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A Comparison of Multiresolution and Classical One-Dimensional Homogenization Schemes

A. C. Gilbert[1,2]

Department of Mathematics, Princeton University, Princeton, New Jersey 08544

E-mail: agilbert@research.att.com

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Homogenization is a collection of methods for extracting or constructing equations for the coarse-scale behavior of solutions to equations which incorporate many scales. This paper compares the classical method of homogenization with the recently developed multiresolution strategy for a particular class of one-dimensional second-order elliptic equations. We also examine several physical examples which highlight the distinctions between the two methods. © 1998 Academic Press

1. INTRODUCTION

There are many important physical problems which incorporate several scales such as wave propagation through periodic or stratified media. The interactions and the fineness of these scales make solving these problems very difficult and expensive. Often, one would be content with the coarse-scale behavior of the solution but the fine scales affect this behavior so one cannot simply ignore these. Instead, it is useful to find a way of extracting or constructing equations for the coarse behavior of the solution which take into account the effect of the fine scales. This amounts to writing an effective equation for the coarse-scale component of the solution, which can be solved much more economically. Alternatively, one might wish to construct simpler fine-scale equations whose solutions have the same coarse properties as the solutions of the complicated systems. These ''simpler'' equations would also be considerably less expensive to solve. This latter procedure is called homogenization.

There are many approaches to homogenization. The classical theory of homogenization, developed in part by Bensoussan *et al.* [1], Murat [9], and Tartar [11], poses

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[2] Current address: ATT&T Laboratories—Research, Room C-280, 180 Park Avenue, Florham Park, New Jersey, 07932-0971

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the problem as follows: Given a family of differential operators L_ϵ, indexed by a parameter ϵ, assume that the boundary value problem

$$L_\epsilon u_\epsilon = f \quad \text{in } \Omega$$

(with u_ϵ subject to the appropriate boundary conditions) is well posed in a Sobolev space H for all ϵ and that the solutions u_ϵ form a bounded subset of H. Then, there is a weak limit u_0 in H of the solutions u_ϵ. The problem of homogenization is to find the differential equation that u_0 satisfies and to construct the corresponding differential operator. We call the homogenized operator L_0 and the equation $L_0 u_0 = f$ in Ω the homogenized equation.

There are several methods for solving this problem. In [8] and [1], the methods of asymptotic expansions and of G-convergence are used to examine families of operators L_ϵ. Murat and Tartar (see [9] and [11]) developed the method of compensated compactness. Coifman *et al.* (see [5]) have recently shown that there are intrinsic links between compensated compactness theory and the tools of classical harmonic analysis (such as Hardy spaces and operator estimates).

A more recent and philosophically different approach is given in [4]. Via a multiresolution approach, Brewster and Beylkin give a procedure for constructing an equation directly for the coarse-scale component of the solution. From this equation, one can determine a simpler equation for the original function with the same coarse-scale behavior.

In this paper, we will compare the classical homogenization theory with the algorithm of Brewster and Beylkin [4] in the case of linear one-dimensional second-order elliptic operators. This is a natural situation to examine because it is the first setting in which classical results are determined. We will examine physical situations where both theories are valid and explore what physical quantities are ''preserved'' with the two methods. We will also investigate several key physical problems (both numerically and theoretically) which highlight the distinctions between classical and multiresolution homogenization.

2. CLASSICAL HOMOGENIZATION THEORY

Let κ be a periodic function (with period one) in $L^\infty([0, 1])$ such that $\kappa(x) \geqslant \nu > 0$ for all $x \in [0, 1]$. We will associate to κ the differential operator

$$L = \frac{d}{dx}\left(\kappa \frac{d}{dx}\right).$$

If we define $\kappa_\epsilon(x) = \kappa(x\epsilon^{-1})$, then we have an associated family of operators

$$L_\epsilon = \frac{d}{dx}\left(\kappa(x\epsilon^{-1}) \frac{d}{dx}\right).$$

We also have a family of solutions u_ϵ in $H_0^1([0, 1])$ which solve the Dirichlet problems

$$L_\epsilon u_\epsilon = \frac{d}{dx}\left(\kappa(x\epsilon^{-1}) \frac{du_\epsilon}{dx}\right) = f. \tag{1}$$

A positive constant κ_0 is the homogenized or effective coefficient for this problem if for any $f \in H^{-1}([0, 1])$, the solutions u_ϵ of the Dirichlet problem (1) have the property

$$u_\epsilon \to u_0 \quad \text{weakly in } H_0^1([0, 1])$$

and

$$\kappa_\epsilon \frac{du_\epsilon}{dx} \to \kappa_0 \frac{du_0}{dx} \quad \text{weakly in } L^2([0, 1]) \text{ as } \epsilon \to 0,$$

where u_0 is the solution of the Dirichlet problem

$$\frac{d}{dx}\left(\kappa_0 \frac{du_0}{dx}\right) = f \quad \text{for } u_0 \in H_0^1([0, 1]).$$

The operator $(d/dx)(\kappa_0(d/dx))$ is called the homogenized operator and the equation $(d/dx)(\kappa_0(du_0/dx)) = f$ is called the homogenized equation. The vector fields $p_\epsilon = \kappa_\epsilon(du_\epsilon/dx)$, $p_0 = \kappa_0(du_0/dx)$ are called flows.

Let us derive the value of κ_0 with three different methods: an asymptotic expansion of the solution u_ϵ in powers of ϵ, a direct examination of the flows p_ϵ, and a heuristic rule used in many physical problems. We want to emphasize that these methods are used for physical problems which have two or more (but finite) distinguished scales. We will show that the multiresolution approach can be applied to physical problems with a continuum of scales and as such is more robust.

2.1. Asymptotic Method

In the problem $(d/dx)(\kappa(x\epsilon^{-1})(du_\epsilon/dx)) = f$, we have two distinguished scales (the scales of x and $x\epsilon^{-1}$) so we seek a two-scale asymptotic expansion of the solution u_ϵ. As a first approximation, one looks for a solution of the form

$$u_\epsilon(x, \epsilon) = u_0(x) + \epsilon u_1(x, y),$$

where $y = x\epsilon^{-1}$ and u_1 is periodic with respect to y. Note that

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \frac{1}{\epsilon}\frac{\partial}{\partial y}.$$

Then

$$f = \frac{d}{dx}\left(\kappa(y)\frac{du_\epsilon}{dx}\right) = \epsilon^{-1}(\kappa_1 u_1 + \kappa_2 u_0) + (\kappa_3 u_0 + \kappa_2 u_1) + \epsilon(\kappa_3 u_1), \qquad (2)$$

where

$$\kappa_1 = \frac{\partial}{\partial y}\left(\kappa(y)\frac{\partial}{\partial y}\right)$$

$$\kappa_2 = \frac{\partial}{\partial y}\left(\kappa(y)\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial x}\left(\kappa(y)\frac{\partial}{\partial y}\right),$$

and

$$\kappa_3 = \kappa(y)\frac{\partial^2}{\partial x^2}.$$

The first term in the right-hand-side of Eq. (2) must equal zero, so

$$\frac{\partial}{\partial y}\left(\kappa(y)\,\frac{\partial u_1}{\partial y}\right) = -\,\frac{d\kappa}{dy}\,\frac{du_0}{dx}\,.$$

This is a periodic boundary value problem in y with the right-hand side depending on x as a parameter. Let N be the solution of

$$\frac{d}{dy}\left(\kappa(y)\,\frac{dN}{dy}\right) = -\,\frac{d\kappa}{dy}\,. \tag{3}$$

Notice that Eq. (3) is equivalent to the problem

$$\frac{d}{dy}\left(\kappa(y)\left(1 + \frac{dN}{dy}\right)\right) = 0. \tag{4}$$

Then $u_1(x, y) = N(y)(du_0/dx)$ and $u_\epsilon(x) = u_0(x) + \epsilon N(y)(du_0/dx)$. Let us use this fact and the second term in Eq. (2) to determine

$$\begin{aligned}\kappa_3 u_0 + \kappa_2 u_1 &= \kappa(y)\,\frac{d^2u_0}{dx^2} + \frac{d}{dy}\,(\kappa(y)N(y))\,\frac{d^2u_0}{dx^2} + \kappa(y)\,\frac{dN}{dy}\,\frac{d^2u_0}{dx^2}\\ &= \frac{d^2u_0}{dx^2}\left(\kappa(y) + \frac{d}{dy}\,(\kappa(y)N(y)) + \kappa\,\frac{dN}{dy}\right).\end{aligned}$$

Averaging this term with respect to y, we get

$$\begin{aligned}\langle \kappa_3 u_0 + \kappa_2 u_1\rangle &= \left\langle \kappa(y) + \kappa(y)\,\frac{dN}{dy}\right\rangle \frac{d^2u_0}{dx^2}\\ &= \kappa_0\,\frac{d^2u_0}{dx^2}\end{aligned}$$

where $\kappa_0 = \langle \kappa(y) + \kappa(y)(dN/dy)\rangle$ is our homogenized coefficient.

From (4) we know that

$$N(y) = -y + \frac{1}{M_1}\int_0^y \frac{ds}{\kappa(s)} \quad \text{where } M_1 = \int_0^1 \frac{ds}{\kappa(s)}$$

and so

$$\kappa_0 = \left\langle \kappa(y) - \kappa(y) + \frac{1}{M_1}\right)\right\rangle = \frac{1}{M_1} = \frac{1}{\left\langle \dfrac{1}{\kappa}\right\rangle},$$

the harmonic average of κ. For justification of this method see [8].

2.2. Flows

In this section we review a different approach using the flows $p_\epsilon(x) = \kappa(x\epsilon^{-1})(du_\epsilon/dx)$. In this one-dimensional case, these are sufficient for us to determine the value of κ_0. Set $B(x) = 1/\kappa(x)$ and $F(x) = \int_0^x f(t)dt$. Then the flow is

$$p_\epsilon(x) = \kappa(x\epsilon^{-1})\frac{du_\epsilon}{dx} = F(x) - c_\epsilon \quad \text{and} \quad \frac{du_\epsilon}{dx} = B(x\epsilon^{-1})(F(x) - c_\epsilon).$$

The constants c_ϵ are determined by our boundary conditions

$$0 = \int_0^1 \frac{du_\epsilon}{dx}\,dx = \int_0^1 B(x\epsilon^{-1})(F(x) - c_\epsilon)dx.$$

To find $\lim_{\epsilon\to 0} c_\epsilon$ we must invoke a simple property of periodic functions that is frequently used in homogenization theory.

THEOREM 1. *Let $g:\mathbf{R}^n \to \mathbf{C}$ be a periodic function whose period cell is a box B with edges directed along the coordinate axes and edge lengths $l_1, l_2, \ldots, l_n$, respectively. We denote the mean value of g by $\langle g\rangle$; i.e.,*

$$\langle g\rangle = \frac{1}{|B|}\int_B g(x)dx,$$

where $|B| = l_1 l_2 \cdots l_n$. The space $L^p(B)$ is the space of periodic functions with finite norm $\langle |g|^p\rangle^{1/p}$ for $p \geqslant 1$. Assume that $g \in L^p(B)$, $p \geqslant 1$. Then $g(x/\epsilon) \to \langle g\rangle$ weakly in $L^p(\Omega)$ as $\epsilon \to 0$, where Ω is an arbitrary bounded domain in $\mathbf{R}^n$; i.e., $g(x/\epsilon) \to \langle g\rangle$ weakly in $L^p_{loc}(\mathbf{R}^n)$.

