MATH 597 Homework 9. Due Monday April 14

Problem 1.

(a) Let $\mathcal{H}$ be a vector space with an inner product $(\cdot,\cdot)$. Prove the induced norm $\|\cdot\|$ satisfies the parallelogram law

$$\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2$$

for all $f, g \in \mathcal{H}$

(b) Show that the L^p-norm $\|\cdot\|_p$ on $L^p(\mathbb{R})$ does not satisfy the parallelogram law for $1 \le p \le \infty$, $p \neq 2$. (Hint: We can find a counterexample using linear functions f, g on $0 \le x \le 1$.)

Problem 2. Consider the sequence of monomials $\{1, x, x^2, x^3, \cdots\}$ in $L^2((-1,1))$. Let $\{p_0(x), p_1(x), p_2(x), \cdots\}$ be the sequence of polynomials ($p_n(x)$ is of degree n) obtained by applying the Gram-Schimdt process to the monomials:

$$\int_{-1}^{1} p_n(x)\overline{p_k(x)}dx = \delta_{nk}.$$

Note that $p_n(x)$ is uniquely determined by the conditions that

$$\int_{-1}^{1} p_n(x)x^k dx = 0, \qquad 0 \le k \le n-1,$$

$\int_{-1}^{1} |p_n(x)|^2 dx = 1$ and the condition that the coefficient of x^n is positive.

(a) Show that $p_0(x) = \frac{1}{\sqrt{2}}$, $p_1(x) = \frac{\sqrt{3}}{\sqrt{2}}x$ and $p_2(x) = \frac{\sqrt{5}}{\sqrt{2}}(\frac{3}{2}x^2 - \frac{1}{2})$.

(b) Show that $p_n(x) = \frac{\sqrt{2n+1}}{\sqrt{2}}L_n(x)$ where

$$L_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\{(x^2-1)^n\}.$$

Problem 3. Prove the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

by using the Plancherel identity applied to the function f defined by $f(x) = x$ for $0 \leq x < 1$ and $f(x+1) = f(x)$ for all $x \in \mathbb{R}$.

Problem 4. Let $H^1(S^1)$ be the space of functions f on S^1 such that f is AC and $f' \in L^2(S^1)$. (Remark 1. Hence f is periodic, $f(x+1) = f(x)$.) (Remark 2. Since f is AC, we have $f' \in L^1$. The condition that $f' \in L^2$ is stronger since any $L^2(S^1)$ function is in $L^1(S^1)$.) We will show that

$$H^1(S^1) = \{f \in L^2(S^1) : \sum_n n^2 |\hat{f}(n)|^2 < \infty\}.$$

This is a different characterization of the 'Sobolev space' $H^1(S^1)$.

(a) Show that if $g \in L^2(S^1)$, then $g \in L^1(S^1)$. (See Remark 2 above.)

(b) Suppose that $f \in H^1(S^1)$. By computing the Fourier coefficients of f' and using the Plancherel's identity, show that $\sum_n n^2 |\hat{f}(n)|^2 < \infty$.

(c) Let f be a function such that $f \in L^2(S^1)$ and $\sum_n n^2 |\hat{f}(n)|^2 < \infty$. We will show that $f \in H^1(S^1)$.

   (i) Using Schwarz inequality, show that the Fourier series $\sum_n \hat{f}(n) e_n(x)$of f actually converges for all points x. (Here $e_n(x) = e^{2\pi i n x}$ as usual.)

   (ii) Let $g \in L^2(S^1)$ be the function given by the Fourier series $\sum_n 2\pi i n \hat{f}(n) e_n(x)$. Note that the sequence $\{2\pi i n \hat{f}(n)\}_n$ is in $\ell^2(\mathbb{Z})$ and hence the Fourier series defines an L^2 function. It remains to show that f is AC and $f' = g$ almost everywhere. This can be shown by proving $f(x) - f(0) = \int_0^x g(t)dt$ for all x. (Recall the fundamental theorem of Calculus in Chapter 3. Note that g is L^1 by (a).) Prove this by using the Parseval's identity (i.e. $(h_1, h_2) = (\hat{h}_1, \hat{h}_2)$) applied to $g(t)$ and $\chi_{(0,x)}(t)$.