MATH 526 Homework 3. Due Tuesday 2/10

Be sure to give explanations to your answers. If you are using a theorem, please indicate it.

Problem 1. Consider the Markov chain with state space $\{1, 2, 3, 4, 5\}$ and the transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/5 & 4/5 \\ 0 & 0 & 0 & 2/5 & 3/5 \\ 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{pmatrix}.$$

(a) Is the chain irreducible?

(b) Find the period of each state.

(c) Compute $p_{1000}(2, 1)$ and $p_{1000}(2, 2)$ approximately.

Problem 2. Let $Y_1, Y_2, Y_3, \cdots$ be the successive values from independent rolls of a standard six-sided fair die. Let $S_n = Y_1 + Y_2 + \cdots + Y_n$.

(a) Compute

$$\lim_{n\to\infty} \mathbb{P}\{S_n \text{ is divisible by } 8\}.$$

(Hint: consider the remainder of S_n after division by 8 as a Markov chain.)

(a) Compute

$$\lim_{n\to\infty} \mathbb{P}\{S_n \text{ is divisible by } 5\}.$$

Problem 3. Let X_n be a Markov chain with state space $\{0, 1\}$ and transition matrix

$$\mathbf{P} = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{pmatrix}.$$

Let Y_n be an independent Markov chain with the same state space and the same transition matrix. Suppose that $X_0 = 0$ and $Y_0 = 1$. Compute

$$\lim_{n\to\infty} \mathbb{P}\{X_n = Y_n\}.$$

(Hint: Consider a new Markov chain $Z_n = (X_n, Y_n)$. This Markov chain has the state space $\{(0,0),(0,1),(1,0),(1,1)\}$. What is the transition matrix for Z_n? For example, the transition probability from $(0,1)$ to $(1,1)$ is given by $\mathbb{P}\{Z_1 = (1,1)|Z_0 = (0,1)\} = \mathbb{P}\{X_1 = 1|X_0 = 0\}\mathbb{P}\{Y_1 = 1|Y_0 = 1\} = p(0,1)p(1,1) = \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{8}$ due to the independence. In the end, re-write the event $\{X_n = Y_n\}$ in terms of an event involving Z_n.)

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Do not hand in the solutions to the following problems.

Problem 4. Consider the Markov chain with state space $\{0,1,2,3,4,5\}$ and the transition matrix

$$\mathbf{P} = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0.9 & 0 \\ 0.25 & 0.25 & 0 & 0 & 0.25 & 0.25 \\ 0 & 0 & 0.7 & 0 & 0.3 & 0 \\ 0 & 0.2 & 0 & 0.2 & 0.2 & 0.4 \end{pmatrix}.$$

What are the communication classes? Find the period of each state.

Problem 5. Consider the Markov chain with state space $\{1,2,3,4,5\}$ and the transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Is the chain irreducible?

(b) Find the period of each state.

(c) Compute $p_{25}(2,1)$ and $p_{25}(2,2)$. (Hint: Note that $25 = 3 \cdot 8 + 1$)

Problem 6. There is a production line where each item has probability p of being defective, independent of each other. Consider the following sampling plan. Initially we do 100% sampling: every item is sampled as it is produced. This continues until N *consecutive* items are non-defective. If this happens then sample only one item out of r items. Continue this until a defective item is found. When this happens, then go back to the 100% sampling and sample every items until N consecutive items are non-defective, etc.

We model this as a Markov chain with state space $\{0, 1, 2, \cdots, N, N+1\}$. The state i, $i = 0, 1, 2, \cdots, N$, denotes that i consecutive items have bee found to be non-defective in the 100% sampling. The state $N + 1$ denotes that the sampling plan is in the second stage (i.e. sampling one out of r items. Hence $X_n = i$, $i = 0, 1, \cdots, N$, means that at time n (i.e. after the immediate time after the nth inspection), i consecutive non-defective items have been found. On the other hand, $X_n = N$ means that at time n (i.e. after the immediate time after the nth inspection), we are at the second state.

(a) Show that

$$\mathbf{P} = \begin{pmatrix} p & 1-p & 0 & 0 & \cdots & 0 & 0 \\ p & 0 & 1-p & 0 & \cdots & 0 & 0 \\ p & 0 & 0 & 1-p & \cdots & 0 & 0 \\ p & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p & 0 & 0 & 0 & \cdots & 0 & 1-p \\ p & 0 & 0 & 0 & \cdots & 0 & 1-p \end{pmatrix}.$$

(b) Compute the limiting distributions $\lim_{n\to\infty} \mathbb{P}\{X_n = i\}$.

Problem 7. Suppose that the weather on a given day depends on the weather conditions for the previous 2 days. More concretely. suppose that if it was sunny today and yesterday, then it will be sunny tomorrow with probability .8; if it was sunny today but cloudy yesterday, then it will be sunny tomorrow with probability .6; if was cloudy today but sunny yesterday, then it will be sunny tomorrow with probability .4; if it was cloudy today and yesterday, then it will be sunny tomorrow with probability .1. Compute the probability that it will be sunny today next year approximately.

(Hint: Model this as a Markov chain with state space $\{(S,S),(S,C),(C,S),(C,C)\}$. For example, $X_n = (S,C)$ means that it was sunny on day $n-1$ and cloudy on day n. Hence for example, $\mathbb{P}\{X_1 = (C,C)|X_0 = (S,C)\} = \mathbb{P}\{\text{it will be cloudy on day 1}|\text{it was sunny on day } -1 \text{ but cloudy on day } 0\} = 1 - .4 = .6$. On the other hand, $\mathbb{P}\{X_1 = (S,C)|X_0 = (S,C)\} = 0$ since conditioned on that it was cloudy on day 0, it can't be sunny on day 0.)

Problem 8.(theoretical) Here we prove Lemma 1.11. An elementary theorem in number theory states the following: if two positive integers m and n are relatively prime (i.e. g.c.d.$\{m,n\} = 1$), then there exist integers (positive or negative) a and b such that

$$am + bn = 1.$$

(a) Suppose that m and n are relative prime. Show that the set

$$\{xm + yn : x \text{ and } y \text{ are } \underline{\text{positive}} \text{ integers}\}$$

contains the set $\{N, N+1, N+2, N+3, \cdots\}$ for some positive integer N.

(Hint: Observe that one of a or b (in $am+bn = 1$) is negative. Suppose that b is negative. Then $Am - Bn = 1$ where A and B are positive integers. Show that we can take $N = Bn^2$.)

(b) Let J be a set of positive integers, and let $d =$ g.c.d.J. Suppose that J is closed under addition (i.e. if $\alpha, \beta \in J$, then $\alpha + \beta \in J$). Show that J contains the set $\{Nd, (N+1)d, (N+2)d, (N+3)d, \cdots\}$ for some positive integer N. Discuss that Lemma 1.11 follows from this result.