CONTROL OF SQUEEZED PHONON AND SPIN STATES

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Abstract: In this paper we analyze the quantum control of squeezed states of harmonic oscillators as well as spin squeezing and its relationship to quadrature squeezed states of other bosonic fields. Squeezing provides a method for reducing noise below the quantum limit and provides an example of the control of under-actuated control systems in the stochastic and quantum context. We consider also the interaction of a squeezed quantum oscillator with an external heat bath and the problem of cancellation of squeezed states. Our controls consist of single or multiple pulses.

Keywords: Quantum Systems, Hamiltonians, Squeezing, Control

1. INTRODUCTION

In this paper we consider the problem of squeezing of harmonic oscillators and spin systems from the point of view of control theory. Squeezing has been suggested as a method for reducing noise in quantum systems below the standard quantum limit. This can be achieved by using laser pulses and in that sense may be viewed as a quantum control problem, although the classical squeezing problem is also of interest. In the latter case one is interested in reducing noise induced by random perturbations.

The quantum control problem has been of great interest recently, see for example (Brockett and Khaneja, 1999), (Khaneja, Brockett and Glaser, 2001) (Lloyd, 1996) and (Warren, Rabitz and Dahleh, 1993), (Ramakrishna, 2001) (Turinici and Rabitz, 2001) (D'Alessandro, 2003), (Altafini, 2003), and references therein.

Here we consider squeezing as a control problem in both the classical and quantum setting. In the classical case we consider a system subject to thermal noise while in the quantum case we consider a system at zero temperature and in the presence of noise. In both cases the control is given by an external electromagnetic field and enters the control equations multiplicatively. A key feature of squeezing is that it results in a redistribution of uncertainty between observables.

In this paper we consider a model for phonon squeezing in solids following the work of (Garret, Rojo, Sood, Whitaker Merlin, 1997) (see also (Hu and Nori, 1996) and references therein for interesting related work), but one can equally well consider the case of photons in quantum optics. The control is via a single pulse on a large ensemble of oscillators and this sense we are considering under-actuated control systems in both the classical and quantum case.

In addition we consider spin squeezing and discuss its relationship to quadrature squeezed states of other bosonic fields.

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We model the effect of dissipation on the classical system and the effect of coupling to a heat bath in the quantum setting. This causes the squeezing effect to gradually moderate.

We remark that squeezed states and their possible moderation due to dissipation have been proposed as a method for secure quantum cryptography, see (Gottesman and Preskill, 2000) and (Gottesman, Kitaev and Preskill, 2001). Ideally for the codes discussed in these papers one want states which are infinitely squeezed in momentum or position which is of course not possible. In these papers dissipation is modeled by a master equation. Here we consider direct coupling to a heat bath modeled by a quantum string.

In the last section of the paper we consider cancellation of squeezed states where we need an additional pulse of appropriate form.

This paper extends work reported in (Bloch and Rojo, 2000).

2. CLASSICAL SQUEEZING OF THE HARMONIC OSCILLATOR

In this section we briefly discuss classical squeezing of a set of identical coupled harmonic oscillators. Denote the position of each oscillator by q^i.

The Hamiltonian for the system is of the form:

$$H = \sum_i \frac{p_i^2}{2} + \sum_{i,j} \frac{K_{ij}}{2} q^i q^j, \qquad (2.1)$$

with K_{ij} the dynamical matrix and the oscillators are assumed to have unit mass and $p_i = \dot{q}^i$.

In order to analyze the system we decompose it into its normal modes. Denoting the normal mode coordinates by Q^i we thus obtain a system of uncoupled harmonic oscillator equations of the form $\ddot{Q}^i + \Omega_i^2 Q^i = 0$.

The main control mechanism we consider here is squeezing by pulses. In this case each oscillator is forced by a pulse at time $t = 0$ which is proportional to its displacement, i.e. we have equations of the form:

$$\ddot{Q}^i + \Omega_i^2 Q^i = 2\lambda Q^i \delta(t) \qquad (2.2)$$

where $\delta(t)$ is the Dirac delta function and λ is a constant which is proportional to the frequency Ω.

Thus we obtain

$$\dot{Q}^i(0^+) = \dot{Q}^i(0^-) + 2\lambda Q^i(0)\,. \qquad (2.3)$$

Thus, if one considers the system subject to white noise,

$$\ddot{Q}^i + \Omega_i^2 Q^i = 2\lambda Q^i \delta(t) + \alpha \dot{w}^i\,, \qquad (2.4)$$

one sees that while one starts with a spherical equilibrium distribution which is invariant in time, after the pulse one has an elliptical distribution which rotates in time at twice the harmonic frequency (by the $\mathbb{Z}_2$ symmetry of the ellipse). (A precise analysis is given below in the course of our treatment of the quantum mechanical case.) Noise reduction is then achieved by viewing the system "stroboscopically" when the noise is low.

