MATH 526 Homework 5. Due Tuesday 3/10

Hand in the solutions to questions 1,2,5,6,7

Problem 1. Consider simple random walk on the graph in Figure 1.

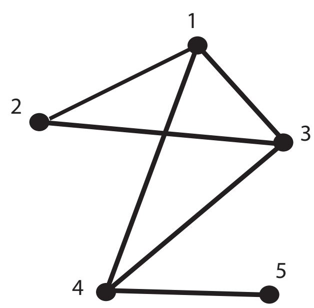

Figure 1: Graph for Problem

(a) Suppose that the walker starts in the vertex 2. What is the expected number of visits to the vertex 1 before it returns to the vertex 2?

(b) Suppose that the walker starts in the vertex 2. What is the probability that the walker reaches the vertex 1 before returning to the vertex 2?

Problem 2. Let $X_1, X_2, X_3, \cdots$ be successive values from independent rolls of a standard 6-sided die. Let $S_n = X_1 + X_2 + \cdots + X_n$. Let

$$T_0 = \min\{n \geq 1 : S_n \text{ is divisible by } 4\}.$$

and

$$T_1 = \min\{n \geq 1 : S_n - 1 \text{ is divisible by } 4\}.$$

Find $\mathbb{E}[T_0]$ and $\mathbb{E}[T_1]$.

Problem 3. Suppose we flip a fair coin repeatedly until we have flipped 4 consecutive heads. What is the expected number of flips that are needed?

(Hint: Consider a Markov chain with state space $\{0, 1, 2, 3, 4\}$.)

Problem 4. Consider the Markov chain with state space $\{0, 1, 2, 3, 4, 5\}$ and the transition matrix

$$\mathbf{P} = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0.9 & 0 \\ 0.25 & 0.25 & 0 & 0 & 0.25 & 0.25 \\ 0 & 0 & 0.7 & 0 & 0.3 & 0 \\ 0.1 & 0.1 & 0 & 0.2 & 0.2 & 0.4 \end{pmatrix}.$$

Suppose that the chain starts in state 5. What is the probability that the chain will be at state 0 after a long run?

Problem 5. Consider a graph with vertices $1, 2, 3, \cdots, N$ such that every pair of distinct points are connected by an edge. (Such a graph is called a complete graph.) Consider a simple random walk X_n on this complete graph. Let T be the first time that the walker reaches the vertex 1.

(a) Find the distribution of T when $X_0 = i$, for each $i = 1, 2, \cdots, N$.

(b) Suppose that $X_0 = 1$. Find the expected number of steps needed until every vertices has been visited at least once.

(Hint: Note that the expected number of steps needed until every vertices has been visited at least once equals $T_1 + T_2 + T_3 + \cdots + T_N$, where T_k is the number of steps until a new vertex is visited, starting from the moment that $k - 1$ vertices have been visited at least once for the first time.)

Problem 6. Consider the following queueing model. Let X_n denote the number of customers waiting in line for some service. The first person in line is being served while all others are waiting their turn. During each time interval there is a probability p that a new customer arrives. Also with probability q, the service for the first person is completed and that customer leaves the queue. There is no restriction on the number of customers waiting in line. This is a Markov chain with state space $S = \{0, 1, 2, 3, \cdots\}$ and transition probabilities

$$p(i, i-1) = q(1-p), \quad p(i,i) = qp + (1-q)(1-p), \quad p(i, i+1) = (1-q)p$$

for $i = 1, 2, 3, \cdots$, and

$$p(0,0) = 1-p, \qquad p(0,1) = p.$$

For which values of p and q is the chain transient?

Problem 7. Let $p > 1/2$. Consider a random walk on $\mathbb{Z}$ with the transition probabilities

$$p(i, i+1) = p, \quad p(i, i-1) = 1-p, \qquad i \in \mathbb{Z},$$

starting at the origin ($X_0 = 0$).

(a) For a positive integer k, let $T_k = \min\{n : X_n = k\}$. Set $e(k) = \mathbb{E}[T_k]$. Explain why $e(k) = ke(1)$.

(b) Compute $e(1)$. (Hint: Use (a).)

(Discussion: In your answer, take the limit $p \downarrow \frac{1}{2}$ and see that $e(1) = \infty$ when $p = \frac{1}{2}$.)

Problem 8. We will check that $\sum_{m=1}^{\infty} \frac{1}{m^\alpha} < \infty$ if and only if $\alpha > 1$. First show using the graph of the function $y = \frac{1}{x^\alpha}$ that

$$\frac{1}{(m+1)^\alpha} \leq \int_m^{m+1} \frac{1}{x^\alpha} dx \leq \frac{1}{m^\alpha}.$$

From this show that

$$\int_1^\infty \frac{1}{x^\alpha} dx \leq \sum_{m=1}^{\infty} \frac{1}{m^\alpha} \leq 1 + \int_1^\infty \frac{1}{x^\alpha} dx.$$

Now show that $\int_1^\infty \frac{1}{x^\alpha} dx < \infty$ if and only if $\alpha > 1$, by directly computing the integral.