MATH 526 Homework 2. Due Tuesday 2/3

Problem 1.

(a) Consider two urns A and B. Initially the urn A contains N black balls and the urn B contains N white balls. At each step, one ball is selected at random from each urn and the two balls interchange. Let X_n denote the number of white balls in the urn A. Determine the transition matrix $\mathbf{P}$.

(b) Consider two urns A and B containing a total N balls together. At each time, a ball is selected at random (all selections are equally likely) from among the totality of N balls. Then an urn is selected at random: A is selected with probability p and B is selected with probability $1-p$. And the ball previous drawn is placed in this urn. Let X_n denote the number of balls in A at time n. Determine the transition matrix $\mathbf{P}$.

Problem 2. Consider the Markov chain with state space $\{1, 2, 3\}$ and the transition matrix

$$\mathbf{P} = \begin{pmatrix} 4/10 & 2/10 & 4/10 \\ 6/10 & 0 & 4/10 \\ 2/10 & 5/10 & 3/10 \end{pmatrix}.$$

Compute

$$\lim_{n\to\infty} \mathbb{P}\{X_n = 1\}.$$

(Note that this limit does not depend on the initial distribution. Why?)

Problem 3. Compute the invariant distribution $\bar{\pi}$ for the transition matrix $\mathbf{P}$ for example 1.1.3 (random walks with reflecting boundaries) when $p \neq \frac{1}{2}$.

(Hint: When you determine the constant at the end from the condition that the sum of components of $\bar{\pi}$ is 1, you may recall the identity $1 + x + x^2 + \cdots + x^m = \frac{1-x^{m+1}}{1-x}$ for $x \neq 1$.)

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Do not hand in the solutions to the following problems.

Problem 4. The newspaper is delivered every morning to a house. Amy reads the newspaper at 8am, and puts it on a pile after reading it. However, if the pile contains 5 newspapers when she new the newspaper, she empty the pile. Also, at 8pm, with probability $\frac{1}{3}$, Amy takes all the papers in the pile and puts them in a trash can. Model this by a Markov chain and write the transition matrix.

Problem 5. Consider a Markov chain with state space $\{0, 1\}$ and transition matrix

$$\mathbf{P} = \begin{pmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{pmatrix}.$$

Suppose that the chain starts in state 0 at time $n = 0$. What is the probability that the chain is in state 1 at time $n = 3$?

Problem 6. Consider the Markov chain with state space $\{1, 2, 3\}$ and the transition matrix

$$\mathbf{P} = \begin{pmatrix} 2/10 & 4/10 & 4/10 \\ 1/10 & 5/10 & 4/10 \\ 6/10 & 3/10 & 1/10 \end{pmatrix}.$$

Compute

$$\lim_{n\to\infty} \mathbb{P}\{X_n = 1\}.$$

(Note that this limit does not depend on the initial distribution. Why?)

Problem 7. Consider the Markov chain with state space $\{1, 2, 3, 4, 5\}$ and the transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Compute the invariant distribution $\bar{\pi}$.

Problem 8. Consider simple random walk on the graph in Figure 1.

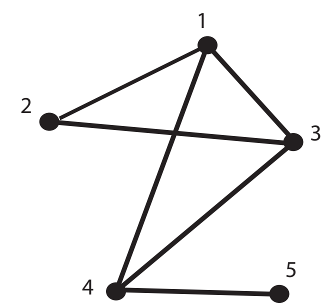

Figure 1: Graph for Problem

(a) Write the transition matrix $\mathbf{P}$.

(b) Compute the invariant distribution $\bar{\pi}$. For this problem you may assume that there is a unique invariant distribution: this will follow from a theorem (Theorem 1.14) to be discussed in the class. Assume that $\bar{\pi}$ is of the form $\bar{\pi} = (a, b, a, a, c)$. Notice that the vertices $1, 3, 4$ have the same number of edges coming out of them. Discuss your answer for $\bar{\pi}$ in terms of the number of edges coming out of each vertex.

Problem 9. (theoretical) In this problem, we prove Theorem 1.4.

(a) Since $\mathbf{P}^n$ is a stochastic matrix with strictly positive entries, due to Theorem 1.3, there is a unique invariant distribution $\hat{\pi}$ *for* $\mathbf{P}^n$. Now as $\mathbf{P}$ is a stochastic matrix, there is a left-eigenvector $\bar{w}$ for the eigenvalue 1: $\bar{w}\mathbf{P} = \bar{w}$. Show that $\bar{w}$ is a left-eigenvector of $\mathbf{P}^n$ corresponding to the eigenvalue 1. Deduce from this that $\bar{w} = c\hat{\pi}$ for some constant $c \neq 0$. This proves that there is a unique invariant distribution $\bar{\pi}$ for $\mathbf{P}$, and it is given by $\bar{\pi} = \hat{\pi}$.

(b) Let $\lambda \neq 1$ be an eigenvalue of $\mathbf{P}$. We would like to show that $|\lambda| < 1$. There is $\bar{v}$ such that $\bar{v}\mathbf{P} = \lambda\bar{v}$. Show that $\bar{v}\mathbf{P}^n = \lambda^n\bar{v}$. Show that if

$\lambda^n = 1$, then $\bar{v} = c'\bar{\pi}$ for some constant c'. Conclude from this that $\lambda^n \neq 1$. Theorem 1.3 applied to $\mathbf{P}^n$ implies that $|\lambda^n| < 1$, and hence we obtain that $|\lambda| < 1$. This completes the proof.