag{3}$$

The controls $u(t)$ and $v(t)$ are related to the applied fields via the equations

$$u(t) = c_1 E_c(t) = 0.25\mu \exp(-\eta^2/2) E_c(t), \tag{4}$$
$$v(t) = c_2 E_r(t) = 0.25\eta\mu \exp(-\eta^2/2) E_r(t). \tag{5}$$

Here η, the so-called Lamb-Dicke parameter, is the product of k, wave vector of the light, and x_0, the amplitude of the zero-point motion of the particle in the harmonic potential (or the spatial extent of the ground state harmonic oscillator wave function). By ordering the eigenstates as $|\uparrow, 0\rangle$, $|\uparrow, 1\rangle$, $\ldots, |\downarrow, 0\rangle$, $|\downarrow, 1\rangle$, $\ldots$, the control matrices are written as

$$B_c = \left(\begin{array}{c|c} 0 & iL_0 \\ \hline iL_0^T & 0 \end{array}\right). \tag{6}$$

$$B_r = \left(\begin{array}{c|c} 0 & L_1 \\ \hline -L_1^T & 0 \end{array}\right). \tag{7}$$

The upper-triangular matrices L_0 and L_1 are defined as

$$L_0 = \begin{pmatrix} L_0(\eta^2) & 0 & 0 & \ldots \\ 0 & L_1(\eta^2) & 0 & \ldots \\ 0 & 0 & L_2(\eta^2) & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{8}$$

$$L_1 = \begin{pmatrix} 0 & L_0^{(1)}(\eta^2) & 0 & \ldots \\ 0 & 0 & L_1^{(1)}(\eta^2) & \ldots \\ 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{9}$$

This structure of the control Hamiltonian precludes the use of adiabatic methods such as those used in Ref. [32]. Unlike the situation encountered in the analysis of the quantum harmonic oscillator algebra, here the formal commutation of the operators B_c and B_r does not lead to a finite-dimensional algebra, suggesting that the model with spin is much more controllable. This is the case, as will be explored in the next subsection.

1) Controllability: Lie Algebra: It is interesting to compare our analysis with the formal calculations suggested by Lie theory. The first thing to do is to determine the formal structure of the Lie algebra, which we now consider.

The control of the trapped-ion system is often studied in two different limiting cases - one in which the extent of zero-point motion of the spin-half particle in the harmonic potential x_0 is much smaller than the wavelength of the applied light $2\pi/k$, i.e., $\eta \ll 1$ (the Lamb-Dicke limit), and the other in which $\eta \simeq 1$ (beyond the Lamb-Dicke limit). The case in which $\eta \simeq 1$ is more general than the case of the Lamb-Dicke limit, but requires a more sophisticated analysis. We study initially the Lamb-Dicke limit in which the Lamb-Dicke parameter $\eta \ll 1$. The terms in equations (8) and (9) are expanded to first order in η. The control Hamiltonians can then be expressed in operator form as

$$B_c = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \text{ and } B_r = \eta \begin{bmatrix} 0 & a \\ -a^\dagger & 0 \end{bmatrix}, \tag{10}$$

where a and $a^\dagger$ denote the annihilation and creation operators of the harmonic oscillator [41]. (Note that B_r is the same Hamiltonian as obtained from the well-known Jaynes-Cummings model [42] that describes the interaction between a quantized cavity field and a two-level atom.)

In order to compute the Lie algebra, let us consider T, an operator acting on a complex Hilbert space. We associate with T a skew-hermitian operator acting on $\mathcal{H} \oplus \mathcal{H}$ defined by

$$J(T) = \begin{bmatrix} 0 & T \\ -T^\dagger & 0 \end{bmatrix}. \tag{11}$$

For convenience, let $K(T)$ be another operator defined in a similar way as

$$K(T) = \begin{bmatrix} T & 0 \\ 0 & -T \end{bmatrix}. \tag{12}$$

Of course, $K(T)$ is skew-hermitian if and only if T is. The control operators we are interested in for the purposes of determining the structure of the Lie algebra are given by $B_c = J(iI)$ and $B_r = \eta J(a)$. We have

Lemma 3: The Lie algebra generated by $J(iI)$ and $J(T)$ includes the operators

$$J(W^{2p});\ p = 1, 2, 3, \cdots\ ;\ K(W^{2p+1});\ p = 0, 1, 2, \cdots\ , \tag{13}$$

where, $W = i(T + T^{\dagger})$.