Actually the above is an idealization: in actuality the oscillator should be viewed as in equilibrium with a heat bath which dissipates energy. In the classical setting one can model this by simple linear dissipation (in the quantum setting one has to introduce a heat bath – see below).

Thus we have a system of the form

$$\ddot{Q}^i + \Omega_i^2 Q^i = -\eta_i \dot{Q}^i + U_i(t) + \alpha \dot{w}^i \qquad (2.5)$$

where η_i is a dissipation constant and $U_i(t)$ is the control which we can choose to be a single pulse or a sequence of pulses. Depending on the dissipation strength an initial squeezing effect will decay away and we need a continual sequences of pulses to keep the system in a squeezed state.

It is worthwhile remarking on how the control enters in our setting: the control is a single pulse applied overall (and in this sense the system is under-actuated) while the effect on each (normal mode) of oscillation is to apply a pulse proportional to displacement (minus the mean displacement). We shall return to the classical squeezing of oscillator by pulses, and in particular a computation of mean square displacement, after a discussion of the quantum case below.

3. SQUEEZING OF THE QUANTUM HARMONIC OSCILLATOR

We now turn to the quantum setting.

Consider the following Hamiltonian

$$H = \frac{P^2}{2m} + \frac{m\omega^2}{2} Q^2 + \lambda \delta(t) Q^2, \qquad (3.1)$$

which reflects an impulsive change in the spring constant and where $\omega = \sqrt{K/m}$, K being the original spring constant.

The variables P and Q, which are operators in the quantum case, obey canonical commutation rules $[P, Q] = i\hbar$. We can rewrite the above Hamiltonian in terms of creation operators a and $a^\dagger$ defined through

$$Q = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \; P = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a), \qquad (3.2)$$

with $[a, a^\dagger] = 1$. Written in terms of the new variables, the Hamiltonian is

$$H = \hbar\omega(a^\dagger a + 1/2) + \lambda\delta(t)(a + a^\dagger)^2. \qquad (3.3)$$

The ground state of the system, for $t \neq 0$, $|0\rangle$, corresponds to the vacuum of a, $(a|0\rangle = 0)$, and the excited states are of the form $(a^\dagger)^2|0\rangle$.

We now want to study the behavior of the system at $t > 0$, given that the system is in its ground state at $t < 0$. The wave function at $t = 0^+$ is of the form $|\psi(t = 0^+)\rangle = \exp(-i\lambda Q^2)|0\rangle$, and for longer times the system evolves with the "unperturbed" Hamiltonian: $|\psi(t > 0)\rangle = \exp(-iH_0 t)e^{-i\lambda Q^2}|0\rangle$. Our first quantity of interest is $\langle\psi(t)|Q^2|\psi(t)\rangle \equiv \langle Q^2(t)\rangle$. Let us compute it using the general method of coherent states. We find

$$\langle Q^2(t)\rangle = \langle 0|e^{i\lambda Q^2}(ae^{-i\omega t} + a^\dagger e^{i\omega t})^2 e^{-i\lambda Q^2}|0\rangle, \tag{3.4}$$

where we have used the fact that $e^{iH_0 t}ae^{-iH_0 t} = ae^{-i\omega t}$, which states that $a^\dagger$ and a respectively destroy and create eigenstates of H_0, and where Q is defined in units of $\sqrt{\hbar/(2m\omega)}$.

Now we introduce a basis of coherent states $|z\rangle$, which satisfy $a|z\rangle = z|z\rangle$, $\langle z|a^\dagger = \langle z|z^*$, and form an overcomplete set of states:

$$1 = \frac{1}{2\pi i}\int dz\, dz^* e^{-zz^*}|z\rangle\langle z|. \tag{3.5}$$

Inserting (3.5) in (3.4) we find

$$\langle Q^2(t)\rangle = \frac{1}{2\pi i}\int\int dz\, dz^* e^{-zz^*}$$
$$(z^2 e^{-2i\omega t} + z^{*2}e^{2i\omega t} + 2zz^* - 1)|\langle 0|e^{i\lambda x^2}|z\rangle|^2.$$

In order to evaluate the last term we need the position representation of the ground state (note that at this point Q is a real number)

$$\langle 0|Q\rangle = \frac{1}{\pi^{\frac{1}{4}}}e^{-Q^2/2} \tag{3.6}$$

and that of the coherent state

$$\langle Q|z\rangle = \frac{1}{\pi^{\frac{1}{4}}}e^{-Q^2/2+\sqrt{2}zQ-z^2/2}. \tag{3.7}$$