Proof. We can restrict ourselves to the situation $\Omega = sB$ where Ω is a dilation of the basic box B with ratio $s \geqslant 1$. Observe that for $f \in L^p(B)$ and $\epsilon \leqslant 1$,

$$\int_\Omega |f(x/\epsilon)|^p dx = \epsilon^n \int_{sB/\epsilon} |f(x)|^p dx \leqslant \epsilon^n([s\epsilon^{-1}] + 1)^n\langle |f|^p\rangle \leqslant c_0\langle |f|^p\rangle$$

for c_0 depending on Ω and for $[s\epsilon^{-1}]$ the greatest integer not larger than $s\epsilon^{-1}$. Let q be a trigonometric polynomial such that $\langle q\rangle = \langle g\rangle$ and $\langle |g - q|^p\rangle \leqslant \delta$. Then for $\epsilon \leqslant 1$, we also have

$$\int_\Omega |g(x/\epsilon) - q(x/\epsilon)|^p dx \leqslant c_0\delta.$$

This estimate shows us that it is sufficient to prove the result for trigonometric polynomials. However, for trigonometric polynomials this is simply a consequence of the Riemann–Lebesgue lemma. ■

Now, let us apply the previous result about the mean value to the relation

$$0 = \int_0^1 \frac{du_\epsilon}{dx}\,dx = \int_0^1 B(x\epsilon^{-1})(F(x) - c_\epsilon)dx.$$

We then determine that $\lim_{\epsilon\to 0} c_\epsilon = \int_0^1 F(x)dx$. We can also determine the weak limits in $L^2([0, 1])$ of the sequences du_ϵ/dx and p_ϵ. We have

$$\lim_{\epsilon\to 0} p_\epsilon(x) = F(x) - \int_0^1 F(x)dx = p_0(x),$$

$$\lim_{\epsilon\to 0} \frac{du_\epsilon}{dx}(x) = \langle B\rangle\left(F(x) - \int_0^1 F(x)dx\right) = \frac{du_0}{dx}(x),$$

and

$$u_0(x) = \int_0^x \frac{du_0}{dx}(t)dt.$$

These formulas show us that

$$p_0(x) = \frac{1}{\langle B\rangle}\frac{du_0}{dx}(x) \quad \text{and} \quad \frac{dp_0}{dx}(x) = f(x),$$

and that u_0 is the solution of the Dirichlet problem

$$\frac{d}{dx}\left(\langle \kappa^{-1}\rangle^{-1}\frac{du_0}{dx}\right) = f \quad \text{with } u_0 \in H_0^1([0, 1]).$$

Therefore, the homogenized coefficient κ_0 is $\langle \kappa^{-1}\rangle^{-1}$.

2.3. Homogenization Rule

The convergence of the flows is important because we can use it to formulate a rule (or a third method) for calculating the homogenized coefficient. This rule is quite simple and has been used repeatedly in physical literature. We discuss this rule because we will compare it to the multiresolution scheme. In particular, we will show which properties of the solution this rule preserves as opposed to the multiresolution strategy.

RULE 1. *If $v \in L^2([0, 1])$ is of the form $v = \lambda + du/dx$, $u \in H_0^1([0, 1])$ (so that $\lambda = \langle v\rangle$) and if $(d/dx)(\kappa v) = 0$, then $\kappa_0\langle v\rangle = \langle \kappa v\rangle$.*

Remark. To prove that this rule fits into the general framework of homogenization (at least for κ of the form $\kappa(\cdot\,\epsilon^{-1})$), set $u_\epsilon(x) = \lambda x + \epsilon u(x\epsilon^{-1})$. Then

$$\frac{d}{dx}(u_\epsilon(x)) = \lambda + \frac{d}{dx}u(x\epsilon^{-1}) = v(x\epsilon^{-1}),$$

which converges weakly to $\langle v\rangle$ in $L^2([0, 1])$ as $\epsilon \to 0$. Also, we know that $u_\epsilon(x\epsilon^{-1})$ converges weakly to $u_0(x) = \lambda x$. Since $(d/dx)(\kappa_\epsilon(du_\epsilon/dx)) = 0$ and $u_\epsilon \rightharpoonup u_0 = \lambda x$, we know that u_0 must solve $(d/dx)(\kappa_0(du_0/dx)) = 0$ and that the flows converge $\kappa_\epsilon(du_\epsilon/dx) \rightharpoonup \kappa_0(du_0/dx)$. However, Theorem 1 tells us that $\kappa_\epsilon(du_\epsilon/dx) = \kappa(x\epsilon^{-1})v(x\epsilon^{-1})$ converges weakly to $\langle \kappa v\rangle$. Therefore, $\langle \kappa v\rangle = \kappa_0\langle v\rangle$.

We note that the value of κ_0 for these one-dimensional elliptic equations holds in a much more general context (although we derived the value in the above, more restricted context). In particular, the operators $L_\epsilon: H_0^1([0, 1]) \to H_0^1([0, 1])^*$ form a sequence of linear operators which are uniformly coercive and uniformly bounded. The sequence of inverses $L_\epsilon^{-1}: H_0^1([0, 1])^* \to H_0^1([0, 1])$ is bounded uniformly so we may extract a subsequence L_ϵ^{-1} which converges (with respect to the weak topology in the space of operators) to a bounded operator $M: H_0^1([0, 1])^* \to H_0^1([0, 1])$. This operator M is coercive and admits an inverse, denoted $L_0: H_0^1([0, 1]) \to H_0^1([0, 1])^*$. It can be shown that the operator L_0 is the homogenized operator with the form $(d/dx)\kappa_0(d/dx)$ with homogenized coefficient $\kappa_0 = \langle \kappa^{-1} \rangle^{-1}$. See [8] for a more detailed discussion.

3. MULTIRESOLUTION HOMOGENIZATION METHOD

Let us first summarize the ideas in [4]. The algorithm for numerical homogenization depends on the general framework or multiresolution analysis (MRA) associated to the construction of a wavelet basis. An MRA is a natural framework in which to discuss the behavior of a solution on both fine and coarse scales. Also, we use a multiresolution analysis to represent operators in a matrix form [2]. For a wide class of operators (e.g., Calderón–Zygmund operators), the MRA representation of the matrix is a sparse matrix and allows us to construct fast algorithms. This MRA representation gives an explicit description of the operator's interactions between different scales and appears to be an appropriate tool for numerical homogenization.

The paper [4] begins with an algorithm for the numerical solution of and homogenization of linear systems of ordinary differential equations (equivalently, linear systems of integral equations in one variable); work by the same authors on non-linear ODEs and equations in more than one variable is in progress.

3.1. Linear Algebraic System

Let us review the main idea of the MRA scheme in [4] by illustrating it with a linear algebraic example. We will use the Haar basis here and in the following more general discussion. Suppose we have a linear algebraic system

$$Kx = b, \tag{5}$$

where K is a matrix of size $2^n \times 2^n$. This system might be a discretization of a linear ODE, for example. We change basis (in an orthogonal way) with the discrete Haar transform by writing

$$s_k = \frac{1}{\sqrt{2}}(x_{2k+1} + x_{2k}) \quad \text{and} \quad d_k = \frac{1}{\sqrt{2}}(x_{2k+1} - x_{2k})$$

for $k = 0 \cdots 2^{n-1} - 1$. The elements of s are essentially averages of neighboring entries in x (they have an extra factor $\sqrt{2}$ when compared with true averages) and the elements of d are differences. We can write the discrete Haar transform as a matrix M_n of size $2^n \times 2^n$:

$$M_n = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 & & & \cdots \\ 0 & 0 & 1 & 1 & 0 & 0 & \cdots \\ & & & \ddots & & & \\ -1 & 1 & 0 & 0 & & & \cdots \\ 0 & 0 & -1 & 1 & 0 & 0 & \cdots \\ & & & \ddots & & & \end{pmatrix}.$$

If we denote the top half of M_n by L_n and the bottom half by H_n, then

$$M_n^* M_n = M_n M_n^* = H_n^* H_n + L_n^* L_n = \mathrm{I} \quad \text{and} \quad H_n H_n^* = \mathrm{I} = L_n L_n^*.$$

We also have $L_n x = s$ and $H_n x = d$.

We will split Eq. (5) into two equations in the two unknowns s and d. If we apply L_n to both sides of (5), we get (dropping subscripts)

$$LKx = (LKL^*)Lx + (LKH^*)Hx = Lb. \tag{6}$$

Similarly, if we apply H_n, we get

$$HKx = (HKL^*)Lx + (HKH^*)Hx = Hb. \tag{7}$$

Let us denote

$$T = LKL^*, \quad C = LKH^*,$$
$$B = HKL^*, \quad A = HKH^*,$$
$$b_s = Lb, \quad \text{and} \quad b_d = Hb.$$

Then, Eqs. (6) and (7) are

$$Ts + Cd = b_s \tag{8}$$
$$Bs + Ad = b_d. \tag{9}$$

Assume that A is invertible so that we can solve Eq. (9) for d in terms of s:

$$d = -A^{-1}Bs + A^{-1}b_d. \tag{10}$$

Let us plug (10) into (8) and we obtain

$$(T - CA^{-1}B)s = b_s - CA^{-1}b_d, \tag{11}$$

a reduced equation for s which exactly determines the averages of x. That is, we have an exact ''effective'' equation for the averages of x which contains the contribution from the fine-scale behavior of x. Since we have a linear system and we have assumed that A is invertible, then we can exactly solve (8) and (9) for s. Note that this reduced equation has half as many unknowns as the original system. We will call this procedure the reduction step.

We should point out that under the reduction step the form of the original equation is preserved. Let $K_n = K$, $b_n = b$, $K_{n-1} = T - CA^{-1}B$, and $b_{n-1} = b_s - CA^{-1}b_d$. Then our Eq. (11) for s has the form $K_{n-1}s = b_{n-1}$, where $s = L_{n-1}x$ is our unknown. This procedure can be repeated up to n times using the recursion formulas

$$K_{j-1} = T_j - C_j A_j^{-1} B_j$$
$$b_{j-1} = L_{n-j}b - C_j A_j^{-1} H_{n-j} b,$$

where

$$T_j = L_{n-j} K_j L_{n-j}^*, \qquad C_j = L_{n-j} K_j H_{n-j}^*,$$
$$B_j = H_{n-j} K_j L_{n-j}^*, \quad \text{and} \quad A_j = H_{n-j} K_j H_{n-j}^*.$$

This recursion process involves only the matrices K_j and the vectors b_j. In other words, we do not have to solve for x at any step in the reduction procedure. If we apply this reduction process n times, we have a scalar equation which we can solve easily. This scalar equation tells us the average of the vector x (up to the normalization $2^{n/2}$), i.e., the ''coarse'' behavior of x. If we are interested in only this coarse behavior of x, then the reduction process gives us a way of determining exactly the average of x without having to solve the original system $Kx = b$ for x and then computing its average. Of course, in this linear algebra example, the MRA technique may not be any faster than the best solvers for such a simple linear algebraic system. However, this example illustrates a technique that may be very useful for complicated physical systems and for linear ODEs where solving the complete system is computationally expensive and where we are interested in only the solution's coarse-scale behavior. Considered only as a tool for the numerical analysis of linear systems, the MRA technique is closely related to several well-known methods for numerical linear algebra (see, e.g., [7]).

Standard homogenization results are really formulated in terms of an ''elevation'' or ''augmentation'' of the reduction step. That is, an equivalent equation is written down where the solution has the same coarse behavior as the original solution. So far we have discussed half of the MRA homogenization ideas, the reduction process. Let us use our simple linear algebraic example to illustrate the numerical augmentation approach. Suppose we have two different equations

$$K_n^1 x^1 = b^1 \quad \text{and} \quad K_n^2 x^2 = b^2 \tag{12}$$

such that after one reduction step the effective matrices and vectors are equal; i.e.,

$$K_{n-1}^1 = K_{n-1}^2 \quad \text{and} \quad b_{n-1}^1 = b_{n-1}^2.$$

Then the solutions s^1 and s^2 must be equal. In other words, the solutions of (12) agree on a coarse scale and differ on only a finer scale. Suppose that one of the equations, say $K_n^2 x^2 = b^2$, has a ''simpler'' form than the other. (For linear systems of variable coefficient ODEs, equations of a ''simpler'' form might be constant coefficient ODEs.) We will exploit this more desirable structure by replacing the first system, $K_n^1 x^1 = b^1$, with the second and we can be confident that the coarse-scale

behavior of the solution is not affected by this replacement. In other words, we have substituted a more desirable equation for a complicated one but the desirable equation has the same coarse properties as the solution of the original equation. We call the simpler equation a homogenized equation and refer to this process of refining or simplifying an equation as homogenization.