Proof: A calculation shows that $[J(T), J(iI)] = K(W)$ and further, $[J(iI), K(W)] = -2iJ(W)$. We can then check that

$$ad^p_{J(W)}(K(W)) = (-2)^p \left\{ \begin{array}{l} J(W^{p+1}),\ \text{if p is odd} \\ K(W^{p+1}),\ \text{if p is even} \end{array} \right\}. \tag{14}$$

These calculations make it clear that if the powers of W are independent then $J(iI)$ and $J(T)$ do not generate a finite-dimensional algebra. Thus if T is nonzero only on the diagonal immediately above the main diagonal (which is true for the operator a), and if every term on this upper-diagonal is nonzero, then the successive powers of W are independent and the algebra is infinite-dimensional. ■

This is the case for the coupled spin-half harmonic oscillator system. Of course, this calculation only shows that this system, unlike the harmonic oscillator, does not generate a finite-dimensional controllability Lie algebra. More work is required to say with precision exactly what the reachable states are. This is precisely the role of Theorem 2 which gives more specific information of which operators play a role in the control process.

Note: In the case where the Lamb-Dicke limit does not apply, the Lie algebra will still be infinite-dimensional but the terms are more complicated.

2) Finite controllability: In this subsection we discuss how finite controllability works in this infinite-dimensional setting.

From Fig. 2, it is seen that the linear span of the sequentially connected eigenstates can be looked at as an infinite set of finite-dimensional subspaces with the ground state $|\downarrow, 0\rangle$ being equal to $\mathcal{H}_1$. Subsequent subspaces $\mathcal{H}_i$ are defined to be the linear span of the up and down spin and harmonic oscillators states up to the harmonic oscillator quantum number i. Further, when operators B_c and B_r are applied *sequentially*, each subspace H_i can be transferred to $\mathcal{H}_{i-1}$. Thus the criteria for finite controllability are met. By sequential application of the two operators, any finite superposition of eigenstates can be transferred to the ground state in finite time.

The application of these statements to the spin-half in quadratic potential example is best understood by writing the control matrices B_c and B_r in a re-ordered basis as follows: The eigenstates can be ordered as $|\uparrow, 0\rangle,\ |\uparrow, 1\rangle,\ \ldots, |\downarrow, 0\rangle,\ |\downarrow, 1\rangle,\ \ldots$. In the interaction picture, the Schrödinger equation is written as

$$\dot{Y} \ = \ (u(t)B_c + v(t)B_r)Y, \tag{15}$$

where $u(t)$ and $v(t)$ are defined as before. Then,

$$B_c \ = \ i \left(\begin{array}{cc|cc|cc|c} 0 & L_0 & 0 & 0 & 0 & 0 & \cdots \\ L_0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \hline 0 & 0 & 0 & L_1 & 0 & 0 & \cdots \\ 0 & 0 & L_1 & 0 & 0 & 0 & \cdots \\ \hline 0 & 0 & 0 & 0 & 0 & L_2 & \cdots \\ 0 & 0 & 0 & 0 & L_2 & 0 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right).$$

$$B_r \ = \ \left(\begin{array}{c|cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \hline 0 & 0 & L_0^{(1)} & 0 & 0 & 0 & \cdots \\ 0 & -L_0^{(1)} & 0 & 0 & 0 & 0 & \cdots \\ \hline 0 & 0 & 0 & 0 & L_1^{(1)} & 0 & \cdots \\ 0 & 0 & 0 & -L_1^{(1)} & 0 & 0 & \cdots \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right).$$

L_i's and $L_i^{(1)}$'s are Laguerre polynomials of the zeroth and first order, all with argument η^2.

