

MATH 526 Homework 10. Do NOT hand in

Problem 1. Let X_t be a Poisson process with rate λ . Compute

$$\mathbb{E}[X_3|X_1], \quad \text{and} \quad \mathbb{E}[X_1|X_3].$$

Problem 2. Let $N \geq 3$. Consider a symmetric random walk X_n on $\{0, 1, \dots, N\}$ with absorbing boundaries at 0 and N . Set $X_0 = i$. Fix θ such that $0 < \theta < \frac{\pi}{N}$. Defined

$$Y_n \equiv \frac{\cos(\theta(X_n - \frac{1}{2}N))}{\cos^n \theta}.$$

Show that $\{Y_n\}$ is a martingale wrt $\{X_n\}$.

Problem 3. Consider a random walk X_n on $\{0, 1, \dots, N\}$ with absorbing boundaries at 0 and N . Assume that $\mathbb{P}\{X_{n+1} = i + 1 | X_n = i\} = p$ and $\mathbb{P}\{X_{n+1} = i - 1 | X_n = i\} = 1 - p$ where $p \neq \frac{1}{2}$. Let

$$X_0 = i.$$

In the class, by considering the martingale $Y_n = \left(\frac{1-p}{p}\right)^{X_n}$, we obtained that

$$\mathbb{P}\{\text{chain reaches } N \text{ before } 0, \text{ starting at } i\} = \frac{1 - \left(\frac{1-p}{p}\right)^i}{1 - \left(\frac{1-p}{p}\right)^N}.$$

Now set $Z_n = X_n - n(2p - 1)$. (Note that $2p - 1$ is the expected value of a jump in one step). Let $T = \min\{k : X_k = 0 \text{ or } X_k = N\}$. Show that $\{Z_n\}$ is a martingale wrt $\{X_n\}$, and show that

$$\mathbb{E}[T] = \frac{N}{2p - 1} \left(\frac{1 - \left(\frac{1-p}{p}\right)^i}{1 - \left(\frac{1-p}{p}\right)^N} - \frac{i}{N} \right).$$

(For the symmetric case $p = \frac{1}{2}$, we obtained $\mathbb{E}[T] = i(N - i)$ in the class.)

Problem 4. Let $\frac{1}{2} < p < 1$. Let X_1, X_2, X_3, \dots be iid random variables with $\mathbb{P}\{X_i = 1\} = p$ and $\mathbb{P}\{X_i = -1\} = 1 - p$. Define

$$m(t) = \mathbb{E}[e^{tX_1}] = pe^t + (1-p)e^{-t}, \quad t \geq 0.$$

(This is the moment generating function of X_1 .) Set $X_0 = 0$, $S_0 = 0$ and $S_n = X_1 + X_2 + \dots + X_n$.

1. Set

$$Y_n = m(t)^{-n} e^{tS_n}.$$

Show that $\{Y_n\}$ is a martingale with respect to $\{X_n\}$.

2. Show that for $t \geq 0$,

$$m(t) \geq 1.$$

3. Fix $b > 0$. For $a > 0$, set

$$T_a \equiv \min\{n : S_n \leq -a \text{ or } S_n \geq b\}.$$

Show that

$$\mathbb{E}[m(t)^{-T_a} e^{tS_{T_a}}] = 1.$$

(This is the special case of Wald's identity for the Bernoulli rv.) Explicitly this implies that

$$e^{tb} \mathbb{E}[m(t)^{-T_a} | S_{T_a} = b] \mathbb{P}\{S_{T_a} = b\} + e^{-ta} \mathbb{E}[m(t)^{-T_a} | S_{T_a} = -a] \mathbb{P}\{S_{T_a} = -a\} = 1.$$

(Hint: Use 2. above to estimate $m(t)^{-T \wedge n}$.)

4. Set

$$T \equiv \min\{n : S_n = b\}.$$

Show, by taking $a \rightarrow +\infty$ in 3. above, that

$$\mathbb{E}[s^T] = \left(\frac{1 - \sqrt{1 - 4p(1-p)s^2}}{2(1-p)s} \right)^b, \quad 0 < s \leq 1.$$

5. Show that (with T as above)

$$\mathbb{E}[T] = \frac{b}{2p-1}, \quad \text{Var}(T) = \frac{4bp(1-p)}{(2p-1)^3}.$$

Problem 5. Consider the Polya's urn. Here X_n denote the number of red balls in the urn after the n th turn. We saw in the class that setting

$$Y_n \equiv \frac{X_n}{n+2},$$

$\{Y_n\}$ is a martingale wrt $\{X_n\}$. (Note that Y_n equals to the proportion of the number of red balls in the urn.) Show that

$$\mathbb{P}\{Y_n \geq \frac{3}{4} \text{ for some } n\} \leq \frac{2}{3}.$$