In many physical situations, one is interested in only the coarse-scale behavior of a solution and so a reduced or effective equation for this behavior is sufficient. One need not use the second half of the MRA strategy to find a homogenized equation. We think that the real advantage in the MRA scheme is a precise algorithm for determining this effective equation. The classical theory provides no such algorithm, only a homogenized equation. On the other hand, we will use the augmentation process to compare the numerical homogenization procedure with the classical results, both theoretically and with physical examples.

3.2. Linear ODEs

Let us now summarize the MRA method for linear ODEs. Consider the differential equation

$$\frac{d}{dt}\Big(G(t)x(t) + q(t)\Big) = F(t)x(t) + p(t), \quad t \in [0, 1],$$

where F and G are bounded matrix-valued functions, and p and q are vector-valued functions (with elements in $L^2([0, 1])$). We will rewrite this differential equation as an integral equation

$$G(t)x(t) + q(t) - \beta = \int_0^t F(s)x(s) + p(s)ds, \quad t \in [0, 1] \tag{13}$$

(where β is a complex or real parameter), since we can preserve the form of this equation under reduction. To express this integral equation in terms of an operator equation on functions in $L^2([0, 1])$, let $\mathbf{F}$ and $\mathbf{G}$ be the operators whose actions on functions is pointwise multiplication by F and G and let $\mathbf{K}$ be the integral operator whose kernel K is

$$K(s, t) = \begin{cases} 1, & 0 \leqslant s \leqslant t \\ 0, & \text{otherwise.} \end{cases}$$

Then (13) can be rewritten as

$$\mathbf{G}x + q - \beta = \mathbf{K}(\mathbf{F}x + p).$$

We will use a general MRA of $L^2([0, 1])$. See the Appendix for definitions. We begin with an initial discretization of our integral equation by applying the projection operator P_n and looking for a solution $x_n^{(n)}$ in V_n. This is equivalent to discretizing our problem at a very fine scale. We have

$$G_n^{(n)} x_n^{(n)} + q_n^{(n)} - \beta = K_n(F_n^{(n)} x_n^{(n)} + p_n^{(n)}),$$

where

$$G_n^{(n)} = P_n \mathbf{G} P_n^*,\ F_n^{(n)} = P_n \mathbf{F} P_n^*,\ p_n^{(n)} = P_n p, \text{ and } q_n^{(n)} = P_n q.$$

As in the linear algebraic example, we want a recursion relation for the operators $G_j^{(n)}$, $F_j^{(n)}$, $p_j^{(n)}$, and $q_j^{(n)}$ for $j = n, \ldots, 0$ so that $x_j^{(n)}$ solves the equation

$$G_j^{(n)} x_j^{(n)} + q_j^{(n)} - \beta = K_j(F_j^{(n)} x_j^{(n)} + p_j^{(n)}).$$

In other words, at each level j, we want an effective equation for the projection of the solution onto V_j and we want to determine this effective equation recursively (as in the previous example). To clarify the notation, $M_j^{(n)}$ is the operator we derive at level j beginning with the discretization level n.

We will proceed by induction (initializing the recursion scheme with $G_n^{(n)} = P_n \mathbf{G} P_n^*$, $F_n^{(n)} = P_n \mathbf{F} P_n^*$, $K_n = P_n \mathbf{K} P_n^*$, $q_n^{(n)} = P_n q$, and $p_n^{(n)} = P_n p$). Let us assume that we have an equation for level $j + 1$:

$$G_{j+1}^{(n)} x_{j+1}^{(n)} + q_{j+1}^{(n)} - \beta = K_{j+1}(F_{j+1}^{(n)} x_{j+1}^{(n)} + p_{j+1}^{(n)}). \tag{14}$$

We rewrite $x_{j+1}^{(n)}$ in terms of its averages and differences

$$x_{j+1}^{(n)} = P_j x_{j+1}^{(n)} + Q_j x_{j+1}^{(n)} = v_j + w_j$$

and plug this into our Eq. (14):

$$G_{j+1}^{(n)}(v_j + w_j) + q_{j+1}^{(n)} - \beta = K_{j+1}(F_{j+1}^{(n)}(v_j + w_j) + p_{j+1}^{(n)}). \tag{15}$$

Next, we apply the operators P_j and Q_j to (15) to split it into two equations (dropping subscripts):

$$\begin{aligned}
&(PGP^*)v + (PGQ^*)w + Pq \\
&\qquad = \mathrm{PKP}^*((PFP^*)v + (PFQ^*)w + Pp) + PKQ^*((QFP^*)v \\
&\qquad\qquad + (QFQ^*)w + Qp) \\
&(QGP^*)v + (QGQ^*)w + Qq \\
&\qquad = \mathrm{QKP}^*((PFP^*)v + (PFQ^*)w + Pp) + QKQ^*((QFP^*)v \\
&\qquad\qquad + (QFQ^*)w + Qp).
\end{aligned}$$

Let us denote

$$\begin{aligned}
T_{O,j} &= P_j O P_j^*, \quad C_{O,j} = P_j O Q_j^*, \\
B_{O,j} &= Q_j O P_j^*, \quad A_{O,j} = Q_j O Q_j^*,
\end{aligned}$$

using the non-standard form [2] to represent an operator O. Then we obtain (again dropping subscripts)

$$T_G v + C_G w + Pq - \beta = T_K(T_F v + C_F w + Pp) + C_K(B_F v + A_F w + Qp) \quad (16)$$

$$B_G v + A_G w + Qq = B_K(T_F v + C_F w + Pp) + A_K(B_F v + A_F w + Qp). \quad (17)$$

Let us assume that

$$R = A_G - B_K C_F - A_K A_F$$

is invertible so that we may solve (17) for w and plug in the result to (16), giving us

$$\begin{aligned}(T_G - C_K B_F - (C_G - C_K A_F)R^{-1}(B_G - B_K T_F - A_K B_F))v \\ + (P_q - C_K Q_p - (C_G - C_K A_F)R^{-1}(Q_q - B_K P_p - A_K Q_p)) - \beta \\ = T_K((T_F - C_F R^{-1}(B_G - B_K T_F - A_K B_F))v + P_p \\ - C_F R^{-1}(Q_q - B_K P_p - A_K Q_p)). \quad (18)\end{aligned}$$

So, the recursion relations for the operators and forcing terms are

$$\begin{aligned}F_j^{(n)} &= T_{F,j} - C_{F,j} R_j^{-1}(B_{G,j} - B_{K,j} T_{F,j} - A_{K,j} B_{F,j}),\\ G_j^{(n)} &= T_{G,j} - C_{K,j} B_{F,j} - (C_{G,j} - C_{K,j} A_{F,j}) R_j^{-1}(B_{G,j} - B_{K,j} T_{F,j} - A_{K,j} B_{F,j}),\\ q_j^{(n)} &= P_j q - C_{K,j} Q_j p - (C_{G,j} - C_{K,j} A_{F,j}) R_j^{-1}(Q_j q - B_{K,j} P_j p - A_{K,j} Q_j p),\\ p_j^{(n)} &= P_j p - C_{F,j} R_j^{-1}(Q_j q - B_{K,j} P_j p - A_{K,j} Q_j p).\end{aligned}$$

These recursion relations allow us to pass from Eq. (14) in V_{j+1} to Eq. (18) in V_j. If we apply them n times, we get an equation in V_0

$$G_0^{(n)} x_0^{(n)} + q_0^{(n)} - \beta = \tfrac{1}{2}(F_0^{(n)} x_0^{(n)} + p_0^{(n)})$$

for the coarse-scale behavior of $x_0^{(n)}$, the solution to the discretized system.

These recursion relations hold for general wavelets. In most of this paper, we shall use the Haar basis. Because the supports of the Haar scaling functions at the same scale are disjoint, many of the matrices involved in the reduction procedure become very simple. However, other wavelets with short support may be used as well. To illustrate that the scheme remains computationally viable with wavelets of short support, we will also work out an example with a different wavelet scheme; in particular, we will use a biorthogonal basis where the analyzing wavelet has three vanishing moments, leading to better approximation properties. If the reduction process is stopped at some level $j > 0$ in order to retain slightly more detail, then with the Haar basis, $P_j x$ is a piecewise constant function with step width 2^{-j}; with the biorthogonal basis (and other wavelet bases in general), $P_j x$ is a smoother function, still an approximation of x with resolution 2^{-j}, but with a higher approximation order.

We are restricting ourselves to a one-dimensional system here for simplicity. (For

N-dimensional systems, the analysis is similar, except that the scalar entries in the matrices below become themselves $N \times N$ matrices.) First, the integral operator $\mathbf{K}$ in the Haar basis has a simple form. The operator $T_{K,n} = P_n \mathbf{K} P_n^* : V_n \to \mathrm{V}_n$ has the matrix form

$$T_{K,n} = \frac{1}{2^n} \begin{pmatrix} \frac{1}{2} & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & 1 & \frac{1}{2} \end{pmatrix}.$$

The operator $C_{K,n} = P_n \mathbf{K} Q_n^* : W_n \to \mathrm{V}_n$ has the matrix form

$$C_{K,n} = \frac{1}{2^{n+2}} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix};$$

i.e., it identifies the space W_n with V_n in the sense that the element $\sum_k c_k \psi_{n,k}$ is mapped to $1/2^{n+2} \sum_k c_k \phi_{n,k}$. Also, $B_{K,n} = Q_n \mathbf{K} P_n^* : V_n \to \mathrm{W}_n$ identifies W_n with V_n and has the matrix form

$$B_{K,n} = \frac{-1}{2^{n+2}} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

The operator $A_{K,n} = Q_n \mathbf{K} Q_n^* : W_n \to \mathrm{W}_n$ is identically zero. The initial operators $F_n^{(n)}$ and $G_n^{(n)}$ have the matrix forms

$$M_n^{(n)} = \begin{pmatrix} M_{n,0} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & M_{n,2^n-1} \end{pmatrix} = \operatorname{diag}\{M_{n,0}, M_{n,1}, \ldots, M_{n,2^n-1}\},$$

where $M_{n,k} = 2^n \int_{2^{-n}k}^{2^{-n}(k+1)} M(x)\,dx$, for $M = F$ or G.

The operators $T_{M,j}$, $C_{M,j}$, $B_{M,j}$, and $A_{M,j}$ also have a simple form in the Haar basis:

$$T_{M,j} = A_{M,j} = \operatorname{diag}\{S_{M,j,0}^{(n)}, \ldots, S_{M,j,2^n-1}^{(n)}\}$$

and

$$C_{M,j} = B_{M,j} = \operatorname{diag}\{D_{M,j,0}^{(n)}, \ldots, D_{M,j,2^n-1}^{(n)}\},$$

where

$$S_{M,j,k}^{(n)} = \tfrac{1}{2}(M_{j+1,2k+1}^{(n)} + M_{j+1,2k}^{(n)})$$

and

$$D_{M,j,k}^{(n)} = \tfrac{1}{2}(M_{j+1,2k+1}^{(n)} - M_{j+1,2k}^{(n)}).$$

So our recursion relations can be written simply as

$$F_{j-1} = S_{F,j} - D_{F,j}R_j^{-1}(D_{G,j} + \delta_j S_{F,j}),$$
$$G_{j-1} = S_{G,j} - \delta_j D_{F,j} + (D_{G,j} - \delta_j S_{F,j})R_j^{-1}(-S_{F,j} - D_{G,j}),$$
$$q_{j-1} = (D_{G,j} - \delta_j S_{F,j})R_j^{-1}(-D_{q,j} - \delta_j S_{p,j}) - \delta_j D_{p,j} + S_{q,j},$$

and

$$p_{j-1} = D_{F,j}R_j^{-1}(-D_{q,j} - \delta_j S_{p,j}) + S_{p,j},$$

where $\delta_j = 2^{-j-1}$.