The model of the trapped-ion qubit highlights the existence of important examples for which it is desirable for the evolution to occur on a non-closed subspace of a Hilbert space, i. e., the space of finitely nonzero elements. In this model, this non-closed subspace consists of vectors in the oscillator representation with finitely many nonzero elements. The B_c and B_r operators and their one parameter groups, leave invariant the subspace l_0 of l_2 consisting of finitely nonzero sequences. The semigroup $e^{\alpha B_c + \beta B_r}$ will not, however, typically have any nontrivial invariant subspace. To satisfy the requirements of the Finite Controllability Theorem one never uses a linear combination of the operators. Further, as we have seen, the key to controllability is that each operator has a different invariant subspace within the set of finite superpositions.

3) Explicit finite controllability scheme: The property that both control vector fields are never used simultaneously is exploited by Law and Eberly [13] and Kneer and Law [14] in order to devise a explicit scheme for the production of a finite superposition of eigenstates from another finite superposition in the control of a spin-half particle coupled to a harmonic oscillator (in the Lamb-Dicke limit). It shows that if x can be transferred to y by a series of such "single nonzero u_i" moves then the transfer from y to x is also possible.

Specifically the Law-Eberly scheme [13] to transfer any eigenstate $|i\rangle$ to any other eigenstate $|j\rangle$ involves the alternate use of transitions generated by spin reversal (π-pulses of E_c) and transitions generated by π-pulses of E_r which convert from a state in which the oscillator has energy E_i and spin down to a state in which the energy of the oscillator in altered by one unit and the spin is flipped as well (see equation (10)). For example suppose we wish to drive a state from the $|\downarrow, n\rangle$ to $|\uparrow, n-2\rangle$ (see Fig. 2). This can be done using B_r to drive the system from $|\downarrow, n\rangle$ to $|\uparrow, n-1\rangle$, B_c to drive the system from $|\uparrow, n-1\rangle$ to $|\downarrow, n-1\rangle$ and finally B_r to go from $|\downarrow, n-1\rangle$ to $|\uparrow, n-2\rangle$.

We note that this scheme works both in the Lamb-Dicke limit and beyond the Lamb-Dicke limit. In the Law-Eberly scheme, the π-pulses of E_c are all of the same time duration because in the Lamb-Dicke limit, all the carrier transitions are equally strong. However, the coupling strengths of the red-sideband transitions are proportional to $\sqrt{n}$, and therefore the π-pulses of E_r are shorter in duration as eigenstates of higher n are addressed. In order to generate an arbitrary superposition of a finite number of eigenstates, starting from another arbitrary superposition, an additional trick is to go through the ground state of the system which acts as a "pass state" [43]. It is possible to provide an explicit algorithm which will drive the system from any finite superposition to any other finite superposition.

To prepare an arbitrary finite superposition, the simplest path is to take the system through the ground state. One assumes that the desired state is the initial state and then designs a sequence of alternating pulses of the E_c and E_r fields that would take this state to the ground state $|\downarrow, 0\rangle$ [14]. The actual sequence that produces the superposition is the time-reversed sequence that was designed. For example, if the desired superposition is $(|\uparrow, 3\rangle + |\downarrow, 2\rangle)/\sqrt{2}$, the sequence of pulses that will transfer this state to the ground state is $E_c^{(1)}(\pi)\, E_r^{(2)}(\phi_2)\, E_c^{(3)}(\phi_3)\, E_r^{(4)}(\phi_4)\, E_c^{(5)}(\phi_5)\, E_r^{(6)}(\phi_6)\, E_c^{(7)}(\phi_7)$. The action of each pulse is the following: $E_c^{(1)}$ is a π pulse of the carrier field that moves the state $|\uparrow, 3\rangle$ to $|\downarrow, 3\rangle$. (Simultaneously, the population in $|\downarrow, 2\rangle$ is transferred to $|\uparrow, 2\rangle$). $E_r^{(2)}$ is a pulse of the red-sideband field that moves between the states $|\downarrow, 3\rangle$ and $|\uparrow, 2\rangle$. Since there is already a superposition of the two states, the duration of the red-sideband field is shorter than that of a π-pulse. Simultaneously, a superposition of $|\downarrow, 2\rangle$ and $|\uparrow, 1\rangle$ is created. The next transition $E_c^{(3)}(\phi_3)$ transfers population between $|\uparrow, 2\rangle$ and $|\downarrow, 2\rangle$, and again is shorter than a π pulse. This sequence progresses till all the population is in $|\downarrow, 0\rangle$. The actual sequence is the time-reversed sequence of the one that is described above — this creates the desired superposition from the initial ground state.