A simple integration gives

$$\langle 0|e^{i\lambda Q^2}|z\rangle = \int dx\langle 0|Q\rangle\langle Q|z\rangle e^{i\lambda Q^2} \tag{3.8}$$
$$= \frac{1}{\sqrt{1-i\lambda}}e^{i\lambda z^2/2(1-i\lambda)}. \tag{3.9}$$

Changing to the variables $z = u + iv$ we have

$$e^{-zz^*}|\langle 0|e^{i\lambda Q^2}|z\rangle|^2$$
$$= \frac{1}{\sqrt{1+\lambda^2}}e^{-[v^2+(2\lambda^2+1)u^2+2\lambda uv]/(1+\lambda^2)} \tag{3.10}$$

and

$$\langle Q^2(t)\rangle = \frac{4}{\pi\sqrt{1+\lambda^2}}\int_{-\infty}^{\infty}du\int_{-\infty}^{\infty}dv$$
$$\left(u^2\cos^2\omega t + v^2\sin^2\omega t + uv\sin 2\omega t - \frac{1}{4}\right)$$
$$\times e^{-[v^2+(2\lambda^2+1)u^2+2\lambda uv]/(1+\lambda^2)} \tag{3.11}$$
$$= 1 + 4\lambda^2\sin^2\omega t + 2\lambda\sin 2\omega t \tag{3.12}$$

It is interesting to compare this with an ensemble of classical oscillators with initial conditions taken from a heat bath. For simplicity let us take $\omega = m = k_B = T = 1$ (k_B is Boltzman's constant). An arbitrary oscillator will evolve as $Q(t) = u\cos t + v\sin t$, with u and v its initial position and velocity. If a pulse is applied at $t = 0$ of the form treated above $Q(t) = u\cos t + (v + 2\lambda u)\sin t$. Now let us average over initial conditions taken from a measure given by (a thermal bath)

$$\langle Q^2(t)\rangle \sim \int du\, dv[u\cos t + (v + 2\lambda u)\sin t]^2 e^{-(u^2+v^2)}$$
$$= 1 + 4\lambda^2\sin^2 t + 2\lambda\sin 2t. \tag{3.13}$$

We note that the two expressions for, respectively, the quantum oscillator at zero temperature and the classical oscillator at finite temperature, are exactly the same. The general time dependence of the variance for a squeezed harmonic oscillator with frequency ω can thus be written in the following form

$$\langle [Q(t)]^2\rangle = \frac{\epsilon_0}{K}\left[1 + \left(\frac{2\lambda}{\omega}\right)\sin 2\omega t + \left(\frac{2\lambda}{\omega}\right)^2\sin^2\omega t\right] \tag{3.14}$$

with $\epsilon_0 = \hbar\omega/2$ for the quantum case and $\epsilon_0 = k_B T$ for the classical oscillator at a temperature T.

The method of coherent states presented above has the advantage of being suitable for calculating other quantities. For example, if the oscillators are atoms within a solid, the scattering amplitude for an X-ray is decreased by a factor (called the Debye-Waller factor – see (Ziman, 1972)) $\sim \langle\exp ikQ(t)\rangle$, with k the wave-vector of the X-ray. We now ask ourselves what is the time evolution of the Debye-Waller factor for a squeezed phonon. This means that we need to compute the following expression

$$I(\lambda, t) = \langle 0|e^{i\lambda Q^2}e^{(ae^{-i\omega t}+a^\dagger e^{-i\omega t})}e^{-i\lambda Q^2}|0\rangle$$
$$= \frac{1}{\sqrt{e}}\frac{1}{\sqrt{1+\lambda^2}}\frac{1}{\pi}\int du\, dv$$
$$e^{2u\cos\omega t+2v\sin\omega t-\frac{-[v^2+(2\lambda^2+1)u^2+2\lambda uv]}{(1+\lambda^2)}}$$
$$= e^{1+4\lambda^2\sin^2\omega t+2\lambda\sin 2\omega t}. \tag{3.15}$$

For the Debye-Waller factor, we obtain the following time dependence

$$\langle e^{ikQ(t)}\rangle = e^{-k^2\langle Q^2(t)\rangle} \tag{3.16}$$

Measurement of the Debye-Waller factor may provide a practical method of detecting the squeezing phenomenon experimentally.

4. SQUEEZING AND DISSIPATION

In this section we consider the squeezing of a quantum oscillator coupled to a an infinite number of oscillators representing a "heat" bath. We show that this causes a decay in the squeezing oscillation for small time and true damping in the limit of a continuum of oscillators. This damping effect of the heat bath is similar to that analyzed classically in (Lamb, 1900), (Komech, 1995), (Sofer and Weinstein, 1999) and (Hagerty, Bloch and Weinstein, 1999). We stress that we are considering a zero temperature case, and the damping effects appear due to a) the coupling of a single variable with a continuum of variables and b) an "asymmetry" in the initial conditions. The applied pulse on the oscillator generates outgoing waves on the continuum system which in turn gives rise to a positive damping (for a detailed discussion of negative versus positive damping see (Keller and Bonilla, 1986)).