We will also work with the (3, 1) biorthogonal wavelet basis, which has analyzing filters

$$\tilde{h}_n = \left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}, \quad \tilde{g}_n = \left\{\frac{-1}{8\sqrt{2}}, \frac{-1}{8\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{1}{8\sqrt{2}}, \frac{1}{8\sqrt{2}}\right\}$$

and synthesizing filters

$$h_n = \left\{\frac{-1}{8\sqrt{2}}, \frac{1}{8\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{8\sqrt{2}}, \frac{-1}{8\sqrt{2}}\right\}, \quad g_n = \left\{\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right\}.$$

Note that this filterbank can be implemented with Sweldens' lifting scheme [10]. In this basis, the operator $T_{K,n} = \tilde{P}_n \mathbf{K} P_n^* : \tilde{V}_n \to \mathrm{V}_n$ has the matrix form

$$T_{K,n} = \mathrm{Tri}\left\{1; \frac{719}{720}, \frac{47}{45}, \frac{1}{2}, -\frac{2}{45}, \frac{1}{720}\right\}.$$

That is, $T_{K,n}$ is a lower triangular matrix, with the entry 1 in the lower triangular region, and $T_{K,n}$ has a diagonal band. The entry $1/2$ lies on the main diagonal, while the entries $719/720$ and $47/45$ (resp., $-2/45$ and $1/720$) lie on the left (resp., right) of the main diagonal. We have written the entries in order from left to right. The operator $C_{K,n} = \tilde{P}_n \mathbf{K} Q_n^* : \tilde{W}_n \to \mathrm{V}_n$ has the matrix form

$$C_{K,n} = \frac{1}{2^{n+2}}\,\mathrm{Band}\left\{\frac{-2}{45}, \frac{49}{45}, \frac{-2}{45}\right\}.$$

That is, $C_{K,n}$ is a banded matrix with the entry $49/45$ along the main diagonal and the entry $-2/45$ to the left and to the right of the main diagonal. The operator $B_{K,n} = \tilde{Q}_n \mathbf{K} P_n^* : \tilde{V}_n \to \mathrm{W}_n$ has the matrix form

$$B_{K,n} = \frac{1}{2^{n+2}}\,\mathrm{Band}\left\{\frac{1}{1440}, \frac{-9}{320}, \frac{47}{160}, \frac{-767}{1440}, \frac{47}{160}, \frac{-9}{320}, \frac{1}{1440}\right\}.$$

The operator $A_{K,n} = \tilde{Q}_n \mathbf{K} Q_n^* : \tilde{W}_n \to \mathrm{W}_n$ has the matrix form

$$A_{K,n} = \frac{1}{2^{n+2}}\,\mathrm{Band}\left\{\frac{1}{180}, \frac{-11}{60}, 0, \frac{-11}{60}, \frac{1}{180}\right\}.$$

All of these matrices must be altered appropriately at the boundaries of the interval. This alteration is made by changing the analyzing and synthesizing filters at the boundaries (see [10] for details). The initial operators $F_n^{(n)}$ and $G_n^{(n)}$ have the matrix forms

$$M_n^{(n)} = \{m_{i,j} | i, j = 0, \ldots, 2^n - 1\},$$

where $m_{i,j} = \langle \tilde{\phi}_{n,i}, M\phi_{n,j} \rangle$ for $M = F$ or G. Unlike the Haar basis, this basis does not yield simplified recursion relations. We must use instead the recursion relations for a general wavelet basis which we have previously derived.

3.3. Augmentation Procedure

After we have reduced the equation

$$G(t)x(t) - \beta = \int_0^t F(s)x(s)ds, \tag{19}$$

we have an effective equation (in V_0)

$$G_0^{(\infty)}\langle x \rangle - \beta = \tfrac{1}{2}F_0^{(\infty)}\langle x \rangle. \tag{20}$$

In this section we will describe how to augment this reduced equation or how to determine the homogenized coefficients F^h and G^h in the integral equation

$$G^h x(t) - \beta = F^h \int_0^t x(s)ds \tag{21}$$

such that applying the same reduction procedure to Eq. (21) produces (20) for all β. In other words, we want to find a constant-coefficient integral equation whose solution has the same average on [0, 1] as the solution of Eq. (19).

The recurrence relations applied to (21) after simplification give us

$$\begin{aligned} F_{j-1}^h &= F_j^h \\ G_{j-1}^h &= G_j^h + (\delta_j)^2 F_j^h (G_j^h)^{-1} F_j^h, \end{aligned}$$

where $\delta_j = 2^{-j-1}$. Since F_j^h remains unchanged at each level of the reduction procedure, the homogenized coefficient is $F^h = F_0^{(\infty)}$. We now have to determine the homogenized coefficient G^h; in general, it is not simply $G_0^{(\infty)}$.

The solution of Eq. (21) is

$$x(t) = -\exp((G^h)^{-1}F^h t)((G^h)^{-1}\beta)$$

and its average is

$$\begin{aligned} \langle x \rangle &= \left(-\int_0^1 \exp((G^h)^{-1}F^h t)dt \right)((G^h)^{-1}\beta) \\ &= (\mathrm{I} - \exp((G^h)^{-1}F^h))((G^h)^{-1}F^h)^{-1}((G^h)^{-1}\beta). \end{aligned} \tag{22}$$

However, we can also solve (20) for the average of x and get

$$\langle x \rangle = (G_0^{(\infty)} - \tfrac{1}{2}F_0^{(\infty)})^{-1}\beta. \tag{23}$$

Because we want to preserve the average of the solution under homogenization, Eq. (22) must equal Eq. (23) for all β. In other words,

$$(G_0^{(\infty)} - \tfrac{1}{2}F^h)^{-1} = (\mathrm{I} - \exp((G^h)^{-1}F^h))((G^h)^{-1}F^h)^{-1}(G^h)^{-1}, \tag{24}$$

where we have replaced $F_0^{(\infty)}$ with F^h. Solving (24) for G^h in terms of $G_0^{(\infty)}$ and F^h, we have

$$G^h = F^h(\tilde{F})^{-1}),$$

where $\tilde{F} = \log(-(\mathrm{I} + (G_0^{(\infty)} - \tfrac{1}{2}F^h)^{-1}F^h))$.

We have derived the augmentation algorithm for zero forcing terms p and q. See [4] for a more detailed discussion. We should also note here that we do not have to preserve simply the average of the solution; we can, instead, preserve a linear functional of the solution (again, see [4]). Also, we do not have to take as our simpler equation one which has constant coefficients. We can choose any equation so long as applying the reduction procedure to this equation produces an effective equation which is equal to the effective equation of the original problem. That is, any equation whose solution has the same average as the solution of our original equation will suffice.

4. SECOND-ORDER ELLIPTIC PROBLEMS

We will now examine the results of applying the MRA homogenization scheme to the one-dimensional second-order ellipic problem which we discussed in the previous section on classical homogenization theory. This approach works only for one-dimensional elliptic problems. For higher-dimensional elliptic problems, we must use the methods presented in [3]. The results in [3] indicate that the MRA homogenization methods for n-dimensional elliptic equations do not preserve the form of differential operators; instead, pseudo-differential operators seem to be the classes of operators to consider. In order to apply the above algorithm to the equation

$$\frac{d}{dx}\left(\kappa \frac{du}{dx}\right) = f,$$

we must rewrite this as a system of first-order differential equations:

$$\frac{du}{dx} = \frac{v}{\kappa} \quad \text{and} \quad \frac{dv}{dx} = f.$$

Again, we assume that κ is a continuous function on $[0, 1]$ which is periodic (with period one) and which is bounded away from zero.

4.1. Reduction Procedure without Forcing Terms

We first discuss the case where f is identically equal to zero which simplifies our calculations, we shall come back to the case where $f \neq 0$ afterwards. Using the notation of Section 3.2, we have

$$G(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, F(t) = \begin{pmatrix} 0 & 1/\kappa(t) \\ 0 & 0 \end{pmatrix}, x(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \text{ and } p(t) = q(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We will derive the operators $G_0^{(n)}$ and $F_0^{(n)}$ for general n and then determine $\lim_{n\to\infty} G_0^{(n)}$ and $\lim_{n\to\infty} F_0^{(n)}$, the effective operators on the space V_0 beginning with an infinitely small discretization.

We begin by simplifying the recursion relations for our two-dimensional system. Let us write $G(t)$ and $F(t)$ in block form so that

$$G(t) = \begin{pmatrix} 1 & \Gamma(t) \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad F(t) = \begin{pmatrix} 0 & \Theta(t) \\ 0 & 0 \end{pmatrix},$$

where $\Gamma(t) = 0$ initially and $\Theta(t) = 1/\kappa(t)$. Because of the structure of $G(t)$ and $F(t)$, the two-dimensional recursion relations for this system are very simple. In particular,

$$G_j = \begin{pmatrix} 1 & \Gamma_j \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad F_j = \begin{pmatrix} 0 & \Theta_j \\ 0 & 0 \end{pmatrix} \tag{25}$$

with

$$\Gamma_j = P_j \Gamma_{j+1} P_j^* - (P_j K Q_j^*)(Q_j \Theta_{j+1} P_j^*)$$

and

$$\Theta_j = P_j \Theta_{j+1} P_j^*.$$

Since these recursion relations change only Γ_j and Θ_j, we will work only with these with the understanding that the operators G_j and F_j are organized as in Eq. (25).

4.1.1. Haar MRA

At this point we must choose a basis in which to evaluate the algorithm. We will use the Haar basis first (see the Appendix for the definitions of the Haar scaling function ϕ and wavelet ψ). We shall extend these results to a biorthogonal basis in the next section.

We will now examine the results of the reduction procedure for one level of resolution. For this it suffices to choose the discretization to be a very coarse one (dividing the unit interval into only two parts); we will reduce the equation by only one level of resolution. The initial discretization of our integral equation is

$$G_1^{(1)} x_1^{(1)} - \beta = K(F_1^{(1)} x_1^{(1)}),$$

where

$$G_1^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad F_1^{(1)} = \begin{pmatrix} 0 & 0 & \theta_{0,0} & \theta_{0,1} \\ 0 & 0 & \theta_{1,0} & \theta_{1,1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K = \frac{1}{4} \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad x_1^{(1)} = \begin{pmatrix} P_1 u \\ P_1 v \end{pmatrix}.$$

The entries $\theta_{l,m}$ in the matrix Θ_1 are defined as inner products of $1/\kappa$ with scaling functions: $\theta_{l,m} = \langle \phi_{1,l}, (1/\kappa)\phi_{1,m} \rangle$ for $l, m = 0, 1$. Note that the matrix Γ_1 is initially zero.