If one were to transfer an arbitrary initial superposition to an arbitrary final superposition of eigenstates, one employs the above algorithm twice. The sequences A and B that take the system from the initial and final superpositions respectively to the ground state are first calculated. Then the sequence A is first applied taking all the population to the ground state. The time time-reversed sequence of B is then applied which takes the population to the desired final superposition. Clearly, this scheme works in finite time only if the initial and final states are both superpositions of a *finite* number of states.

Note that finite superpositions are dense in the Hilbert space of all possible states. Hence from our Lie algebra analysis and the use of the Law-Eberly algorithm we have

Proposition 4: The span of the Lie algebra generated by the operators B_c and B_r for the quantum control system in Eq. (3) is infinite-dimensional and the reachable set, which is dense in the Hilbert space of all states, includes all finite superpositions.

Note that this provides an explicit dense subspace controllability result which is hard to prove by abstract methods (see [25] and [23]). Note also that the proof of controllability that Law and Eberly give of what they term "arbitrary control" might be more accurately described as demonstrating that any state in l_0 (finite linear superpositions) can be mapped to any other state in l_0, staying within l_0.

Other examples such as the red and blue sideband controlled trapped-ion qubit and the control of an N-level atom in a harmonic potential are discussed in Ref. [41].

C. System 3: Spin-half particle coupled to two harmonic oscillators

In this section we consider another paradigm of quantum computing - the trapped-electron qubit, which can be well-modelled as a spin-half particle coupled to two harmonic oscillators. We show that this is a system that is less controllable than the spin-half particle coupled to one harmonic oscillator. In this system, it is not possible to make a transfer from any finite superposition of states to any other finite superposition. This implies in particular that the system cannot be finitely controllable. As we illustrate below, the natural control sequence does not allow one to remain within any one of the nested invariant subspaces spanned by the eigenstates. In addition the nested sequence of invariant subspaces is infinite-dimensional.

As detailed in Ref. [44], the energy eigenstates of a spin-half particle coupled to two harmonic oscillators (called the cyclotron oscillator and axial oscillator respectively) can be written as $|nlj\rangle$ where n refers to the number state of the cyclotron oscillator, l refers to the number state of the axial oscillator, and j refers to the spin state (up or down). The system is addressed by three control fields. The spin qubit is controlled using a field of angular frequency ω_s that connects states $|n\ l\ \downarrow\rangle$ and $|n\ l\ \uparrow\rangle$. The spin-axial transition is controlled by field of angular frequency ω_{sa} that connects states $|n\ l\ \downarrow\rangle$ and $|n\ l+1\ \uparrow\rangle$. A spin-cyclotron transition is controlled by a field of angular frequency ω_{sc} that connects states $|n\ l\ \downarrow\rangle$ and $|n-1\ l\ \uparrow\rangle$. The transfer graph in Fig.3 clearly indicates that the three fields transitively connect all the eigenstates of the spin-axial-cyclotron system.