We start with a general formulation, and at the end of this section discuss a specific continuum example.

The Hamiltonian of the system consists of three parts: H_0 describing the original oscillator:

$$H_0 = \frac{p_0^2}{2m} + \frac{m\omega_0^2}{2}q_0^2\,, \tag{4.1}$$

the Hamiltonian H_e of the environment:

$$H_e = \sum_\alpha \left[\frac{p_\alpha^2}{2m} + \frac{m\omega_\alpha^2}{2}q_\alpha^2\right]\,, \tag{4.2}$$

and a linear coupling between the two

$$H_{\rm int} = \sum_\alpha \xi_\alpha q_\alpha q_0\,. \tag{4.3}$$

Formally, the total Hamiltonian $H = H_0 + H_e + H_{\rm int}$ can be written in terms of its normal mode coordinates X_ν and P_ν:

$$H = \sum_\nu \left[\frac{P_\nu^2}{2m} + \frac{m\omega_\nu^2}{2}X_\nu^2\right]\,, \tag{4.4}$$

and we will consider a situation in which the initial (before the pulse) wave function corresponds to all the modes in the ground state:

$$\Psi_0 = \prod_\nu \left(\frac{\omega_\nu}{\pi\hbar}\right)^{1/4} e^{-\omega_\nu X_\nu^2/2\hbar}. \tag{4.5}$$

At $t = 0$ a pulse is applied to the (original) oscillator, the wave function immediately after the pulse given by:

$$\Psi_0(t=0^+) = e^{i\lambda q_0^2}\Psi_0 \tag{4.6}$$

$$= e^{i\lambda\sum_{\mu\nu} U_{0\mu}U_{0\nu}X_\mu X_\nu}\Psi_0\,, \tag{4.7}$$

where $U_{\mu\nu}$ is the matrix transforming from the original (uncoupled) modes to the coupled system ($q_0 = \sum_\nu U_{0\nu}X_\nu$).

As in previous sections, we are interested in the fluctuations of the variance of q_0, given in this case by

$$\langle q_0^2(t)\rangle = \sum_{\mu\nu} U_{0\mu}U_{0\nu}\langle X_\mu X_\nu\rangle(t)\,, \tag{4.8}$$

and that we will compute by solving the equation of motion obeyed by the correlations $\langle X_\mu X_\nu\rangle(t)$. Since X_μ and X_ν correspond to harmonic coordinates, using the quantum mechanical commutation relations we compute the equations of motion to be:

$$\begin{aligned}
&\frac{d}{dt}\langle X_\mu X_\nu\rangle = \frac{1}{m}\langle(P_\mu X_\nu + P_\nu X_\mu)\rangle\\
&\frac{d^2}{dt^2}\langle X_\mu X_\nu\rangle = -(\omega_\mu^2+\omega_\nu^2)\langle X_\mu X_\nu\rangle + \frac{2}{m^2}\langle P_\mu P_\nu\rangle\\
&\frac{d}{dt}\langle P_\mu P_\nu\rangle = -m\left(\omega_\mu^2\langle X_\mu P_\nu\rangle + \omega_\nu^2\langle X_\nu P_\mu\rangle\right)\\
&\frac{d^2}{dt^2}\langle P_\mu P_\nu\rangle = -(\omega_\mu^2+\omega_\nu^2)\langle P_\mu P_\nu\rangle + 2m^2\omega_\mu^2\omega_\nu^2\langle X_\mu X_\nu\rangle\,.
\end{aligned}$$

Note that the above equations are identical to those of classical harmonic oscillators for the quantities $X_\mu(t)X_\nu(t)$ etc., with initial conditions given by the values of the correlations evaluated for the quantum wave function:

$$\begin{aligned}
&\langle X_\mu X_\nu\rangle(0^+) = \delta_{\mu\nu}\frac{\hbar}{2m\omega_\mu},\\
&\langle P_\mu P_\nu\rangle(0^+) = \delta_{\mu\nu}\frac{\hbar m\omega_\mu}{2}\\
&+2\hbar^2\lambda^2(1+\delta_{\mu\nu})\frac{U_{0\mu}}{m\omega_\mu}\frac{U_{0\nu}}{m\omega_\nu}q_0^2\\
&\langle(X_\mu P_\nu + P_\nu X_\mu)\rangle(0^+)\\
&= 4\lambda\hbar U_{0\mu}U_{0\nu}\frac{\hbar}{2m}\left(\frac{1}{\omega_\mu}+\frac{1}{\omega_\nu}\right)
\end{aligned}$$

with $q_0^2 \equiv \langle q_0^2(0^-)\rangle = \sum_\alpha \hbar U_{0\alpha}^2/2m\omega_\alpha$.