Using the reduction scheme, we can write Θ_0 and Γ_0 in operator form:

$$\Theta_0 = P_0 \Theta_1 P_0^* \quad \text{and} \quad \Gamma_0 = -\tfrac{1}{4} Q_0 \Theta_1 P_0^*.$$

In ''function'' form, the reduced operators are given by

$$\Theta_0 = \left\langle \phi_0, \frac{1}{\kappa} \phi_0 \right\rangle \quad \text{and} \quad \Gamma_0 = \left\langle -\frac{1}{4} \psi_0, \frac{1}{\kappa} \phi_0 \right\rangle.$$

Before we proceed to a reduction spanning more than one level, let us introduce several definitions. Assume that we begin the reduction process at resolution level n and reduce l levels so that we are at resolution $n - l$. We define a composition of the operators P_j and Q_j (for j ranging from n to $n - l$) as $(T_{\lambda_k})_{n-l} = P_{n-l} \cdots P_{n-l+(k-1)} Q_{n-l+k}$, where the multi-index λ_k has the form

$$\lambda_k = \underbrace{0, \ldots, 0}_{k}, 1.$$

Note that the following three relations hold for $(T_{\lambda_k})_{n-l}$:

$$(T_{\lambda_0})_{n-l} = (T_1)_{n-l} = Q_{n-l} \tag{26}$$

$$(T_{\lambda_k 0})_{n-l} = (T_{\lambda_k})_{n-l} \quad (\text{equivalently, } Q_{j-1} P_j = Q_{j-1}). \tag{27}$$

$$(T_0)_{n-l} = P_{n-l}. \tag{28}$$

In terms of the scaling and wavelet functions, we are simply defining a special type of wavelet packet. Recall the definition of a wavelet packet $\psi_{n-l;\epsilon_1,\ldots,\epsilon_{n-l}}$, where $\epsilon_j = 0$ or 1 (see [6]). Then the Fourier transform of the wavelet packet is given by

$$\hat{\psi}_{n-;l\epsilon_1,\ldots,\epsilon_{n-l}}(\xi) = \prod_{j=1}^{n-l} m_{\epsilon_j}(\xi/2^j)\hat{\phi}(\xi/2^{n-l+1}).$$

Using the same notation λ_k as above, we will work with the wavelet packets $\psi_{n-l;\lambda_k}$. Notice that the ''function'' forms of the three relations (26)–(28) also hold:

$$\psi_{n-l;\lambda_0}(x) = \psi_{n-l}(x)$$

$$\psi_{n-l;\lambda_k,0}(x) = \psi_{n-l;\lambda_k}(x)$$

$$\psi_{n-l;0}(x) = \phi_{n-l}(x).$$

For $n - l = 0$, these special wavelet packets in the Haar basis are simply Walsh functions. For simplicity we will drop the subscript $n - l$ on both ψ_{λ_k} and T_{λ_k} when $n - l = 0$. When we say that ψ_{λ_k} is the ''function'' form of the operator T_{λ_k}, we mean that the composition of projections applied to a function f is the inner product of the wavelet packet with f; i.e.,

$$(T_{\lambda_k})f = \langle \psi_{\lambda_k}, f \rangle \psi_{\lambda_k}.$$

We may now write the result of our first calculation in this form

$$\Theta_0 = P\Theta_1 P^* = \left\langle \phi_0, \frac{1}{\kappa}\phi_0 \right\rangle \quad \text{and} \quad \Gamma_0 = -\frac{1}{4} T_{\lambda_0}\Theta_1 P_0^* = \left\langle -\frac{1}{4}\psi_0, \frac{1}{\kappa}\phi_0 \right\rangle$$

with

$$G_0^{(1)} = \begin{pmatrix} 1 & \Gamma_0^{(1)} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad F_0^{(1)} = \begin{pmatrix} 0 & \Theta_0^{(1)} \\ 0 & 0 \end{pmatrix}.$$

We want to find the form of the effective operators $G_0^{(n)}$ and $F_0^{(n)}$ for arbitrary n. We proceed by induction. Assume that for level n we have

$$G_0^{(n)} = \begin{pmatrix} 1 & \Gamma_0^{(n)} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad F_0^{(1)} = \begin{pmatrix} 0 & \Theta_0^{(n)} \\ 0 & 0 \end{pmatrix},$$

where

$$\Gamma_0^{(n)} = -\frac{1}{4}\sum_{k=0}^{n-1} 2^{-k} T_{\lambda_k}\Theta_n^{(n)} P_0^* \quad \text{and} \quad \Theta_0^{(n)} = P_0\Theta_n^{(n)}P_0^*. \tag{29}$$

If we start at level $n + 1$ and reduce n steps, then we have (dropping superscripts)

$$\Gamma_1 = -\frac{1}{8}\sum_{k=0}^{n-1} 2^{-k}(T_{\lambda_k})_1\Theta_{n+1}P_1^* \quad \text{and} \quad \Theta_1 = P_1\Theta_{n+1}P_1^*.$$

We now apply the recursion relation for Θ_j to Θ_1 to obtain

$$\Theta_0 = P_0\Theta_1 P_0^* = P_0(P_1\Theta_{n+1}P_1^*)P_0^* = P_0\Theta_{n+1}P_0^*.$$

For Γ_0 we have

$$\begin{aligned}\Gamma_0 &= P_0\Gamma_1 P_0^* - \frac{1}{4} Q_0\Theta_1 P_0^* \\ &= P_0\left(- \frac{1}{8} \sum_{k=0}^{n-1} 2^{-k}(T_{\lambda_k})_1\Theta_{n+1}P_1^*\right)P_0^* - \frac{1}{4} Q_0(P_1\Theta_{n+1}P_1^*)P_0^* \\ &= - \frac{1}{4} Q_0\Theta_{n+1}P_0^* - \frac{1}{8}\sum_{k=0}^{n-1} 2^{-k}P_0(T_{\lambda_k})_1\Theta_{n+1}P_0^* \\ &= - \frac{1}{4}\sum_{k=0}^{n} 2^{-k}T_{\lambda_k}\Theta_{n+1}P_0^*.\end{aligned}$$

This gives us the general form of $\Gamma_0^{(n)}$ and $\Theta_0^{(n)}$ and proves (29) for all n. In the limit as $n \to \infty$, we find

$$\Theta_0^{(\infty)} = \left\langle \phi_0, \frac{1}{\kappa}\phi_0 \right\rangle = \int_0^1 \frac{dt}{\kappa(t)}$$

for κ any continuous function which is bounded away from zero.

Let us now determine the limiting behavior of $\Gamma_0^{(n)}$. We claim that for the Haar basis

$$\lim_{n\to\infty} \Gamma_0^{(n)} = \lim_{n\to\infty} - \frac{1}{4}\sum_{k=0}^{n-1} 2^{-k}T_{\lambda_k}\Theta_n^{(n)}P_0^* = \int_0^1 \frac{t - 1/2}{\kappa(t)}\, dt.$$

Proof. Let $\Omega(t) = -\frac{1}{4}\sum_{k=0}^{\infty} 2^{-k}\psi_{\lambda_k}(t)$. Observe that Ω is an infinite (but pointwise convergent) sum of Walsh functions, supported on $[0, 1]$. The Fourier transform of Ω is given by

$$\begin{aligned}\hat{\Omega}(\xi) &= - \frac{1}{4}\sum_{k=0}^{\infty} 2^{-k}\hat{\psi}_{\lambda_k}(\xi) \\ &= - \frac{1}{4} m_1(\xi/2)\hat{\phi}(\xi/2) - \frac{1}{4}\sum_{k=1}^{\infty} 2^{-k}\prod_{j=1}^{k} m_0(\xi/2^j)m_1(\xi/2^{k+1})\hat{\phi}(\xi/2^{k+1}).\end{aligned}$$

We now multiply $\hat{\Omega}(\xi/2)$ by $m_0(\xi/2)$ and obtain

$$\begin{aligned}m_0(\xi/2)\hat{\Omega}(\xi/2) &= - \frac{1}{4} m_0(\xi/2)m_1(\xi/4)\hat{\phi}(\xi/4) \\ &\quad - \frac{1}{4}\sum_{k=0}^{\infty} 2^{-k}\prod_{j=1}^{k} m_0(\xi/2)m_0(\xi/2^{j+1})m_1(\xi/2^{k+2})\hat{\phi}(\xi/2^{k+2}) \\ &= 2\hat{\Omega}(\xi) + \frac{1}{2}\hat{\psi}(\xi).\end{aligned}$$

For the Haar basis $m_0(\xi/2)$ is given by $m_0(\xi/2) = 1/2 + 1/2e^{-i\xi/2}$, so we can rewrite the product of $m_0(\xi/2)$ and $\hat{\Omega}(\xi/2)$ as

$$\tfrac{1}{2}\hat{\Omega}(\xi/2) + \tfrac{1}{2}e^{-i\xi/2}\hat{\Omega}(\xi/2) = 2\hat{\Omega}(\xi) + \tfrac{1}{2}\hat{\psi}(\xi). \tag{30}$$

If we take the inverse Fourier transform of Eq. (30), we see that Ω must satisfy the relation

$$\Omega(2t) + \Omega(2t - 1) = 2\Omega(t) + \tfrac{1}{2}\psi(t). \tag{31}$$

Because $\Omega(t)$ is restricted to the unit interval $[0, 1]$, the weight function $\Omega(t) = t - 1/2$ does indeed satisfy Eq. (31). On the other hand, suppose $\Omega^{\#}(t)$ were another solution of Eq. (31), also bounded and supported on $[0, 1]$. Then $\omega(t) = \Omega(t) - \Omega^{\#}(t)$ would satisfy $\hat{\omega}(\xi) = \frac{1}{4}(1 + e^{-i\xi/2}\hat{\omega}(\xi/2))$. Since $\prod_{j=1}^{\infty}(1 + e^{-i2^{-j}\xi})/4 = 0$ for all ξ, it follows that $\omega = 0$. This shows that Eq. (31) determines Ω uniquely. Thus, we have proven our claim. ■

Finally, the limiting behavior of $G_0^{(n)}$ and $F_0^{(n)}$ is

$$G_0^{(\infty)} = \begin{pmatrix} 1 & M_2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad F_0^{(\infty)} = \begin{pmatrix} 0 & M_1 \\ 0 & 0 \end{pmatrix},$$

where

$$M_1 = \int_0^1 \frac{1}{\kappa(t)}\,dt \quad \text{and} \quad M_2 = \int_0^1 \frac{t - 1/2}{\kappa(t)}\,dt.$$

4.1.2. Biorthogonal MRA

We turn now to the (3, 1) biorthogonal basis and evaluate the reduction algorithm in this basis. For a biorthogonal basis the recursion relations are given by

$$\Gamma_j = \tilde{P}_j\Gamma_{j+1}P_j^* - (\tilde{P}_jKQ_j^*)(\tilde{Q}_j\Theta_{j+1}P_j^*)$$

and

$$\Theta_j = \tilde{P}_j\Theta_{j+1}P_j^*.$$

Using arguments similar to those for the Haar basis, we find the general form of $\Gamma_0^{(n)}$ and $\Theta_0^{(n)}$ to be

$$\Theta_0^{(n)} = \tilde{P}_0\Theta_n^{(n)}P_0^*$$

and

$$\Gamma_0^{(n)} = -\sum_{k=0}^{n-1}(\tilde{P}_0\cdots\tilde{P}_nKR_{\lambda_k}^*)(\tilde{Q}_{n-(k+1)}\Theta_n^{(n)}P_0^*).$$

The operator $R^*_{\lambda_k}$ is similar to the operator T_{λ_k} in that it is a composition of the operators P^*_j and Q^*_j (for j ranging from n to $n-k$):

$$R^*_{\lambda_k} = P^*_n \cdots P^*_{n-k} Q^*_{n-(k+1)}.$$

We can write $\Gamma^{(n)}_0$ in ''function'' form as

$$\Gamma^{(n)}_0 = \left\langle \sum_{k=0}^{n-1} \Xi_{n,k}, \frac{1}{\kappa}\phi_0 \right\rangle \quad \text{where } \Xi_{n,k}(t) = -\sum_{j=0}^{2^{n-(k+1)}-1} r_{k,j}\tilde{\psi}_{n-(k+1),j}(t).$$

The coefficients $r_{k,j}$ are the entries in the $1 \times 2^{n-(k+1)}$ matrix $\tilde{P}_0 \cdots \tilde{P}_n K R^*_{\lambda_k}$.