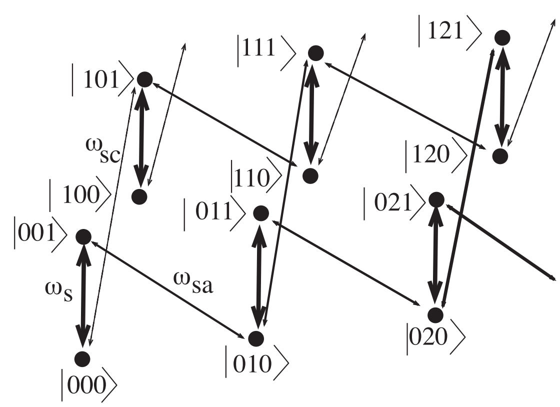

Fig. 3. Transfer graph of the controlled trapped-electron. States are denoted $|nlj\rangle$, where n is the cyclotron harmonic oscillator number state, l is the axial harmonic oscillator number state, and j is the spin state. Lines marked ω_s, ω_{sa}, and ω_{sc} indicate spin, spin-axial and spin-cyclotron transitions, respectively.

As in the case of the trapped-ion, the evolution equation is clearer in the interaction picture and is derived in Ref. [41]. Writing down the control matrices explicitly is not trivial, and doesn't provide significant insights into the problem. Instead, one can look at the structure of these matrices as below. The control matrices can be written in the basis of trapped-electron eigenstates denoted by $|nlj\rangle$, where n the cyclotron state, l the axial state, and j is the spin state. Qualitatively, S, A and C indicate spin, spin-axial and spin-cyclotron transitions, respectively, and the prefactors indicate the relative strengths. The eigenstates are ordered as: $|l=0,n=0,\downarrow\rangle$, $|l=0,n=0,\uparrow\rangle$, $|l=0,n=1,\downarrow\rangle$, $|l=0,n=1,\uparrow\rangle$, $|l=0,n=2,\downarrow\rangle$, $|l=0,n=2,\uparrow\rangle$, $\ldots,|l=1,n=0,\downarrow\rangle$, $|l=1,n=0,\uparrow\rangle$, $|l=1,n=1,\downarrow\rangle$, $|l=1,n=1,\uparrow\rangle$, $|l=1,n=2,\downarrow\rangle$, $|l=1,n=2,\uparrow\rangle$, $\ldots$.

The control matrices that describes the spin-flip and spin-axial transitions have the form $B_s = i\mathrm{diag}(X_S, X_S, \ldots, X_S)$ and $B_{sa} = i\mathrm{diag}(X_A, X_A, \ldots, X_A$ respectively.
Here X_S and X_A are the infinite block matrices

$$X_S = \left[\begin{array}{cc|cc|cc|c} 0 & S & 0 & 0 & 0 & 0 & \cdots \\ S & 0 & 0 & 0 & 0 & 0 & \cdots \\ \hline 0 & 0 & 0 & S & 0 & 0 & \cdots \\ 0 & 0 & S & 0 & 0 & 0 & \cdots \\ \hline 0 & 0 & 0 & 0 & 0 & S & \cdots \\ 0 & 0 & 0 & 0 & S & 0 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right], \tag{16}$$

$$X_A = \left[\begin{array}{cc|cc|cc|c} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & A & 0 & 0 & 0 & \cdots \\ \hline 0 & A & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & A\sqrt{2} & 0 & \cdots \\ \hline 0 & 0 & 0 & A\sqrt{2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right]. \tag{17}$$

respectively.

The control matrix describing the spin-cyclotron transition has the form

$$B_{sc} = i\left[\begin{array}{c||c||c||c} 0 & X_C & 0 & \cdots \\ \hline\hline X_C^\dagger & 0 & X_C & \cdots \\ \hline\hline 0 & X_C^\dagger & 0 & \cdots \\ \hline\hline \vdots & \vdots & \vdots & \ddots \end{array}\right], \tag{18}$$

where X_C is an infinite block matrix of the form

$$X_C = \left[\begin{array}{cc|cc|cc|c} 0 & C & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \hline 0 & 0 & 0 & C & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \hline 0 & 0 & 0 & 0 & 0 & C\sqrt{2} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right]. \tag{19}$$