Collecting the above equations we obtain

$$\langle q_0^2(t)\rangle = q_0^2\left\{1 + 4\lambda^2 S^2(t) + \frac{\lambda}{q_0^2}C(t)S(t)\right\}, \tag{4.9}$$

with

$$S(t) = \sum_\mu \frac{\hbar U_{0\mu}^2}{m\omega_\mu}\sin\omega_\mu t\;\; C(t) = \sum_\mu \frac{\hbar U_{0\mu}^2}{m\omega_\mu}\cos\omega_\mu t.$$

All the information of the evolution of the variance is contained in the function $J(\omega)$, the physical interpretation of which is that of a local density of states of the oscillator, defined as

$$J(\omega) = \sum_{\mu} \frac{\hbar U_{0\mu}^2}{m\omega_\mu} \delta(\omega - \omega_\mu), \tag{4.11}$$

from which

$$S(t) = \int d\omega J(\omega) \sin \omega t, \quad C(t) = \int d\omega J(\omega) \cos \omega t. \tag{4.12}$$

Note that $J(\omega)$ is a sum over delta functions, giving rise to a superposition of oscillations with the frequencies ω_ν for both $S(t)$ and $C(t)$. In the limit of an infinite system, and when the modes are spatially extended over all space, $J(\omega)$ becomes a continuous function. In that case the oscillatory behavior acquires a damped component, the detailed time dependence being given by the frequency spectrum of $J(\omega)$. A lorenzian shape for $J(\omega)$ will give an exponentially damped oscillation for both $S(t)$ and $C(t)$. As an illustration of this point we consider a model for which $J(\omega)$ can be computed explicitly – see the classical analysis of Lamb and Komech. Consider a one-dimensional string coupled to our oscillator. The string is described by a "transverse" displacement $u(x,t)$. The classical equations of motion of the system are

$$\begin{aligned} u_{tt}(x,t) &= c^2 u_{xx}(x,t) \\ M d^2 q_0(t)/dt^2 &= -V q_0(t) + T[u_x(0+,t) - u_x(0-,t)] \\ q_0(t) &= u(0,t). \end{aligned} \tag{4.13}$$

The normal modes consist of even and odd (in x) solutions. The odd solutions do not involve q_0 and are of the form $u_{q,o}(x,t) = e^{icqt} \sin qx$, whereas the even solutions are of the form $u_{q,e}(x,t) = e^{icqt} \cos(q|x| + \delta_q)$, with δ_q a phase shift (to be found). The wave vectors q label the normal modes, and play the role of the index μ in the above discussion: $\omega_\mu = cq$, and $U_{\mu 0}^2 = \cos^2(\delta_q)$ (up to a normalization constant) in the present case. Substituting this expression in (4.13) we obtain ($\omega_0^2 = V/M$)

$$\tan \delta_q = \frac{Mc}{2T} \frac{(\omega_0^2 - \omega_q^2)}{\omega_q}, \tag{4.14}$$

from which $U_{\mu 0}^2 = \cos^2 \delta_q$ is given by

$$U_{\mu 0}^2 = \frac{\alpha^2 \omega_q^2}{\alpha^2 \omega_q^2 + (\omega_q^2 - \omega_0^2)^2} \equiv U_q^2, \tag{4.15}$$

where we have defined $\alpha = 2T/Mc$. Note that U_q represents the transformation matrix that has to be normalized and since the frequencies form a continuum we normalize $U_q(\omega_q)$ to its integral over ω_q. Omitting the index q in ω_q, we obtain

$$U(\omega) = \frac{2\alpha}{\pi} \frac{\omega^2}{\alpha^2 \omega^2 + (\omega^2 - \omega_0^2)^2} = \frac{m\omega}{\hbar} J(\omega). \tag{4.16}$$

Substituting (4.16) in (4.12) we obtain

$$S(t) = \frac{\hbar}{m\omega_0} e^{-\Gamma t} \sin \Omega_0 t, \; C(t) = \frac{\hbar}{m\omega_0} e^{-\Gamma t} \cos \Omega_0 t, \tag{4.17}$$

with

$$\Omega_0 = \omega_0 \left(1 + [\alpha/\omega_0]^2\right)^{1/4} \cos \delta/2, \tag{4.18}$$

$$\Gamma = \omega_0 \left(1 + [\alpha/\omega_0]^2\right)^{1/4} \sin \delta/2, \tag{4.19}$$

where $\delta = \tan^{-1} \alpha/\omega_0$.