Once again we find that in the limit as $n \to \infty$

$$\Theta^{(\infty)}_0 = \left\langle \tilde{\phi}_0, \frac{1}{\kappa}\phi_0 \right\rangle = \int_0^1 \frac{dt}{\kappa(t)}.$$

We claim that the limiting behavior of $\Gamma^{(n)}_0$ for the (3, 1) basis is the same behavior as for the Haar basis. That is,

$$\lim_{n\to\infty} \Gamma^{(n)}_0 = \lim_{n\to\infty} \left(-\sum_{k=0}^{n-1} (\tilde{P}_0 \cdots \tilde{P}_n K R^*_{\lambda_k})(\tilde{Q}_{n-(k+1)} \Theta^{(n)}_n P^*_0) \right) = \int_0^1 \frac{t - 1/2}{\kappa(t)}\, dt.$$

Proof. Since we know that for the Haar basis $\Omega(t) = -1/4 \sum_{k=0}^{\infty} 2^{-k}\psi_{\lambda_k}(t) = t - 1/2$, it suffices to show that the difference between $\sum_{k=0}^{n-1} \Xi_{n,k}(t)$ and $\Omega(t)$ goes to zero as n tends to infinity.

We begin with the nth ($n \geqslant 2$) partial sum $\sum_{k=0}^{n-1} \Xi_{n,k}(t) = -\sum_{k=0}^{n-1} \left(\sum_{j=0}^{2^{n-(k+1)}-1} r_{k,j}\psi_{n-(k+1),j}(t) \right)$. One can show that the coefficients $r_{k,j}$ (for $k \geqslant 2$) are given by

$$r_{k,0} = r_{k,2^n-1} = 2^{-3(n/2+1)} \frac{77}{1920}$$

$$r_{k,1} = r_{k,2^n-2} = 2^{-3(n/2+1)} \frac{187}{5760}$$

$$r_{k,2} = \cdots = r_{k,2^n-3} = 2^{-3(n/2+1)} \frac{1}{32}.$$

For $k = 0$ and 1, we have $r_{0,0} = 1/3$ and $r_{1,0} = r_{1,1} = 119\sqrt{2}/1440$. The ''boundary'' coefficients are different from the interior coefficients (just as the boundary wavelets are different from the interior ones) because we are working on the interval [0, 1]. See [10] for the construction of these boundary wavelets.

One can also show that the difference $\sum_{k=0}^{n-1} \Xi_{n,k}(t) - \Omega(t)$ is zero for the ''interior'' of the interval and is non-zero at the ''boundary.'' More specifically, one can show that the difference is a piecewise constant function which takes on the values

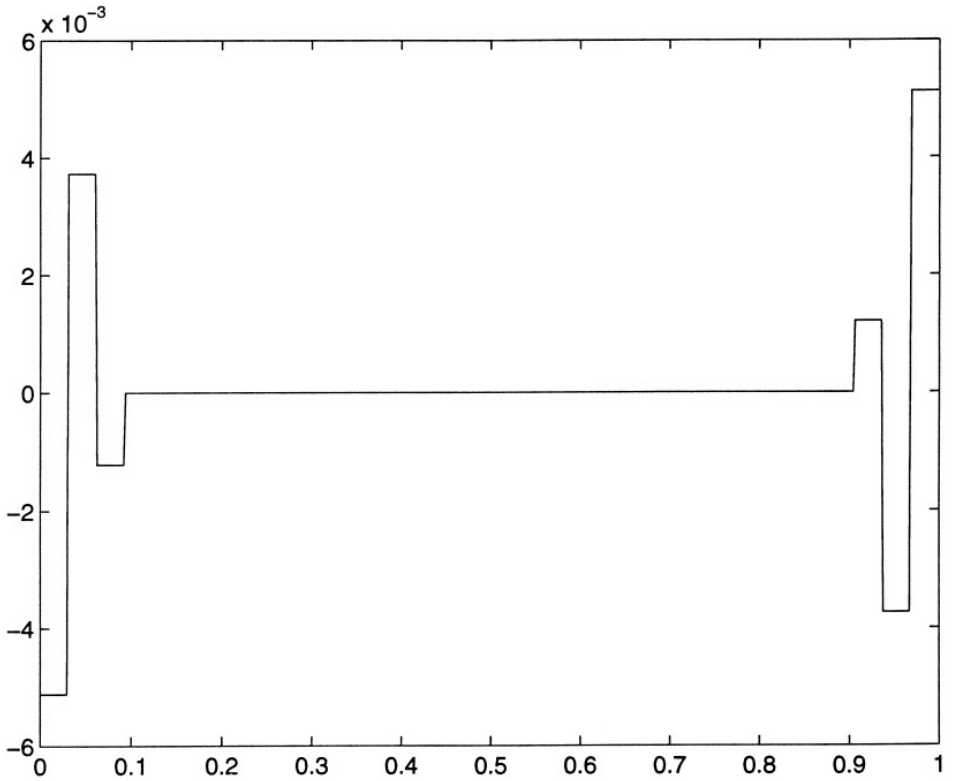

FIG. 1. The difference between the fourth partial sum of (3, 1) biorthogonal wavelets and the weight function in the Haar basis, $\Sigma_{k=0}^{4-1}\,\Xi_{4,k}(t) - \Omega(t)$.

$$\sum_{k=0}^{n-1} \Xi_{n,k}(t) - \Omega(t) = 2^{-n+2} \begin{cases} \dfrac{-59}{2880} & t \in \left[0, \dfrac{1}{2^{n+1}}\right), \\ \dfrac{59}{2880} & t \in \left(1 - \dfrac{1}{2^{n+1}}, 1\right], \\ \dfrac{43}{2880} & t \in \left[\dfrac{1}{2^{n+1}}, \dfrac{2}{2^{n+1}}\right), \\ \dfrac{-43}{2880} & t \in \left(1 - \dfrac{2}{2^{n+1}}, 1 - \dfrac{1}{2^{n+1}}\right], \\ \dfrac{-7}{1440} & t \in \left[\dfrac{2}{2^{n+1}}, \dfrac{3}{2^{n+1}}\right), \\ \dfrac{7}{1440} & t \in \left(1 - \dfrac{3}{2^{n+1}}, 1 - \dfrac{2}{2^{n+1}}\right], \\ 0 & \text{otherwise.} \end{cases}$$

See Fig. 1. That is, the difference between the nth partial sum $\sum_{k=0}^{n-1} \Xi_{n,k}(t)$ and $\Omega(t)$

is non-zero on an interval of length $3/2^n$ and has largest magnitude $2^{-n+2}(59)/2880$. We can then conclude that $\sum_{k=0}^{n-1} \Xi_{n,k}(t)$ converges pointwise to $\Omega(t) = t - 1/2$, proving our claim. ■

4.2. Homogenization via Augmentation

We now apply the augmentation procedure of Section 3.3 to our effective equation. The corresponding homogenized integral equation

$$G^h x(t) - \beta = F^h \int_0^t x(s)ds$$

has homogenized coefficients

$$G^h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad F^h = \begin{pmatrix} 0 & M_1 - 2M_2 \\ 0 & 0 \end{pmatrix}.$$

This integral equation corresponds to the differential equations

$$\begin{cases} \dfrac{du}{dt} = (M_1 - 2M_2)v \\ \dfrac{dv}{dt} = 0. \end{cases} \tag{32}$$

Notice that these are different from the homogenized equations for the classical theory:

$$\begin{cases} \dfrac{du}{dt} = M_1 v \\ \dfrac{dv}{dt} = 0. \end{cases}$$

However, the first system of differential equations (32) is consistent with the goal of the wavelet-based homogenization. The averages of the solutions to the original equations (the non-constant coefficient case) are now

$$\langle v \rangle = \beta_2 \quad \text{and} \quad \langle u \rangle = \beta_1 + \beta_2 \int_0^1 \left(\int_0^t \frac{1}{\kappa(s)}\, ds \right) dt;$$

note that the initial condition is $\beta = (\beta_1, \beta_2) \in \mathbf{R}^2$. To compare this with the averages of u and v as determined by G^h and F^h, given by

$$\langle v\rangle = \beta_2 \quad \text{and} \quad \langle u\rangle = \beta_1 + \beta_2(\tfrac{1}{2}M_1 - M_2),$$

notice that

$$\frac{1}{2}M_1 - M_2 = \frac{1}{2}\int_0^1 \frac{dt}{\kappa(t)} - \int_0^1 \frac{t - 1/2}{\kappa(t)}\,dt$$
$$= \int_0^1 \frac{1 - t}{\kappa(t)}\,dt;$$

if we now integrate this last integral by parts, we see that it is exactly $\int_0^1 (\int_0^t (ds/\kappa(s))dt$.

4.3. Reduction Procedure with Forcing Terms

Let us now apply the MRA scheme to the general problem given by

$$\frac{d}{dx}\left(\kappa\,\frac{du}{dx}\right) = f, \tag{33}$$

where f is no longer taken to be identically equal to zero. Let f be a continuous function on $[0, 1]$. We now have to include forcing terms p and q in our reduction procedure. With the same notation as in Eq. (13), Eq. (33) corresponds to the initial choices

$$q(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad p(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

The operators $G_0^{(n)}$ and $F_0^{(n)}$ and their limits $G_0^{(\infty)}$ and $F_0^{(\infty)}$ remain unchanged.

Using similar techniques as those above, we calculate the general form of the vectors $p_0^{(n)}$ and $q_0^{(n)}$ and determine that the limiting behavior of these quantities is

$$\lim_{n\to\infty} p_0^{(n)} = \begin{pmatrix} p_1 \\ m_1 \end{pmatrix} \quad \text{and} \quad \lim_{n\to\infty} q_0^{(n)} = \begin{pmatrix} q_1 \\ m_2 \end{pmatrix},$$

where

$$p_1 = \int_0^1 \frac{1}{\kappa(t)} \int_0^t s f(s)\,ds\,dt + \int_0^1 (s - 1) f(s) \int_0^s \frac{1}{\kappa(t)}\,dt\,ds$$
$$q_1 = \int_0^1 (t - 1/2)\,\frac{1}{\kappa(t)} \int_0^t s f(s)\,ds\,dt + \int_0^1 (1 - s) f(s) \int_0^s (1/2 - t)\,\frac{1}{\kappa(t)}\,dt\,ds$$
$$m_1 = \int_0^1 f(t)\,dt \quad \text{and} \quad m_2 = \int_0^1 (t - 1/2) f(t)\,dt.$$

The reduced equations for the averages $\langle u \rangle$ and $\langle v \rangle$ are

$$\langle u \rangle = \beta_1 + \beta_2 \left(\frac{1}{2} M_1 - M_2 \right) + \left(\frac{1}{2} m_1 - m_2 \right) \left(\frac{1}{2} M_1 - M_2 \right) + \frac{1}{2} p_1 - q_1 \quad (34)$$

$$\langle v \rangle = \beta_2 + \frac{1}{2} m_1 - m_2. \quad (35)$$

If we simplify the expressions (34) and (35), we have

$$\begin{aligned}
\langle u \rangle &= \beta_1 + \beta_2 \int_0^1 \int_0^t \frac{ds}{\kappa(s)}\, dt + \int_0^1 \int_0^1 (1-t) f(t)(1-s) \frac{1}{\kappa(s)}\, ds dt \\
&\quad + \int_0^1 (1-t) \frac{1}{\kappa(t)} \left(\int_0^t s f(s) ds - \int_t^1 (1-s) f(s) ds \right) dt \\
&= \beta_1 + \beta_2 \int_0^1 \int_0^t \frac{ds}{\kappa(s)}\, dt + \int_0^1 \int_0^1 (1-t) f(t)(1-s) \frac{1}{\kappa(s)}\, ds dt \\
&\quad + \int_0^1 (1-t) \frac{1}{\kappa(t)} \left(\int_0^1 (s-1) f(s) ds + \int_0^t f(s) ds \right) dt \\
&= \beta_1 + \beta_2 \int_0^1 \int_0^t \frac{ds}{\kappa(s)}\, dt + \int_0^1 (1-t) \frac{1}{\kappa(t)} \int_0^t f(s) ds dt \\
&= \beta_1 + \beta_2 \int_0^1 \int_0^t \frac{ds}{\kappa(s)}\, dt + \int_0^1 \int_0^x \int_0^t \frac{f(s)}{\kappa(t)}\, ds dt dx
\end{aligned}$$

and

$$\langle v \rangle = \beta_2 + \int_0^1 \int_0^t f(s) ds dt.$$

In other words, the equations (34) and (35) are indeed the averages of the solutions to (33) given by

$$u(x) = \beta_1 + \int_0^x \frac{v(t)}{\kappa(t)}\, dt \quad \text{and} \quad v(x) = \beta_2 + \int_0^x f(t) dt.$$