When the three transitions are applied sequentially, it is possible to transfer population from subspace $\mathcal{H}_i$ to subspace $\mathcal{H}_{i-1}$ only under certain conditions. Trivially, if one wishes to transfer population within one of the harmonic oscillator states, the problem is the same as that of the trapped ion. Also, it is easy to see that the levels are connected in such a way that the system is eigenstate controllable in the sense that the population can be coherently transferred from any eigenstate to any other eigenstate. For example, consider the set of eigenstates illustrated in Fig.3. The condition for eigenstate controllability is that the pulses of frequency ω_{sa}, ω_s and ω_{sc} must be applied *sequentially*, and not simultaneously. For example, let us say we want to transfer the $|000\rangle$ state to the $|111\rangle$ state. We can do so by the pulse sequence: $p_s(\pi)$ that transfers to $|001\rangle$, $p_{sa}(\pi)$ that transfers to $|010\rangle$, and then $p_{sc}(\pi)$ that transfers to $|111\rangle$.

Superficially this system appears to be very similar to the trapped-ion system. However, it is seen that this system is not finitely controllable; that is, even though the system is eigenstate controllable, it is not possible to transfer a superposition of eigenstates to the ground state even with sequential applications of the field. This can be seen visually from the control graph in Fig.3, or by examining the control matrices as follows.

For example, let us say we want to transfer the $|000\rangle+|111\rangle$ state to the $|000\rangle$ state. The pulse sequence: $p_s(\pi)$, $p_{sa}(\pi)$, $p_{sc}(\pi)$ that transfers the $|000\rangle$ state to the $|111\rangle$ state also transfers the $|111\rangle$ state to the $|000\rangle$ state. Explicitly, $p_s(\pi)$ transfers $|111\rangle$ to $|110\rangle$, but also transfers $|000\rangle$ to $|001\rangle$; $p_{sa}(\pi)$ transfers $|110\rangle$ to $|101\rangle$, but also transfers $|001\rangle$ to $|010\rangle$; and $p_{sc}(\pi)$ transfers $|101\rangle$ to $|000\rangle$, but also transfers $|010\rangle$ to $|111\rangle$. The pulse sequence that transfers population down one of the harmonic oscillator ladders, transfers population up the other harmonic oscillator ladder. Thus we cannot apply the Finite Controllability Theorem using the above basis.

This lack of controllability can be attributed to the fact that the invariant subspaces are not finite-dimensional. In fact this system presents an infinite number of nested *infinite* -dimensional subspaces.

Thus we see that eigenstate controllability is a much weaker condition than subspace controllability. Simply having vector fields that sequentially connect the eigenstates is sufficient for eigenstate controllability. This is in contrast to the controllability of finite-dimensional quantum systems where eigenstate controllability implies controllability of finite superpositions.

This analysis provides important input into the the development of quantum gates in the trapped-electron system — it is not possible to use Law-Eberly type methods to construct arbitrary quantum gates in this system.

IV. Summary

Many of the novel questions that arise in laying the ground work for quantum computing can be thought of as questions about the controllability of Schrödinger's equation. In this paper we prove a Finite Controllability Theorem which is useful for analyzing certain infinite-dimensional quantum control problems. We discuss several related physical systems that are models for quantum computing.

Among the more interesting paradigms of quantum computing is the trapped ion modeled as a spin-half particle coupled to a quantum harmonic oscillator. In this paper, we discuss a general setting for this type of problem based on infinite-dimensional differential equations and Lie groups acting on a Hilbert space. This allows us to explore the Lie algebraic approach to controllability in this setting. In particular, we show that even though the formal Lie algebra associated with the Jaynes-Cummings model is infinite-dimensional, explicit finite controllability results can be determined. We also establish generalized controllability criteria for improving the controllability of some important infinite-dimensional quantum systems.

For the specific quantum computing system of the trapped-ion qubit, we showed that the Law-Eberly control scheme does not require the system to be in the Lamb-Dicke limit. We also showed that finite controllability cannot be achieved in the trapped-electron quantum computing paradigm, thus limiting the type of quantum operations (gates) that can be executed in this system.

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