In the realistic limit $\alpha \ll \omega_0$ which corresponds to a "weak" coupling to the environment) these expressions take the form:

$$\begin{aligned} S(t) &\cong (\hbar/(m\omega_0) \exp(-Tt/Mc) \sin \omega_0 t, \\ C(t) &\cong (\hbar/(m\omega_0) \exp(-Tt/Mc) \cos \omega_0 t. \end{aligned}$$

Note that in this model, and in the limit of weak coupling, the initial variance q_0^2 of the reference oscillator is unchanged due to the coupling to the environment, and is given by $q_0^2 = \hbar/2m\omega_0$. Our final result for this section is then

$$\langle q_0^2(t) \rangle \cong q_0^2 \left\{ 1 + e^{-2(T/Mc)t} \left[\left(\frac{2\lambda\hbar}{m\omega_0} \right) \sin 2\omega_0 t + \left(\frac{2\lambda\hbar}{m\omega_0} \right)^2 \sin \omega_0^2 t \right] \right\} \tag{4.20}$$

which reduces simply to (3.13) in the uncoupled case of $T = 0$.

In summary we have shown in this section that the coupling to the environment can be included in general, giving rise to dissipation, and that the squeezing effect in the presence of dissipation can be computed explicitly for the Lamb model.

5. SPIN SQUEEZING

The mechanism of squeezing by the application of non-linear pulses extends to spin systems, where the quantum nature of the spatial components S_i is reflected in the commutation relations $[S_i, S_j] = i\hbar\epsilon_{ijk} S_k$. Squeezing for spin systems is of topical interest in quantum information, where quantum processing protocols require manipulation of entangled systems.

One realization of a string of quantum bits is an ensemble of two-level atoms, where each atom can be treated as a spin 1/2. Wineland et. al. (Wineland, Bollinger, Itano and Moore, 1992) showed that the resolution in spectroscopic experiments on N two-level atoms is determined by the factor

$$\xi = \frac{\Delta S_\perp}{|\langle \mathbf{S} \rangle|}, \tag{5.1}$$

which measures the quantum noise in a direction perpendicular to the mean value of the total spin. Note that ξ measures the precision of a measurement on the rotation of a spin. In this section we establish a parallel between the squeezed states of the harmonic oscillator and those of spin systems, focusing on the definition of Eq. (5.1). Note that, depending on the context, other definitions of spin squeezing can also be used: starting from the uncertainty relation $\Delta S_x \Delta S_y \geq |\langle S_z \rangle/2|$ (and cyclic permutations), a possibility is to define states satisfying $\Delta^2 S_i < |\langle S_j \rangle/2|$ as spin-squeezed(Wodkiewicz and Eberly, 1985). However, these states don't have in general a noise reduced in the direction perpendicular to the mean spin, and therefore are not relevant to quantum information.

Consider an ensemble of identical N two-level atoms with energy splitting $\hbar\omega_0$. We define the corresponding spin quantization axis in the x direction so that

$$H_0 = \omega_0 \sum_{i=1}^{N} S_{x,i} = \omega_0 S_x, \tag{5.2}$$

where $\mathbf{S}_i$ is the spin of atom i. The equations of motion for $S_z(t)$ and $S_y(t)$ are very similar to those of x and p for a harmonic oscillator of frequency ω_0:

$$\begin{aligned} \dot{S}_z(t) &= \omega_0 S_y \\ \dot{S}_y(t) &= -\omega_0 S_z, \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} \frac{d}{dt} S_z^2(t) &= \omega_0 (S_y S_z + S_z S_y) \\ \frac{d}{dt}(S_y S_z + S_z S_y) &= 2\omega_0 (S_y^2 - S_z^2) \\ \frac{d}{dt} S_y^2(t) &= -\omega_0 (S_y S_z + S_z S_y) \end{aligned} \tag{5.4}$$

which have also the same structure as the corresponding operators for the harmonic oscillator. Note here that $S_y^2(t) + S_z^2(t)$ is conserved along the flow of the equations (5.4).

The solutions for the expectation values are:

$$\langle S_z^2(t) \rangle = \frac{\langle S_z^2 \rangle_0 + \langle S_y^2 \rangle_0}{2} + \left[\frac{\langle S_z^2 \rangle_0 - \langle S_y^2 \rangle_0}{2} \right] \cos 2\omega_0 t - \frac{X_0}{2} \sin 2\omega_0 t,$$

with $X_0 = \langle S_z S_y \rangle_0 + \langle S_x S_y \rangle_0$. If the initial state $|\Psi\rangle$ is an eigenstate of S_x, for example $S_x|\Psi\rangle = -(N/2)|\Psi\rangle$, then $\langle S_z^2 \rangle_0 = \langle S_y^2 \rangle_0 = N/4$, $X_0 = 0$ and

$$\xi(t) = \frac{\sqrt{\langle S_z^2(t) \rangle}}{\langle S_x \rangle} = \frac{1}{\sqrt{N}}. \tag{5.5}$$

