MATH 526 Homework 4. Do Not Hand in.

Problem 1. Consider a Markov chain with $S = \{1, 2, 3, 4, 5, 6\}$ and

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

Suppose that initially the chain is either at state 1 or at 6 with equal probability. Compute

$$\lim_{n\to\infty} \mathbb{P}(X_n = i). \qquad \text{for } i = 1, 2, 3, 4, 5, 6.$$

Problem 2. Consider a random walk on $S = \{0, 1, 2, 3, 4\}$ with the reflecting boundary condition:

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Here $p \neq \frac{1}{2}$. Suppose that $X_0 = 0$.

(a) Compute

$$\lim_{n\to\infty} \mathbb{P}\{X_{2n} = i\}, \qquad \text{for } i = 0, 1, 2, 3, 4.$$

(b) If the chain starts at state 2, what is the expected number of steps until the chain returns to 2?

Problem 3. Consider a Markov chain with $S = \{1, 2, 3, 4\}$ and

$$\mathbf{P} = \begin{pmatrix} 0 & p & 0 & 1-p \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ p & 0 & 1-p & 0 \end{pmatrix}.$$

If the chain starts at i, what is the expected number of time steps until the chain returns to i, $i = 1, 2, 3, 4$.

Problem 4. Two containers A and B are placed adjacent to each other and gas is allowed to pass through a small aperture joining them. A total of M gas molecules is distributed between two containers. We assume that at each discrete time, one molecule, picked uniformly randomly from the M available, passes the aperture. Let X_n denote the number of molecules in container A at time n. Show that the invariant distribution is given by

$$\bar{\pi}(i) = \binom{M}{i} 2^{-M}, \qquad i = 0, 1, \cdots, M.$$

Problem 5. Consider a Markov chain with $S = \{0, 1\}$ and

$$\mathbf{P} = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

where $0 < \alpha, \beta < 1$. Show that $\bar{\pi} = (\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta})$.

Problem 6. Consider the following approach to shuffle a desk of N cards. Start with any initial ordering of the cards. At each time, a card is chosen uniformly at random, and is put on top of the deck. Repeat this procedure. Show that in the limit $n \to \infty$ (here n is the time), the deck is perfectly shuffled in the sense that the resultant ordering is equally likely to be any of the $N!$ possible orderings.

Problem 7.(theoretical) We prove Theorem 1.15 when $d = 2$. Let $S = A_1 \cup A_2$ such that the chain moves either from a state in A_1 to a state in

A_2, or from a state in A_2 to a state in A_1 at each time step. Note that by re-arranging the states we can assume that $\mathbf{P}$ is of the block-form,

$$\mathbf{P} = \begin{pmatrix} 0 & \mathbf{P}_1 \\ \mathbf{P}_2 & 0 \end{pmatrix}.$$

Consider a new Markov chain obtained from the original chain by observing only the evn times. This chain has state space S and the transition matrix

$$\hat{\mathbf{P}} = \mathbf{P}^2 = \begin{pmatrix} \mathbf{P}_1\mathbf{P}_2 & 0 \\ 0 & \mathbf{P}_2\mathbf{P}_1 \end{pmatrix}.$$

Even though the chain is irreducible, this new Markov chain has two communication classes A_1 and A_2, and the period of each state is 1. Hence there are unique invariant distributions $\bar{\pi}^1$ for $\mathbf{P}_1\mathbf{P}_2$ and $\bar{\pi}^2$ for $\mathbf{P}_2\mathbf{P}_1$.

(a) Show that $\bar{\pi}^1 = \bar{\pi}^2\mathbf{P}_2$ and $\bar{\pi}^2 = \bar{\pi}^1\mathbf{P}_1$. Conclude that $\bar{\pi} = (\frac{1}{2}\bar{\pi}^1, \frac{1}{2}\bar{\pi}^2)$ is an invariant distribution of $\mathbf{P}$. (Hint: You may want to use the fact that $\sum_j(\bar{\pi}^1\mathbf{P}_1)(j) = 1$.)

(b) Suppose that $\bar{v} = (\bar{v}^1, \bar{v}^2)$ is another probability distribution satisfying $\bar{v} = \bar{v}\mathbf{P}$. Show that $\bar{v} = \bar{\pi}$.

(c) Show that

$$\lim_{n\to\infty} \mathbf{P}^{2n} = \lim_{n\to\infty} \begin{pmatrix} (\mathbf{P}_1\mathbf{P}_2)^n & 0 \\ 0 & (\mathbf{P}_2\mathbf{P}_1)^n \end{pmatrix} = \begin{pmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{pmatrix}$$

where

$$\Pi_i = \begin{pmatrix} \bar{\pi}^1 \\ \vdots \\ \bar{\pi}^1 \end{pmatrix}, \qquad i = 1, 2.$$

Also show that

$$\lim_{n\to\infty} \mathbf{P}^{2n+1} = \lim_{n\to\infty} \begin{pmatrix} 0 & (\mathbf{P}_1\mathbf{P}_2)^n\mathbf{P}_1 \\ (\mathbf{P}_2\mathbf{P}_1)^n\mathbf{P}_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Pi_2 \\ \Pi_1 & 0 \end{pmatrix}.$$