The corresponding homogenized integral equation

$$G^h x(t) - \beta = F^h \int_0^t x(s) + p^h ds \quad (36)$$

has homogenized coefficients G^h and F^h as above and the coefficient p^h is

$$p^h = \begin{pmatrix} \dfrac{\frac{1}{2} p_1 - q_1}{\frac{1}{2} M_1 - M_2} + \dfrac{1}{3} \left(\dfrac{1}{2} m_1 - m_2 \right) \\ \\ m_1 - 2m_2 \end{pmatrix}.$$

One can verify that the solutions of Eq. (36) given by

$$u(t) = \beta_1 + (M_1 - 2M_2)\int_0^t v(s) + \frac{\frac{1}{2}p_1 - q_1}{\frac{1}{2}M_1 - M_2} + \frac{1}{3}\left(\frac{1}{2}m_1 - m_2\right)ds$$

$$v(t) = \beta_2 + \int_0^t (m_1 - 2m_2)ds$$

have the same averages as the solutions of Eq. (33).

We conclude that the homogenized coefficients G^h and F^h do not depend on the forcing term f, only the homogenized coefficient p^h depends on our choice of f. Furthermore, the MRA scheme produces a homogenized equation for the general problem (33) which preserves the averages of the solution.

5. ANALYSIS OF HOMOGENIZATION RULE

Recall the homogenization rule from the Section 2.3. The function $v = \lambda + du/dx$ satisfies

$$\frac{d}{dx}\left(\kappa\left(\lambda + \frac{du}{dx}\right)\right) = 0 \quad \text{where } u(0) = u(1) = 0. \tag{37}$$

We wish to determine what quantities (e.g., $\langle v\rangle$, $\langle u\rangle$, etc.) are preserved by this homogenization rule. In other words, if we replace the original equation $(d/dx)(\kappa v) = 0$ with $(d/dx)(\kappa_0 v_0) = 0$, what properties will be shared by v and v_0, or by u and u_0?

Let us first solve (37) and show that

$$\langle u\rangle = \frac{-\lambda M_2}{M_1} \quad \text{and} \quad \langle v\rangle = \lambda.$$

If we rewrite Eq. (37) as a system of ODEs, we have

$$\frac{du}{dx} = \frac{\tilde{v}}{\kappa} - \lambda \quad \text{and} \quad \frac{d\tilde{v}}{dx} = 0 \text{ with } u(0) = u(1) = 0.$$

The solutions u and v are given by

$$u(x) = -\lambda x + \frac{\lambda}{M_1}\int_0^x \frac{ds}{\kappa(s)} \quad \text{and} \quad v(x) = \frac{\lambda}{M_1\kappa(x)}.$$

Clearly, the averages are

$$\langle u\rangle = \frac{-\lambda}{2} + \frac{\lambda}{M_1}\left(\frac{M_1}{2} - M_1\right) = \frac{-\lambda M_2}{M_1}$$

$$\langle v\rangle = \lambda.$$

As before,

$$M_1 = \int_0^1 \frac{ds}{\kappa(s)} \quad \text{and} \quad M_2 = \int_0^1 \frac{s - 1/2}{\kappa(s)}\, ds.$$

We will now solve the homogenized equation (as determined by the homogenization rule) given by

$$\frac{d}{dx}\left(\frac{1}{M_1} v_0\right) = \frac{d}{dx}\left(\frac{1}{M_1}\left(\lambda + \frac{du_0}{dx}\right)\right) = 0 \quad \text{with } u_0(0) = u_0(1) = 0.$$

We can easily determine that $u_0(x) = 0$ and that $v_0(x) = \lambda$. Therefore, $\langle u_0 \rangle = 0$ and $\langle v_0 \rangle = \lambda$. It is clear that $\langle v \rangle = \langle v_0 \rangle$ but that $\langle u \rangle \neq \langle u_0 \rangle$; in other words, with this homogenization rule, the average of v is preserved while the average of u is not. We contrast this with the discussion in Section 4, where we show that the MRA homogenization process preserves the average value of the solution. If we were to use the MRA procedure to homogenize Eq. (37), we would preserve the averages $\langle u \rangle$ and $\langle v \rangle$.

6. PHYSICAL EXAMPLES

In this section we will present three examples which illustrate the differences between the classical and the MRA homogenization methods. We will show that the MRA method is more physically robust, meaning with this method we can handle many more physical situations.

The physical problem which we will look at is the steady-state heat distribution in a rod of length one. We will assume that the temperature T satisfies $T(0) = 0$ and $T(1) = \lambda$. We also assume that the average temperature gradient $\langle dT/dx \rangle = \int_0^1 (dT/dx)(s)ds = \lambda$. Our heat equation is then

$$\frac{d}{dx}\left(\kappa \frac{dT}{dx}\right) = 0$$

with the conditions $T(0) = 0$, $T(1) = \lambda$, and $\langle dT/dx \rangle = \lambda$. Also, the thermal conductivity κ is a continuous function, bounded away from zero, and periodic on $[0, 1]$. We will homogenize this problem for several different functions κ.

First, we will look at a family of thermal conductivities

$$\kappa_n(x) = \kappa(2^n x).$$

Each function κ_n models a material composed of period cells (of length 2^{-n}) and we want to know the effective thermal conductivity of the material as $n \to \infty$ (or as the length of each period cell shrinks to zero). This is the physical motivation for the classical theory. Using the MRA strategy, we will homogenize the problem

$$\frac{d}{dx}\left(\kappa_n \frac{dT_n}{dx}\right) = 0 \tag{38}$$

for each n and then take the limit as $n \to \infty$ of the homogenized coefficients κ_n^h.

Again, rewriting (38) as a system of ODEs gives us

$$\frac{dT_n}{dx} = \frac{v_n}{\kappa_n} \text{ and } \frac{dv_n}{dx} = 0 \text{ with } T_n(0) = 0 \text{ and } v_n(x) = \langle v_n \rangle = \frac{\lambda}{M_{1,n}},$$

where $M_{1,n} = \int_0^1 ds/\kappa_n(s)$. We know from the results of Section 4 that

$$\langle T_n \rangle = T_n(0) + v_n(0)\left(\frac{M_{1,n}}{2} - M_{2,n}\right) = \frac{\lambda}{2} - \frac{\lambda M_{2,n}}{M_{1,n}}$$

$$\langle v_n \rangle = \langle v_n(0) \rangle = \frac{\lambda}{M_{1,n}}.$$

Here

$$M_{2,n} = \int_0^1 \frac{s - 1/2}{\kappa_n(s)}\, ds.$$

Furthermore, the homogenized equations are

$$T_n^h(x) = \int_0^x v_n^h(s)(M_{1,n} - 2M_{2,n})ds$$

$$v_n^h(x) - \frac{\lambda}{M_{1,n}} = 0,$$

or, in differential form,

$$\frac{d}{dx}\left(\kappa_n^h \frac{dT_n^h}{dx}\right) = 0 \quad \text{with } T_n^h(0) = 0 \quad \text{and} \quad \frac{dT_n^h}{dx}(0) = \frac{\lambda}{M_{1,n}}.$$

The effective coefficient is given by

$$\kappa_n^h = \frac{1}{M_{1,n} - 2M_{2,n}}.$$

In the limit as n goes to infinity, we have

$$\lim_{n\to\infty} M_{1,n} = \lim_{n\to\infty} \int_0^1 \frac{ds}{\kappa(2^n s)} = \left\langle \frac{1}{\kappa} \right\rangle$$

and

$$\lim_{n\to\infty} M_{2,n} = \lim_{n\to\infty} \int_0^1 \frac{s - 1/2}{\kappa(2^n s)}\, ds = \left\langle \frac{1}{\kappa} \right\rangle \int_0^1 s - 1/2 ds = 0$$

by Theorem 1. In general, we can conclude that

$$\lim_{n\to\infty} \kappa_n^h = \lim_{n\to\infty} \frac{1}{M_{1,n} - 2M_{2,n}} = \frac{1}{\left\langle \frac{1}{\kappa} \right\rangle},$$

or that our homogenized coefficient is simply the harmonic average of κ (the same as the classical theory!).

We will now examine a specific family of thermal conductivities. Let $1/\kappa_n(x) = 2 - \sin(2\pi 2^n x)$. The moments $M_{1,n}$ and $M_{2,n}$ are

$$M_{1,n} = \int_0^1 \frac{dx}{\kappa_n(x)} = \int_0^1 2 - \sin(2\pi 2^n x)dx = 2$$

$$M_{2,n} = \int_0^1 \frac{x - 1/2}{\kappa_n(x)}\, dx = \int_0^1 (x - 1/2)(2 - \sin(2\pi 2^n x))dx = \frac{1}{2\pi 2^n}.$$

So,

$$\langle T_n^h \rangle = \frac{\lambda}{2}\left(1 - \frac{1}{2\pi 2^n}\right) \quad \text{and} \quad T_n^h(x) = \lambda\left(1 - \frac{1}{2\pi 2^n}\right)x.$$

Furthermore, our homogenized coefficient is

$$\kappa_n^h = \frac{1}{M_{1,n} - 2M_{2,n}} = \frac{1}{2\left(1 - \frac{1}{2\pi 2^n}\right)}.$$

The classical theory tells us that our homogenized problem is

$$\frac{d}{dx}\left(\frac{1}{M_1}\frac{dT_0}{dx}\right) = \frac{d}{dx}\left(\frac{1}{2}\frac{dT_0}{dx}\right) = 0$$

with $T_0(0) = 0$ and $T_0(1) = \lambda$. We get $T_0(x) = \lambda x$ and $\langle T_0 \rangle = \lambda/2$. Observe that in the limit as $n \to \infty$ the two methods agree; i.e.,

$$\lim_{n\to\infty} \kappa_n^h = \lim_{n\to\infty} \frac{1}{2\left(1 - \frac{1}{2\pi 2^n}\right)} = \frac{1}{2} = \frac{1}{M_1}$$

$$\lim_{n\to\infty} T_n^h(x) = \lim_{n\to\infty} \lambda\left(1 - \frac{1}{2\pi 2^n}\right)x = \lambda x.$$

This example prompts a question. Does the MRA strategy provide a higher-order correction term in the asymptotic expansion derived in the classical theory? The answer to the question is no, the MRA scheme is not simply a higher-order term in

the asymptotic expansion of the classical theory. Recall that the classical theory tells us

$$T(x) \approx T_\epsilon(x) = T_0(x) + \epsilon N(y)\,\frac{dT_0}{dx}\,,$$

where we take $\epsilon = 2^{-n}$ and N solves $(d/dx)(\kappa(y)(1 + dN/dy)) = 0$ with $y = 2^n x$ and $N(0) = N(1) = 0$. Here, $T_0(x) = \lambda x$ and $N(y) = -y + 1/M_1 \int_0^y ds/\kappa(s)$. Therefore,

$$\begin{aligned} T(x) \approx T_{2^{-n}}(x) &= T_0(x) + 2^{-n}N(y)\,\frac{dT_0}{dx} = \lambda x + \lambda 2^{-n}\left(-y + \frac{1}{M_1}\int_0^y \frac{ds}{\kappa(s)}\right) \\ &= \frac{\lambda}{2^n M_1}\int_0^{2^n x} 2 - \sin(2\pi s)ds \\ &= \lambda x + \frac{\lambda}{2}\left(\frac{\cos(2\pi 2^n x)}{2\pi 2^n} - \frac{1}{2\pi 2^n}\right). \end{aligned}$$

So, the correction term is

$$\frac{\lambda}{2}\left(\frac{\cos(2\pi 2^n x)}{2\pi 2^n} - \frac{1}{2\pi 2^n}\right).$$

The MRA algorithm gives

$$T_n^h(x) = \lambda x - \frac{\lambda x}{2\pi 2^n}\,.$$

It is clear that the MRA scheme does not give us simply a more accurate approximation to the true solution T. The solution T_n^h is a linear function which has the same average as the true solution T_n but which tends pointwise to T_0 as n goes to infinity.