This time independent value of ξ corresponds to the unsqueezed state, and we are interested in decreasing its value, bringing it as close as possible to the Heisenberg limit $\xi = 1/N$ (Kitagawa and Ueda, 1993). Proceeding in analogy with the harmonic oscillator, we consider the effect of a pulse acting on the ground state of H_0

$$H' = \delta(t) \lambda S_z^2. \tag{5.6}$$

The wave function right after the pulse is

$$|\Psi(t = 0^+)\rangle = e^{i\lambda S_z^2} |\Psi\rangle_0, \tag{5.7}$$

and the quasiprobability distribution in the (S_z, S_y) plane is modified as in Figure 5.1. In order to compute the modified initial conditions in (5.5) we consider the case of large N. We define boson creation operators a and $a^\dagger$, with $[a, a^\dagger] = 1$, in terms of which

$$H_0 = \hbar\omega_0 \left(\frac{N+1}{2} - a^\dagger a \right), \tag{5.8}$$

in such a way that the spin projections in the x direction correspond to the occupation number of the new bosons. The transformation to the S^+ and S^- operators from these bosons is the well known Holstein-Primakov transformation(Kittel, 1987)

$$\begin{aligned} S^+ &= S_z + iS_y \\ &= N^{1/2}(1 - a^\dagger a/N)^{1/2} a \simeq N^{1/2} a \\ S^- &= S_z - iS_y \\ &= N^{1/2} a^\dagger (1 - a^\dagger a/N)^{1/2} a^\dagger \simeq N^{1/2} a^\dagger. \end{aligned}$$

where the approximation is valid as long as the relative variations of the spin projection are small:

$$\langle a^\dagger a \rangle / N \ll 1. \tag{5.9}$$

The operator equivalence between the bosons and spins implies the following correspondence:

$$\begin{aligned} S_z &\to \sqrt{\frac{N}{2}} x, \\ S_y &\to i\sqrt{\frac{N}{2}} \frac{d}{dx} \\ S_x &\to -\frac{N+1}{2} + \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 \right), \end{aligned} \tag{5.10}$$

where x is a new variable, in terms of which the ground state of H_0 is $|\Psi\rangle_0 = \pi^{1/4} \exp -x^2/2$, and

$$\Psi(t = 0^+)\rangle \equiv |\Psi_\lambda\rangle = \frac{1}{\pi^{1/4}} e^{iN\lambda x^2/2} e^{-x^2/2}. \tag{5.11}$$

The mapping allows us to compute the initial values:

$$\langle S_z^2\rangle_0 = \frac{N}{2}\langle\Psi_\lambda|x^2|\Psi_\lambda\rangle = \frac{N}{4}$$
$$\langle S_y^2\rangle_0 = -\frac{N}{2}\langle\Psi_\lambda|\frac{d^2}{dx^2}|\Psi_\lambda\rangle = \frac{N}{4}\left(1+(\lambda N)^2\right)$$
$$\langle S_z S_y\rangle_0 + \langle S_x S_y\rangle_0 = \frac{N}{2}\langle\Psi_\lambda|\left(ix\frac{d}{dx}+i\frac{d}{dx}x\right)|\Psi_\lambda\rangle$$
$$= -\frac{\lambda N^2}{2}$$
$$\langle S_x\rangle_0 = \frac{N}{2}\left(1-\frac{N\lambda^2}{2}\right),$$

where we stress that these values are exact provided $\lambda < 1/\sqrt{N}$ (for larger λ the response is periodic in λ). Notice that the quasiprobability distribution, which before the pulse is a circle of radius $\sqrt{N/2}$ in the (S_z, S_y) plane, now becomes an ellipse as shown in Figure 5.1. With the above initial values, for $t > 0$ the distorted distribution rotates at frequency ω_0 and the squeezing factor evolves as

$$\xi(t) = \frac{1}{\sqrt{N}}\frac{\left[1+(\lambda N\sin\omega_0 t)^2 - \lambda N\sin 2\omega_0 t\right]^{1/2}}{1-N\lambda^2/2}. \quad (5.12)$$

If we call $\lambda = \alpha_0/\sqrt{N}$, with $\alpha_0 < 1$, we obtain the minimum squeezing

$$\xi_{\min} = \frac{1}{N}\frac{1}{[1-\alpha_0^2/2]\alpha_0}, \quad (5.13)$$

which scales as $1/N$ and is reached twice during the cycle of the rotation of the ellipse of Figure 5.1.

From the development above we can see that the analysis of dissipation in Section 4 above extends essentially without change to the spin setting provided the number of spins N is large.

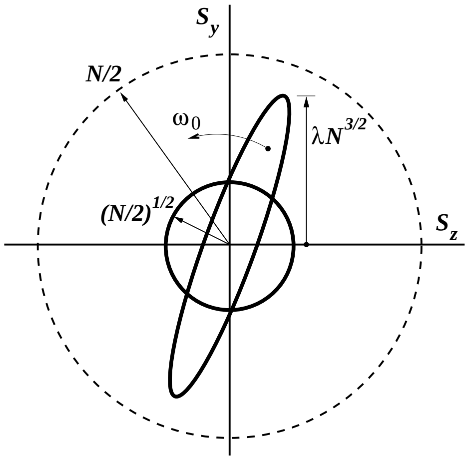

Fig. 5.1. Quasiprobability distribution in the (S_z, S_y) plane for N spins, before and after a pulse $H' = \delta(t)\lambda S_z^2$ is applied on the lowest eigenstate of $H_0 = \omega_0 S_x$. The response is equivalent to the harmonic oscillator case, with the proviso that the distribution is bounded by a circle of radius $N/2$.

6. CANCELLATION OF SQUEEZED STATES

Here we compare the cancellation of squeezing to that of coherent states as generated by impulsively excited coherent phonons, as observed in the experiment by Hase *et al.*(Hase, Mizoguchi, Harima, Nakashima, Tani, Sakai and Hangyo, 1996). See also (Merlin and Zhou, 2003). In the case of coherent states, a second pulse can cancel the coherent state if the separation time between pulses is matched to the period of the phonon oscillation. For squeezed states the separation between the pulses has to be adjusted to the intensity of the pulse. This can be easily seen graphically or from the the structure of the propagator of the harmonic oscillator(Kleinert, 1985):

$$\Psi(x,t) = \int dx' G(x,x';t)\Psi(x',t=0^+) \quad (6.1)$$

with

$$G(x,x';t) = \frac{1}{\sqrt{2\pi i\hbar\sin(\omega t)/m\omega}} \exp\left\{\frac{im\omega}{2\hbar\sin\omega t}\left[(x^2+(x')^2)\cos\omega t - 2xx'\right]\right\}$$

and where $\Psi(x',t=0^+) = \exp(i\lambda(x')^2)\Psi_0(x')$ and $\Psi_0(x') = \Psi(x',t=0^-)$ is the initial (ground) state of the oscillator.

Note that at times $t = t_n$ with

$$\frac{m\omega}{2\hbar}\cot\omega t_n = -\frac{\lambda}{2}, \quad \frac{m\omega}{2\hbar\sin\omega t_n} = \sqrt{\left(\frac{\lambda}{2}\right)^2 + \left(\frac{m\omega}{2\hbar}\right)^2} \quad (6.2)$$

we have

$$\Psi(x,t_n) \propto \exp\left(-i\lambda x^2/2\right)\int dx' \exp\left\{\left(i\frac{\lambda}{2} - \frac{m\omega}{2\hbar}\right)(x')^2 - 2ixx'\sqrt{\left(\frac{\lambda}{2}\right)^2 + \left(\frac{m\omega}{2\hbar}\right)^2}\right\},$$

which gives

$$\Psi(x,t_n) = \exp i\delta \exp\left(-i\lambda x^2\right)\Psi_0(x), \quad (6.3)$$

with δ an overall phase factor.

This means that a second pulse with the same intensity applied at t_n restores the wave function to the ground state. See Figure 6.1 for a graphical illustration.

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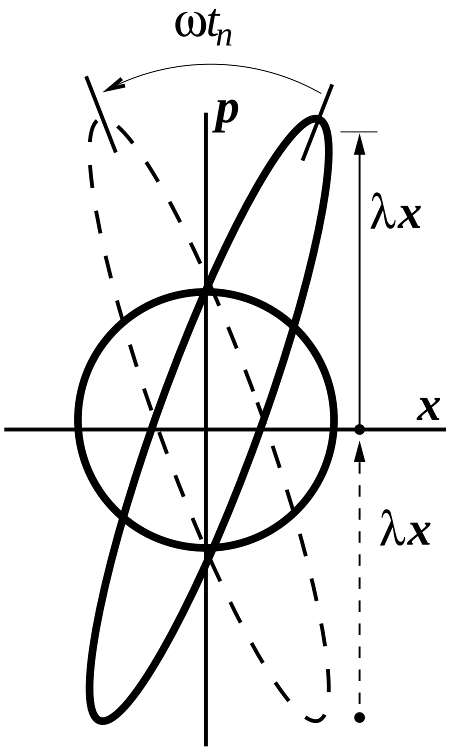

Fig. 6.1. Graphical rendition of the squeezing cancellation produced by a second pulse, of equal amplitude as the first, applied at a time t_n such that $\frac{m\omega}{2\hbar}\cot\omega t_n = -\frac{\lambda}{2}$.

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