If we graph the difference of T_0 and the two functions $T_{2^{-n}}$ and T_n^h (see Fig. 2), we see that the approximate solution $T_{2^{-n}}$ oscillates just below the line $T_0(x) = \lambda x$ and as n tends to infinity these oscillations increase in frequency and decrease in amplitude. The function T_n^h is a straight line from the origin to the point $(1, \lambda(1 - 1/2\pi 2^n))$ with its average value exactly equal to the average of T_n. Also, in the limit T_n^h is the line λx.

As we discussed previously, the MRA scheme is more physically robust than the classical theory. The next example will illustrate a situation where the classical theory would fail and yet, physically, this is an important case we would like to homogenize.

This example is a problem with a continuum of scales-the kind of problem classical homogenization theory cannot solve. Let

$$\frac{1}{\kappa(x)} = 2 - \sin\left(2\pi \tan\left(\frac{\pi}{2}\,x\right)\right)$$

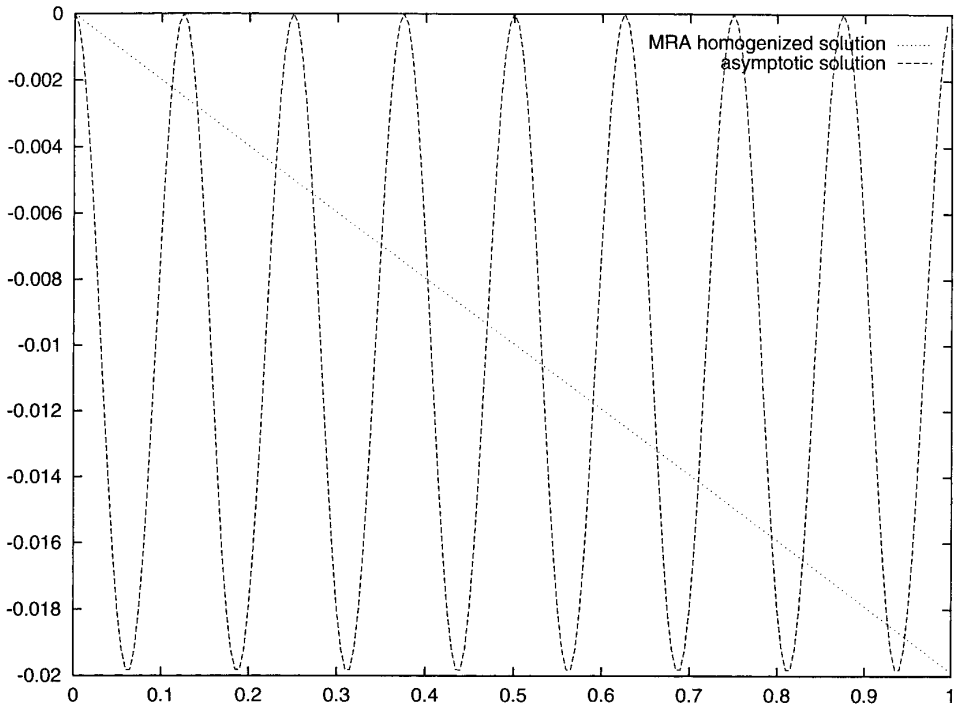

FIG. 2. A comparison of the difference between the MRA homogenized solution and the true solution $T_n^h(x) - \lambda x$ (in the first physical example) on one hand, and of the difference between the asymptotic solution and the true solution $T_{2^{-n}}(x) - \lambda x$. Here $n = 3$ and $\lambda = 1$. Both of the functions T_n^h and $T_{2^{-n}}$ correspond to the temperature in a rod with period cells of length 2^{-n}.

(see Fig. 3). This conductivity corresponds to a material composed of period cells but which has been stressed or distorted at one end. We emphasize that there is no small parameter ϵ (or family of thermal conductivities $\kappa_n(x) = \kappa(2^n x)$) unlike the previous examples. We can calculate

$$M_1 = \int_0^1 2 - \sin\left(2\pi \tan\left(\frac{\pi}{2} x\right)\right) dx \approx 1.89173$$

and

$$M_2 = \int_0^1 (x - 1/2)\left(2 - \sin\left(2\pi \tan\left(\frac{\pi}{2} x\right)\right)\right) dx \approx 0.05225.$$

These quantities allow us to determine the average temperature distribution $\langle T \rangle$ and to write an homogenized equation for this example even though there is no small parameter in which we could do an asymptotic expansion as in the classical theory.

7. CONCLUSIONS

The MRA strategy for numerical homogenization consists of two algorithms: a procedure for extracting the effective equation for the average or for the coarse-

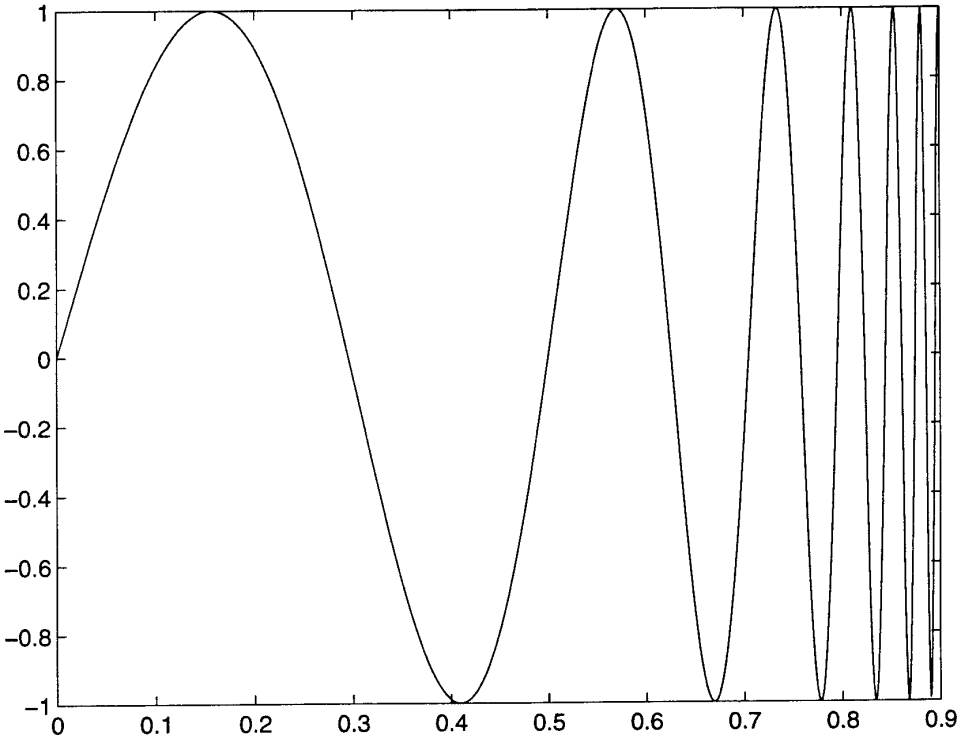

FIG. 3. A plot of the thermal conductivity $1/\kappa(x) = 2 - \sin(2\pi \tan(\pi/2\, x))$. This function ''contains'' a continuum of scales.

scale behavior of the solution (the reduction process) and a method for augmenting this effective equation (the augmentation process). In other words, once one has determined what the average behavior of the solution is, one can construct a simpler equation whose solution has the same average behavior. For physical problems in which one wants to determine only the average behavior of the solution, the reduction process is very useful and is not part of the classical theory of homogenization. In some applications, this step suffices. On the other hand, the augmentation procedure yields effective material parameters (or homogenized coefficients) just as the classical theory does; however, the MRA procedure produces a homogenized equation which preserves important physical characteristics of the original solution, such as its average value.

The MRA method is more physically robust in that it can be applied to many more situations than the classical theory can. For example, the MRA strategy can be applied to problems which have a continuum of scales while the classical theory may be applied to problems with only a finite number of distinguished scales. Moreover, for those two-scale problems for which the classical theory was developed the MRA results agree with the results of classical homogenization in one dimension.

This paper is the first of a series of papers. Currently, we are extending the MRA for homogenization to nonlinear integral equations and equations in more than one variable. In particular, we are examining diffusion problems and Maxwell equations. We are also examining the links between the theory of compensated compactness, the MRA methods, and the tools of classical harmonic analysis.

APPENDIX

A multiresolution analysis of $L^2([0, 1])$ is a decomposition of the space into a chain of closed subspaces

$$V_0 \subset V_1 \subset \cdots \subset V_n \cdots$$

such that

$$\overline{\bigcup_{j \geq 0} V_j} = L^2([0, 1]) \quad \text{and} \quad \bigcap_{j \geq 0} V_j = V_0.$$

If we let P_j denote the orthogonal projection operator onto V_j, then $\lim_{j \to \infty} P_j f = f$ for all $f \in L^2([0, 1])$. We have the additional requirements that each subspace V_j $(j > 0)$ is a rescaled version of the base space V_0 and that the base space V_0 is invariant under integer translation:

$$f \in V_j \Leftrightarrow f(2^{-j} \cdot) \in V_0 \text{ and } f \in V_0 \Rightarrow f(\cdot - n) \in V_0 \text{ for all } n \in \mathbf{Z}.$$

Finally, we require that there exists $\phi \in V_0$ (called the scaling function) so that ϕ and all of its integer translates form an orthonormal basis of V_0. We can conclude that the set $\{\phi_{j,k} | k = 0, \ldots, 2^j - 1\}$ is an orthonormal basis for each subspace V_j. Here $\phi_{j,k}$ denotes a translation and dilation of ϕ:

$$\phi_{j,k} = 2^{j/2} \phi(2^j x - k).$$

As a consequence of the above properties, there is an orthonormal wavelet basis $\{\psi_{j,k} | j \geq 0, k = 0, \ldots, 2^j - 1\}$ of $L^2([0, 1])$, $\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k)$, such that for all f in $L^2([0, 1])$

$$P_{j+1} f = P_j f + \sum_{k=0}^{2^j - 1} \langle f, \psi_{j,k} \rangle \psi_{j,k}.$$

If we define W_j to be the orthogonal complement of V_j in V_{j+1}, then

$$V_{j+1} = V_j \oplus W_j.$$

We have, for each fixed j, an orthonormal basis $\{\psi_{j,k} | k = 0, \ldots, 2^j - 1\}$ for W_j. Finally, we may decompose $L^2([0, 1])$ into a direct sum

$$L^2([0, 1]) = V_0 \bigoplus_{j \geq 0} W_j.$$

The operator Q_j is the orthogonal projection operator onto the space W_j.

The Haar wavelet ψ and its associated scaling function ϕ are defined as

$$\phi(x) = \begin{cases} 1, & x \in [0, 1) \\ 0, & \text{elsewhere} \end{cases} \quad \text{and} \quad \psi(x) = \begin{cases} 1, & x \in [0, 1/2) \\ -1, & x \in [1/2, 1) \\ 0, & \text{elsewhere.} \end{cases}$$

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